LAMA Université Savoie Mont Blanc

Optimal shapes maximizing the Steklov eigenvalues

Dorin Bucur

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Workshop on geometric spectral theory

The Steklov problem

Let $\Omega \subseteq \mathbb{R}^d$, bounded, Lipschitz.

$$\begin{cases} -\Delta u = & 0 \ \text{in} \ \ \Omega \\ \frac{\partial u}{\partial n} = & \sigma_k(\Omega) u \ \ \partial\Omega \end{cases}$$

Rayleigh quotient :

$$\sigma_k(\Omega) = \min_{S \in S_{k+1} \in H^1(\Omega)} \max_{u \in S \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_{\partial\Omega} |u|^2 d\mathcal{H}^{d-1}}$$

$$0 = \sigma_0(\Omega) \le \sigma_1(\Omega) \le \dots \le \sigma_k(\Omega) \le \dots \to +\infty.$$

Relationship with Robin boundary conditions

For $\beta \in \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^d$ bounded, Lipschitz

$$\begin{cases} -\Delta u = & \lambda u \ \text{ in } \ \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \ \ \partial\Omega \end{cases}$$

The mapping

$$\beta \mapsto \lambda_k(\beta)$$

is well defined, increasing, and at some point will cross zero, for some negative β.

Relationship with Neumann to Dirichlet operator

One can define $T : L^2(\partial\Omega)_{/\mathbb{R}} \mapsto L^2(\partial\Omega)_{/\mathbb{R}}$, by

$$T(f) = u|_{\partial\Omega}$$

where

$$\begin{cases} -\Delta u = & 0 \ \text{ in } \ \Omega \\ \frac{\partial u}{\partial n} = & f \ \ \partial\Omega \end{cases}$$

Crucial in inverse problems...

Eigenvalues of the disc

Let $\Omega = B(0,1)$, then

$$\sigma(\Omega) = 0, 1, 1, 2, 2, \dots, k, k, \dots,$$

with eigenfunctions

$$r^k \cos(k\theta), r^k \sin(k\theta).$$

Scaling

$$\sigma_k(t\Omega) = \frac{1}{t}\sigma_k(\Omega).$$

Shape optimization problems

Natural questions :

$$\max\{\sigma_k(\Omega) : |\Omega| = c\} \qquad \max\{\sigma_k(\Omega) : Per(\Omega) = c\}$$

or

$$\max\{F(\sigma_1(\Omega), ..., \sigma_k(\Omega)) : |\Omega|, Per(\Omega) = c\}$$

- Minimization is trivial and does not have any signification.
- For $k = 1$ this is related to the best constant in extension theorems.
- In geometry, the natural constraint is the *perimeter* : optimization of metrics on Riemannian manifolds.

Results : we stay in $\mathbb{R}^d$

- Weinstock 1954, $\sigma_1(\Omega)\mathcal{H}^1(\partial\Omega)$ is maximal for the disc, in the family of simply connected, smooth sets in $\mathbb{R}^2$.
- Hersch, Payne, Schiffer 1974, $\sigma_k(\Omega)\mathcal{H}^1(\partial\Omega)$ is maximal for the union of k discs, in the family of *simply connected*, smooth sets in $\mathbb{R}^2$. Equality for k disjoint disks : Girouard and Polterovich.
- Hersch, Payne, Schiffer 1974, upper bounds for $\sigma_k(\Omega)\sigma_j(\Omega)[\mathcal{H}^1(\partial\Omega)]^2$ (not all sharp).
- Brock 2001, the ball minimizes the sum $\sum_{k=1}^{d} \frac{1}{\sigma_k(\Omega)}$, among all smooth sets of $\mathbb{R}^d$ of prescribed measure.
- The same, under perimeter and convexity constraints in $\mathbb{R}^d$ (2017, B.-Ferone, Nitsch, Trombetti - ongoing work).

