Asymptotic behaviour of cuboids optimising Laplacian eigenvalues

Katie Gittins[1] Simon Larson[2]

[1]Université de Neuchâtel

[2]KTH Royal Institute of Technology

Geometric Spectral Theory, Neuchâtel
19th June 2017

Dirichlet eigenvalues

Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be an open set of finite Lebesgue measure $|\Omega| < \infty$. **Dirichlet** eigenvalues of the Laplacian on Ω, $\lambda_k(\Omega) \in \mathbb{R}$, satisfy

$$\begin{cases} -\Delta u_k(x) = \lambda_k(\Omega) u_k(x) & x \in \Omega, \\ u_k(x) = 0 & x \in \partial\Omega, \end{cases}$$

and form a non-decreasing sequence, counted with multiplicities,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \ldots$$

Goal

If, for each $k \in \mathbb{N}$, an open set $\Omega_k^* \subset \mathbb{R}^m$ exists such that, for prescribed $c > 0$,

$$\lambda_k(\Omega_k^*) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ open}, |\Omega| = c\},$$

then determine the asymptotic behaviour of $(\Omega_k^*)_k$ as $k \to \infty$.

Dirichlet eigenvalues on cuboids of unit measure

Let $R = \prod_{i=1}^{m}(0, a_i) \subset \mathbb{R}^m$ where $a_1, a_2, \ldots, a_m \in \mathbb{R}$ such that $0 < a_1 \leq a_2 \leq \cdots \leq a_m$, $\prod_{i=1}^{m} a_i = 1$.

Dirichlet eigenvalues on cuboids of unit measure

Let $R = \prod_{i=1}^{m}(0, a_i) \subset \mathbb{R}^m$ where $a_1, a_2, \ldots, a_m \in \mathbb{R}$ such that $0 < a_1 \leq a_2 \leq \cdots \leq a_m$, $\prod_{i=1}^{m} a_i = 1$.

For $R \subset \mathbb{R}^m$, $\lambda_k(R)$ obeys the two-term asymptotic formula

$$\lambda_k(R) = 4\pi\Gamma\left(\frac{m}{2} + 1\right)^{2/m} k^{2/m} + \frac{2\pi\Gamma(\frac{m}{2} + 1)^{1+1/m}}{m\Gamma(\frac{m+1}{2})}|\partial R|k^{1/m} + o(k^{1/m}),$$

as $k \to \infty$.

Dirichlet eigenvalues on cuboids of unit measure

Let $R = \prod_{i=1}^{m}(0, a_i) \subset \mathbb{R}^m$ where $a_1, a_2, \ldots, a_m \in \mathbb{R}$ such that $0 < a_1 \leq a_2 \leq \cdots \leq a_m$, $\prod_{i=1}^{m} a_i = 1$.

For $R \subset \mathbb{R}^m$, $\lambda_k(R)$ obeys the two-term asymptotic formula

$$\lambda_k(R) = 4\pi\Gamma\left(\frac{m}{2}+1\right)^{2/m} k^{2/m} + \frac{2\pi\Gamma(\frac{m}{2}+1)^{1+1/m}}{m\Gamma(\frac{m+1}{2})}|\partial R| k^{1/m} + o(k^{1/m}),$$

as $k \to \infty$.

Antunes and Freitas (2012): any sequence of minimising rectangles $(R_k^*)_k$ of unit area for λ_k converges to the unit square as $k \to \infty$.

van den Berg and Gittins (2016): corresponding result holds for $m = 3$.

Dirichlet eigenvalues on cuboids of unit measure

Dirichlet eigenvalues of the Laplacian on $R = \prod_{i=1}^{m}(0, a_i)$ have the form

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \cdots + \frac{\pi^2 i_m^2}{a_m^2}, \quad i_1, i_2, \ldots, i_m \in \mathbb{N}.$$

Dirichlet eigenvalues on cuboids of unit measure

Dirichlet eigenvalues of the Laplacian on $R = \prod_{i=1}^{m}(0, a_i)$ have the form

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \cdots + \frac{\pi^2 i_m^2}{a_m^2}, \quad i_1, i_2, \ldots, i_m \in \mathbb{N}.$$

