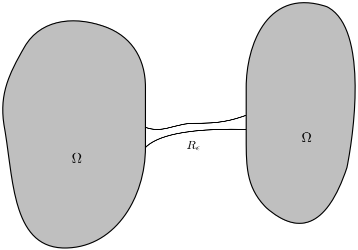
Spectral analysis of a Neumann biharmonic operator on a planar dumbbell domain

Francesco Ferraresso
(based on a joint work with J.M. Arrieta and P.D. Lamberti)

Workshop on geometric spectral theory
Neuchatêl - 22.06.2017

Università degli Studi di Padova

The dumbbell



(a) *Dumbbell domain*.

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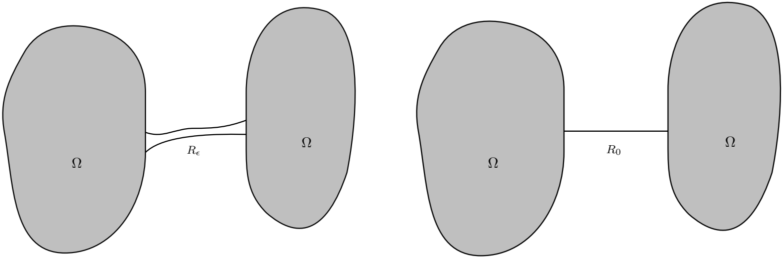

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Aim: better understanding of this "pathological" cases.

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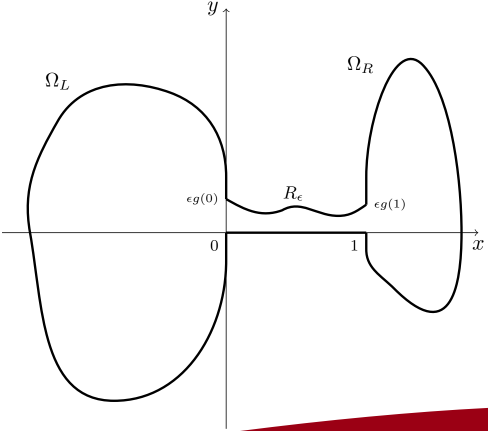

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We denote by $(\varphi_n^\epsilon, \lambda_n(\Omega_\epsilon))$ the eigenpairs $\forall n \in \mathbb{N}$.

Spectral convergence

Definition

Given the operator T_ϵ on $L^2(\Omega_\epsilon)$, we say that T_ϵ is spectrally converging to T_0 on $L^2(\Omega_0)$, with $\Omega_0 \subset \overline{\Omega_\epsilon}$ for all $\epsilon > 0$

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Arrieta's condition $\Rightarrow$ Spectral convergence

In our case, Arrieta's condition is related to the limit as $\epsilon \to 0$ of

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Concentration phenomenon $\Rightarrow$ Spectral instability

There must be extra eigenvalues in the limit!

Main problem

Question: can we characterize these extra eigenvalues?

Two auxiliary problems

We introduce the eigenpairs $(\varphi_k^{\Omega}, \omega_k)$ of

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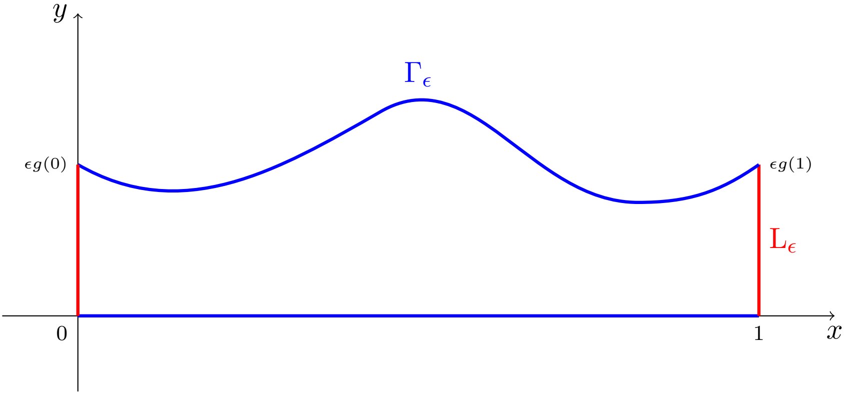
and the eigenpairs $(\gamma_l^{\epsilon}, \theta_l^{\epsilon})$ of