Our problem

$$\max\{\sigma_k(\Omega) : \Omega \subseteq \mathbb{R}^d, |\Omega| = c\}$$

or

$$\max\{F(\sigma_1(\Omega), \ldots, \sigma_k(\Omega)) : \Omega \subseteq \mathbb{R}^d, |\Omega| = c\}$$

Equivalent constraint

$$|\Omega| \geq c.$$

Questions

- Existence of solution.
- Qualitative properties.
- Numerical computations.

Theorem (Bogosel, B., Giacomini)

Problem

$$\max\{F(\sigma_1(\Omega),\ldots,\sigma_k(\Omega)) : \Omega \subseteq \mathbb{R}^d, |\Omega| = c\},$$

has a solution (F increasing, u.s.c.)

- *in the class of measurable sets of $\mathbb{R}^d$ with finite perimeter. The maximizer is bounded, the perimeter and diameter are controlled.*
- *in $\mathbb{R}^2$ in the class of open sets. The maximizer is union of at most k disjoint, bounded, Jordan domains, with topological boundary of finite length and controlled diameter.*

For the perimeter constraint, it is suspected that no existence occurs, in general.

Bogosel, B., Giacomini 2016

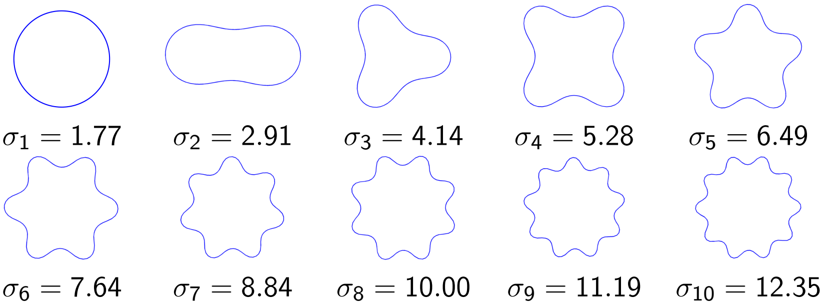


FIGURE: Shapes which maximize the k-th Steklov eigenvalue under area constraint, $k = 1, 2, 3, ..., 10$.

Existence of maximizers : "relaxed spectrum"

- on an arbitrary bounded open set $\Omega \subseteq \mathbb{R}^d$

$$\sigma_k(\Omega) = \inf_{S \in S_{k+1} \in H^1(\mathbb{R}^d)} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} |u|^2 d\mathcal{H}^{d-1}}$$

- on a measurable set of finite perimeter $\Omega \subseteq \mathbb{R}^d$

$$\sigma_k(\Omega) = \inf_{S \in S_{k+1} \in H^1(\mathbb{R}^d)} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial^*\Omega} |u|^2 d\mathcal{H}^{d-1}}$$

Remarks on the relaxed spectrum

- If Ω is not smooth, the risk we take is that $\sigma_k(\Omega)$ is small, possibly zero !
- A priori, the *inf* is not known to be attained. At optimal sets, it is expected to have smoothness.
- Cracks are not seen neither by the measurable sets nor by the definition we have on open sets ! This is not a problem, since one can easily prove that cracked domains are excluded from optimality (e.g. the Dirichlet problem).
- Brock's result holds true as well in both the frameworks of measurable sets or open sets, above.

How to prove existence of a solution ?

- take a maximizing sequence (Ω_n)
- by some compactness identify a limit set Ω^*
- verify that the measure constraint is satisfied
- prove that the spectrum is upper semicontinuous

$$\sigma_k(\Omega^*) \geq \limsup_{n\to\infty} \sigma_k(\Omega_n).$$

Then

$$F(\sigma_1(\Omega^*), \ldots, \sigma_k(\Omega^*)) \geq \limsup_{n\to\infty} F(\sigma_1(\Omega_n), \ldots, \sigma_k(\Omega_n)).$$

1^{st} key ingredient : isoperimetric control of the spectrum

- Hersch, Payne, Schiffer 1974 : simply connected, planar domains

$$\sigma_k(\Omega)\mathcal{H}^1(\partial\Omega) \le 2\pi k.$$

- Colbois, Girouard, El Soufi 2011
  - arbitrary Lipschitz domain in $\mathbb{R}^d$

$$\sigma_k(\Omega)(\mathcal{H}^{d-1}(\partial\Omega))^{\frac{1}{d-1}} \le C(d)k^{\frac{2}{d}}.$$