Let $\Lambda \geq 0$ and let $a_1, a_2, \ldots, a_m \in \mathbb{R}$ be such that $\prod_{i=1}^{m} a_i = 1$, $0 < a_1 \leq a_2 \leq \cdots \leq a_m$. Define

$$E(\Lambda, R) := \left\{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \sum_{j=1}^{m} \frac{x_j^2}{a_j^2} \leq \frac{\Lambda}{\pi^2} \right\}.$$

Dirichlet eigenvalues on cuboids of unit measure

Dirichlet eigenvalues of the Laplacian on $R = \prod_{i=1}^{m}(0, a_i)$ have the form

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \cdots + \frac{\pi^2 i_m^2}{a_m^2}, \quad i_1, i_2, \ldots, i_m \in \mathbb{N}.$$

Let $\Lambda \geq 0$ and let $a_1, a_2, \ldots, a_m \in \mathbb{R}$ be such that $\prod_{i=1}^{m} a_i = 1$, $0 < a_1 \leq a_2 \leq \cdots \leq a_m$. Define

$$E(\Lambda, R) := \left\{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \sum_{j=1}^{m} \frac{x_j^2}{a_j^2} \leq \frac{\Lambda}{\pi^2} \right\}.$$

Dirichlet eigenvalues $\lambda_1(R), \ldots, \lambda_k(R)$ correspond to integer lattice points with positive coordinates that lie inside or on the ellipsoid $E(\lambda_k(R), R)$.

Dirichlet eigenvalues on cuboids of unit measure

Theorem (G - Larson, 2017)

Let $m \geq 4$. For $k \in \mathbb{N}$, let $R_k^* = R_{a_{1,k}^*, a_{2,k}^*, \ldots, a_{m,k}^*}$ denote an m-dimensional cuboid which minimises λ_k where $0 < a_{1,k}^* \leq a_{2,k}^* \leq \cdots \leq a_{m,k}^*$, $\prod_{i=1}^m a_{i,k}^* = 1$. Then, as $k \to \infty$,

$$a_{m,k}^* = 1 + O(k^{(\theta_m - (m-1))/2m}),$$

where θ_m is any exponent such that for all $a_1, \ldots, a_m \in \mathbb{R}_+$

$$\#\{(i_1, \ldots, i_m) \in \mathbb{Z}^m : a_1^{-2} i_1^2 + \cdots + a_m^{-2} i_m^2 \leq r^2\} - \omega_m r^m \prod_{i=1}^m a_i = O(r^{\theta_m}),$$

as $r \to \infty$ uniformly for a_i on compact subsets of $\mathbb{R}_+$ (ω_m is the measure of a ball in $\mathbb{R}^m$ with radius 1).

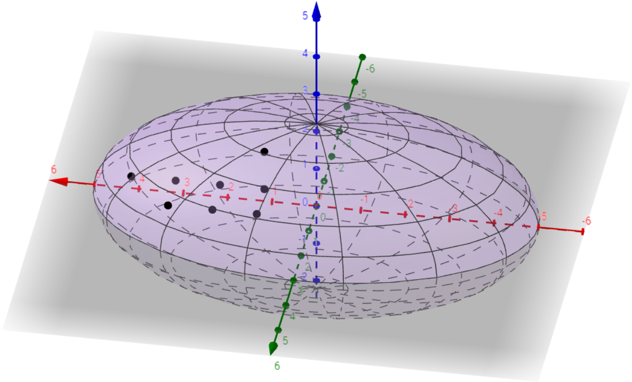
Smallest exponents known are $\theta_4 = \frac{12}{5}$, and $\theta_m = m - 2$ for $m \geq 5$.

Geometric convergence

Dirichlet counting function on R is

$$N(\Lambda, R) := \#\{(i_1, i_2, \dots, i_m) \in \mathbb{N}^m \cap E(\Lambda, R)\}.$$

Express $N(\Lambda, R)$ in terms of lattice-point sums over successive lower-dimensional ellipsoids.



