SHAPE OPTIMISATION FOR THE ROBIN LAPLACIAN: AN OVERVIEW

We consider the eigenvalue problem for the Laplacian with Robin boundary conditions

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, sufficiently smooth domain, ν is the outer unit normal to $\partial\Omega$, and $\alpha \in \mathbb{R}$ is a constant. This recalls, and in many ways behaves like, the Dirichlet Laplacian if $\alpha > 0$ and the Neumann Laplacian if $\alpha < 0$.

In particular, we may pose the usual shape optimisation questions: for example, which domain minimises a given eigenvalue (if $\alpha > 0$) or maximises it (if $\alpha < 0$), assuming a volume constraint? Even for the lowest eigenvalues, this turns out to be a difficult problem, as most of the standard arguments used to treat the Dirichlet or Neumann Laplacians, such as symmetrisation, no longer work. Indeed, only in the last ten years or so has substantial progress been made.

We shall give a survey of known results as well as current open problems, focussing on the above-mentioned isoperimetric problem of optimising a fixed eigenvalue under a volume constraint. For the first eigenvalue an inequality of Faber–Krahn type holds if $\alpha > 0$, but for $\alpha < 0$ the question of the maximiser has not been entirely resolved. For the higher eigenvalues (and $\alpha > 0$), new phenomena emerge due to the behaviour of the Robin eigenvalues under homothetic scaling of the domain: the minimiser will, in general, no longer be independent of $\alpha > 0$, and numerical work suggests an increasingly complex picture as the number of the eigenvalue increases, with potential multiple minimisers for some particular values of α.