  - if $d \ge 3$, a stronger inequality holds

$$\sigma_k(\Omega)(\mathcal{H}^{d-1}(\partial\Omega))^{\frac{1}{d-1}} \le C(d)k^{\frac{2}{d}} \frac{|\Omega|^{\frac{d-2}{d}}}{\mathcal{H}^{d-1}(\partial\Omega)^{\frac{d-2}{d-1}}}.$$

Proof of Colbois, Girouard, El Soufi

Based on a measure theoretic result.

Theorem (Grigor'yan, Netrusov, Yau 2004)

Let μ be a non atomic non negative measure on $\mathbb{R}^d$ and $X \subseteq \mathbb{R}^d$. There exists a constant $c > 0$ such that for every $k \in \mathbb{N}$, there exists a family of k annuli $(A_i)_{i=1}^k$ such that

- *the annuli $2A_i$ are mutually disjoint, where* $2A_i = K(x_i, \frac{r_i}{2}, 2R_i)$
- $$\mu(A_i \cap X) \geq c\frac{\mu(X)}{k}.$$

Conclusion of the isoperimetric control

For a maximizing sequence for σ_k, the perimeters of $(\Omega_n)_n$ are uniformly bounded.

This naturally leads to work with measurable sets with uniformly bounded perimeter : we prove that the same isoperimetric control holds for the relaxed values.

Compactness holds in L^1 provided the minimizing sequence is equi-bounded.

2^{nd} key ingredient : isodiametric control of the spectrum

We want to get for every connected set Ω

$$\sigma_k(\Omega) diam(\Omega) \leq C(d, k)$$

True for convex sets (Bogosel 2015) ! Was unknown for arbitrary sets.

Theorem (Bogosel, B., Giacomini 2016)

There exists a constant $C(d)$ such that for every $k \in \mathbb{N}$ and every measurable set with finite perimeter we have either

$$\sigma_k(\Omega) diam(\Omega) \leq C(d) k^{\frac{2}{d}+1}$$

or Ω is (non trivially) contained in two disjoint, concentric annuli lying at positive distance (i.e. disconnected).

Proof

- Uniform relative isoperimetric inequality in annuli.
- Analysis argument, in the style of Caffarelli.

Either the set is disconnected (so situation 2 occurs)

or it has a "long connected piece" sufficiently narrow somewhere such that a local function with low Rayleigh can be built.

Open problem : find the sharp growth

$$\sigma_k(\Omega) diam(\Omega) \leq C(d) k^{\alpha}$$

Probably k is given by the growth power in the Weyl formula ($\alpha = \frac{2}{d}$? ? ?).

Conclusion of the isodiametric control

For a maximizing sequence for σ_k, either the diameters of $(\Omega_n)_n$ are uniformly bounded, or else, they are disconnected at positive distance.

If they are disconnected, there are at most k connected components, and each diameter is controlled.

Existence of a solution occurs

- we have compactness in L^1
- the limit set is measurable, has finite perimeter and satisfies the measure constraint
- it remains to prove the upper semicontinuity : quite easy and standard
  - continuity of the gradient norms for finite dimension spaces
  - lower semicontinuity of the boundary norm

The two dimensional case : open sets

The minimiser is union of at most k disjoint, bounded, Jordan domains, with topological boundary of finite length.

- we start with arbitrary open sets
- we use a monotonicity property and replace them with unions of simply connected sets

$$\text{if } \Omega_1 \subseteq \Omega_2, \partial\Omega_2 \subseteq \partial\Omega_1 \text{then } \forall k \in \mathbb{N}, \sigma_k(\Omega_1) \leq \sigma_k(\Omega_2).$$

  or (weaker)

$$\sigma_k(\Omega_1)|\Omega_1|^{\frac{1}{2}} \leq \sigma_k(\Omega_2)|\Omega_2|^{\frac{1}{2}}.$$

- we prove that the diameter and perimeters are uniformly bounded
- identify the Hausdorff limits and prove that they are Jordan domains (quite technical)

Is our max the good one ?