Geometric convergence

Dirichlet counting function on R is

$$N(\Lambda, R) := \#\{(i_1, i_2, \ldots, i_m) \in \mathbb{N}^m \cap E(\Lambda, R)\}.$$

Lemma (G - Larson, 2017)

For $m \geq 2$ and $R = \prod_{i=1}^m (0, a_i) \subset \mathbb{R}^m$,

$$N(\Lambda, R) = L_{0,m}^{cl} |R| \Lambda^{m/2} - \frac{L_{0,m-1}^{cl}}{4} |\partial R| \Lambda^{(m-1)/2} + O(\Lambda^{\theta_m/2}),$$

as $\Lambda \to \infty$, where θ_m is any exponent such that for all $a_1, \ldots, a_m \in \mathbb{R}_+$

$$\#\{(i_1, \ldots, i_m) \in \mathbb{Z}^m : a_1^{-2} i_1^2 + \cdots + a_m^{-2} i_m^2 \leq r^2\} - \omega_m r^m \prod_{i=1}^m a_i = O(r^{\theta_m}),$$

as $r \to \infty$ uniformly for a_i on compact subsets of $\mathbb{R}_+$.

Smallest exponents known are $\theta_2 = \frac{2}{3}$, and $\theta_3 = \frac{3}{2}$.

Geometric convergence

Since $\lambda_k(Q) \geq \lambda_k^* = \lambda_k(R_k^*)$, for $\varepsilon > 0$, $N(\lambda_k^* - \varepsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

Geometric convergence

Since $\lambda_k(Q) \geq \lambda_k^* = \lambda_k(R_k^*)$, for $\varepsilon > 0$, $N(\lambda_k^* - \varepsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

$$L^{\mathrm{cl}}_{0,m}(\lambda_k^* - \varepsilon)^{m/2} - \frac{L^{\mathrm{cl}}_{0,m-1}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(m-1)/2} - O((\lambda_k^* - \varepsilon)^{\theta_m/2})$$
$$\leq L^{\mathrm{cl}}_{0,m}(\lambda_k^*)^{m/2} - \frac{L^{\mathrm{cl}}_{0,m-1}}{4}|\partial R_k^*|(\lambda_k^*)^{(m-1)/2} + O((\lambda_k^*)^{\theta_m/2}).$$

Geometric convergence

Since $\lambda_k(Q) \geq \lambda_k^* = \lambda_k(R_k^*)$, for $\varepsilon > 0$, $N(\lambda_k^* - \varepsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

$$L_{0,m}^{\mathrm{cl}}(\lambda_k^* - \varepsilon)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(m-1)/2} - O((\lambda_k^* - \varepsilon)^{\theta_m/2})$$

$$\leq L_{0,m}^{\mathrm{cl}}(\lambda_k^*)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial R_k^*|(\lambda_k^*)^{(m-1)/2} + O((\lambda_k^*)^{\theta_m/2}).$$

Rearranging and choosing $\varepsilon = O(1)$,

$$|\partial R_k^*| - |\partial Q| \leq O((\lambda_k^*)^{(\theta_m-(m-1))/2}) = O(k^{(\theta_m-(m-1))/m}),$$

Geometric convergence

Since $\lambda_k(Q) \geq \lambda_k^* = \lambda_k(R_k^*)$, for $\varepsilon > 0$, $N(\lambda_k^* - \varepsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

$$L_{0,m}^{\mathrm{cl}}(\lambda_k^* - \varepsilon)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(m-1)/2} - O((\lambda_k^* - \varepsilon)^{\theta_m/2})$$
$$\leq L_{0,m}^{\mathrm{cl}}(\lambda_k^*)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial R_k^*|(\lambda_k^*)^{(m-1)/2} + O((\lambda_k^*)^{\theta_m/2}).$$

Rearranging and choosing $\varepsilon = O(1)$,

$$|\partial R_k^*| - |\partial Q| \leq O((\lambda_k^*)^{(\theta_m-(m-1))/2}) = O(k^{(\theta_m-(m-1))/m}),$$

which implies that

$$|\partial R_k^*| = \sum_{i=1}^{m-1} \frac{2}{a_{i,k}^*} + \frac{2}{a_{m,k}^*} \leq 2m + O\big(k^{(\theta_m-(m-1))/m}\big).$$