$$\begin{cases} \Delta^2 v - \tau \Delta v + v = \theta_l^{\epsilon}\, v, & \text{in } R_{\epsilon} \\ (1-\sigma)\frac{\partial^2 v}{\partial n^2} + \sigma \Delta v = 0, & \text{on } \Gamma_{\epsilon} \\ \tau \frac{\partial v}{\partial n} - (1-\sigma)\, \mathrm{div}_{\Gamma_{\epsilon}}(D^2 v \cdot n)_{\Gamma_{\epsilon}} - \frac{\partial(\Delta v)}{\partial n} = 0, & \text{on } \Gamma_{\epsilon}, \\ v = 0 = \frac{\partial v}{\partial n}, & \text{on } L_{\epsilon}. \end{cases}$$

where $\Gamma_{\epsilon} = \{(x, y) : 0 < x < 1, y = \epsilon g(x)\} \cup \{(x, 0) : 0 < x < 1\}$ and $L_{\epsilon} = \partial R_{\epsilon} \setminus \Gamma_{\epsilon}$.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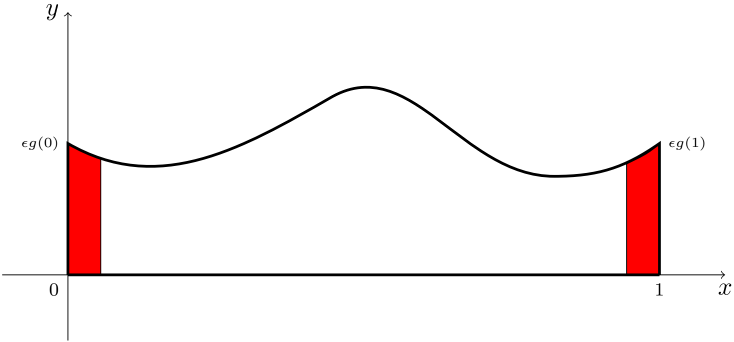
This condition is usually called H-Condition:



Asymptotic spectral decomposition

Recall that $\lambda_n(\Omega_\epsilon)$ are the eigenvalues of

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$$|\lambda_n(\Omega_\epsilon) - \lambda_n^\epsilon| \to 0, \qquad \text{as } \epsilon \to 0.$$

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- Abstract result: compact convergence of resolvent operators implies spectral convergence.

The limit problem

One can prove that

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compact converges to a suitable rescaling of

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We denote by (h_i, θ_i) the eigenpairs associated with this ODE.

Notation

Define the operator $\mathcal{E}_\epsilon : H^2(0,1) \to H^2(R_\epsilon)$ by

$$\mathcal{E}_\epsilon v(x,y) = v(x)$$

for all $(x,y) \in R_\epsilon$. Moreover let $N(\cdot)$ be the counting function defined by

$$N(x) = \#\{\lambda_i : i \in \mathbb{N}, \lambda_i \leq x\}$$

Final convergence result

Theorem (Arrieta, F., Lamberti, 2016)

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$$\left\| \varphi_n^\epsilon|_\Omega - \sum_{i=1}^{N(\omega_k)} (\varphi_n^\epsilon, \varphi_i^\Omega)_{L^2(\Omega)} \varphi_i^\Omega \right\|_{H^2(\Omega)} \to 0$$

as $\epsilon \to 0$*.*

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Theorem (Arrieta, F., Lamberti, 2016)

Let $\Omega_\epsilon \subset \mathbb{R}^2$ be a dumbbell domain satisfying the (H)-condition. The eigenvalues $\lambda_n(\Omega_\epsilon)$ converge either to ω_k or to θ_l. Moreover, if $\lambda_n(\Omega_\epsilon) \to \omega_k$ for some $k \in \mathbb{N}$, then $\left\|\varphi_n^\epsilon|_\Omega\right\|_{L^2(\Omega)} \to 1$, and

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$$\left\|\varphi_n^\epsilon - \sum_{i=1}^{N(\theta_l)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)}\epsilon^{-1/2}\mathcal{E}_\epsilon h_i\right\|_{L^2(R_\epsilon)} \to 0$$

as $\epsilon \to 0$.

Principal references

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Thank you
for your attention

Open questions

- 1. Identification of the limit behavior of the eigenfunction φ_n^ϵ in $[0,1]$ when $\lambda_n(\Omega_\epsilon) \to \omega_k$.

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- 2. Replacement of the $L^2(R_\epsilon)$ norm with a stronger Sobolev norm in the final convergence result.

Partial results are available in the case $\sigma = 0$, the case $\sigma \neq 0$ being open at the moment.