- Are the maximum in the class of measurable sets and the supremum in the class of Lipschitz sets equal ?
- Yes, for the Brock inequality...
- Otherwise, this is a regularity result...
- We can not prove it in general : need control of the L^∞ norm of the eigenfunctions with respect to the eigenvalues.
  - For Dirichlet b.c.

$$\|u\|_\infty \leq C(d)\lambda^{\frac{d}{4}}\|u\|_2$$

  - For Steklov, such an inequality fails ...

$$\exists \Omega_\varepsilon, |\Omega_\varepsilon| \approx \pi, |\partial\Omega_\varepsilon| \approx 2\pi, \sigma_1(\Omega_\varepsilon) = \frac{1}{2}, \frac{\|u_\varepsilon\|_\infty}{\|u_\varepsilon\|_2} \to +\infty$$

Numerical tests

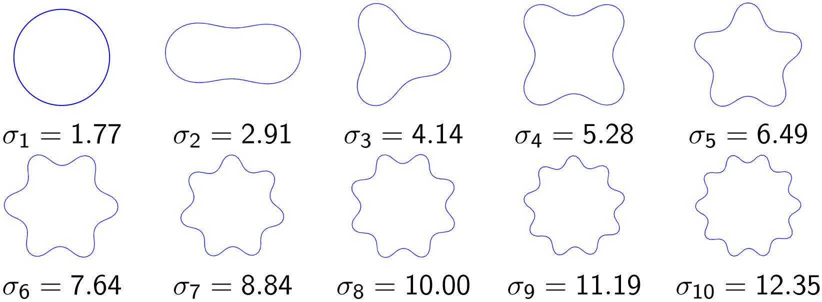
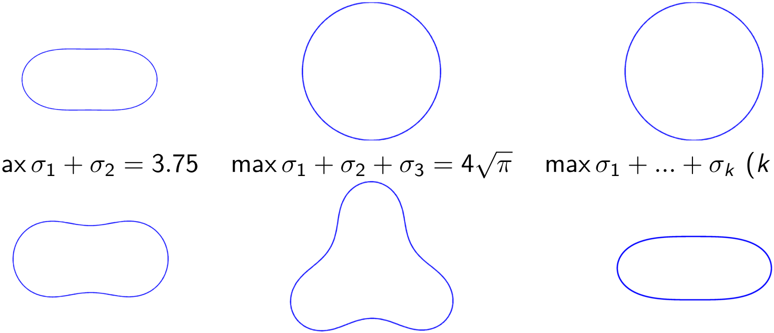


FIGURE: Shapes which maximize the k-th Steklov eigenvalue under area constraint, $k = 1, 2, 3, ..., 10$.

Numerical tests



$\max \sigma_1 + \sigma_2 = 3.75$ $\max \sigma_1 + \sigma_2 + \sigma_3 = 4\sqrt{\pi}$ $\max \sigma_1 + ... + \sigma_k \ (k \leq 10)$

$\max \sigma_2 \cdot \sigma_3 = 8.69$ $\max \sigma_3 \cdot \sigma_4 = 17.18$ $\max \sigma_2 \cdot \sigma_3 \cdot \sigma_4 = 29.59$

FIGURE: Numerical optimizations of functionals depending on the Steklov spectrum

- **(Conjecture)** the maximizers of σ_k with area constraint are connected and have the symmetry of a regular k-gons. Furthermore, we observe that at the optimum the eigenvalues are multiple, the multiplicity cluster starts at k and has length 3 when k is odd and 2 when k is even.
- **(Conjecture)** the product $\sigma_1\sigma_2...\sigma_k$ is maximized by the disk.
- **(Conjecture)** $\sum_{k\in A} \frac{1}{\sigma_k}$ is minimized by the disk if $1 \in A$ and $2k \in A \Rightarrow 2k-1 \in A$. Example

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_3} + \frac{1}{\sigma_4}$$

- **(Conjecture)** Hersch-Payne-Schiffer $\sum_{k=1}^{n} \frac{1}{\sigma_{2k-1}\sigma_{2k}}$ is minimized by the disk.

Spectral optimization problems, Robin eigenvalue with negative parameter $-\beta < 0$

$\Omega \subseteq \mathbb{R}^N$, bounded, open, Lipschitz

$$\begin{cases} -\Delta u = & \lambda u \text{ in } \Omega \\ \frac{\partial u}{\partial n} - \beta u = & 0 \ \partial\Omega \end{cases}$$

$$\lambda_1(\Omega, -\beta) \le \lambda_2(\Omega, -\beta) \le \lambda_3(\Omega, -\beta) \le \ldots \mapsto +\infty.$$

Spectral optimization problems, Robin eigenvalue with negative parameter $-\beta < 0$

$$\lambda_k(\Omega, \beta) = \min_{S_k \subseteq H^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx - \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\int_\Omega u^2 dx}$$

- $$\lambda_1(\Omega, \beta) \leq -\beta \frac{|\partial\Omega|}{|\Omega|} < 0$$
- For k large, the eigenvalues become positive.

Spectral optimization problems, Robin eigenvalue with negative parameter $-\beta < 0$

$$\max_{|\Omega|=c} \lambda_k(\Omega, \beta)$$

Results :

- $k = 1$, Freitas-Krejcirik $-\beta$ close to 0 in $\mathbb{R}^2$, the solution is the ball
- $k = 1$, Freitas-Krejcirik $-\beta$ close to $-\infty$ in $\mathbb{R}^2$, the solution is NOT the ball

Existence of solutions

Relaxation on measurable sets of with finite perimeter : cracks are not desirable

$$\lambda_k(\Omega, \beta) = \min_{S_k \subseteq H^1(\mathbb{R}^d)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx - \beta \int_{\partial^* \Omega} |u|^2 d\mathcal{H}^{N-1}}{\int_\Omega u^2 dx}$$

Theorem (ongoing work with S. Cito 2017)

A solution exists in $\mathbb{R}^d$, for every $k \in \mathbb{R}^d$.

Proof : two key estimates.

Isodiametric control

Let $\Omega \subseteq \mathbb{R}^d$ measurable of finite perimeter and $A \in \mathbb{R}$ s.t.

$$\lambda_k(\Omega, -\beta) \geq -A$$

Then

$$\exists N = N(|\Omega|, \beta, d, k, A), \quad D = D(|\Omega|, \beta, d, k, A)$$

such that

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n, \quad n \leq N$$

and

$$\text{dist}\,(\Omega_i, \Omega_j) > 0, \qquad \text{diam}\,(\Omega_i) \leq D$$

Attention : n might be larger than k ! ! !

- If $\lambda_k(\Omega, \beta) > 0$, then $N = k$.
- No better control of the connected components when $\lambda_k(\Omega, -\beta) < 0$.
- We can not control by k, even for $k = 1$.
- J. Kennedy : if the optimal shape were smooth for σ_1, then it has to be connected.

Isoperimetric control

In the spirit of Colbois, Girouard, El Soufi.

Theorem (ongoing work with S. Cito 2017)

Exists $C_1, C_2 > 0$ dimensional constants, s.t.

$$\lambda_k(\Omega, -\beta) \le -C_2 \Big(\frac{k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1} |\partial^* \Omega|}{|\Omega|} \Big)^- + \\ + \frac{C_1 k}{2|\Omega|^{\frac{2}{d}}} \Big(k^{\frac{2}{d}} |\Omega|^{\frac{d-2}{d}} - \frac{\beta}{C_1} |\partial^* \Omega| \Big)^+$$

Proof of existence

A maximizing sequence for $\lambda_k(\Omega, -\beta)$ has

- uniformly bounded number of connected components
- uniformly bounded diameters
- uniformly bounded perimeters

So existence by the direct method.

Thank you for your attention !