Geometric convergence

Since $\lambda_k(Q) \geq \lambda_k^* = \lambda_k(R_k^*)$, for $\varepsilon > 0$, $N(\lambda_k^* - \varepsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

$$L_{0,m}^{\mathrm{cl}}(\lambda_k^* - \varepsilon)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(m-1)/2} - O((\lambda_k^* - \varepsilon)^{\theta_m/2})$$

$$\leq L_{0,m}^{\mathrm{cl}}(\lambda_k^*)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial R_k^*|(\lambda_k^*)^{(m-1)/2} + O((\lambda_k^*)^{\theta_m/2}).$$

Rearranging and choosing $\varepsilon = O(1)$,

$$|\partial R_k^*| - |\partial Q| \leq O((\lambda_k^*)^{(\theta_m-(m-1))/2}) = O(k^{(\theta_m-(m-1))/m}),$$

which implies that

$$|\partial R_k^*| = \sum_{i=1}^{m-1} \frac{2}{a_{i,k}^*} + \frac{2}{a_{m,k}^*} \leq 2m + O(k^{(\theta_m-(m-1))/m}).$$

By the arithmetic – geometric means inequality, with $a_{m,k}^* = 1 + \delta_k > 1$,

$$(m-1)(1+\delta_k)^{m/(m-1)} + 1 \leq m + m\delta_k + O(k^{(\theta_m-(m-1))/m}).$$

Geometric convergence

Since $\lambda_k(Q) \geq \lambda_k^* = \lambda_k(R_k^*)$, for $\varepsilon > 0$, $N(\lambda_k^* - \varepsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

$$L_{0,m}^{\mathrm{cl}}(\lambda_k^* - \varepsilon)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(m-1)/2} - O((\lambda_k^* - \varepsilon)^{\theta_m/2})$$

$$\leq L_{0,m}^{\mathrm{cl}}(\lambda_k^*)^{m/2} - \frac{L_{0,m-1}^{\mathrm{cl}}}{4}|\partial R_k^*|(\lambda_k^*)^{(m-1)/2} + O((\lambda_k^*)^{\theta_m/2}).$$

Rearranging and choosing $\varepsilon = O(1)$,

$$|\partial R_k^*| - |\partial Q| \leq O((\lambda_k^*)^{(\theta_m-(m-1))/2}) = O(k^{(\theta_m-(m-1))/m}),$$

which implies that

$$|\partial R_k^*| = \sum_{i=1}^{m-1} \frac{2}{a_{i,k}^*} + \frac{2}{a_{m,k}^*} \leq 2m + O(k^{(\theta_m-(m-1))/m}).$$

By the arithmetic – geometric means inequality, with $a_{m,k}^* = 1 + \delta_k > 1$,

$$(m-1)(1+\delta_k)^{m/(m-1)} + 1 \leq m + m\delta_k + O(k^{(\theta_m-(m-1))/m}).$$

Deduce that $\delta_k = O(k^{(\theta_m-(m-1))/(2m)})$.

Are minimising cuboids uniformly bounded?

Dirichlet counting function on R is

$$N(\Lambda, R) := \#\{(i_1, i_2, \ldots, i_m) \in \mathbb{N}^m \cap E(\Lambda, R)\}.$$

Observe that

$$N(\Lambda, R) \leq \sum_{i_1, \ldots, i_{m-1} \in \mathbb{N}} \left(\left(\frac{a_m^2 \Lambda}{\pi^2} - \sum_{j=1}^{m-2} \frac{a_m^2 i_j^2}{a_j^2} \right) - i_{m-1}^2 \right)_+^{1/2},$$

Are minimising cuboids uniformly bounded?

Dirichlet counting function on R is

$$N(\Lambda, R) := \#\{(i_1, i_2, \dots, i_m) \in \mathbb{N}^m \cap E(\Lambda, R)\}.$$

Observe that

$$N(\Lambda, R) \leq \sum_{i_1,\dots,i_{m-1}\in\mathbb{N}} \left(\left(\frac{a_m^2 \Lambda}{\pi^2} - \sum_{j=1}^{m-2} \frac{a_m^2 i_j^2}{a_j^2} \right) - i_{m-1}^2 \right)_+^{1/2},$$

Proofs of the 2D and 3D cases use that the function $(y - i^2)^{n/2}$ is concave on $[0, y^{1/2}]$ for $n = 1, 2$.

Are minimising cuboids uniformly bounded?

Dirichlet counting function on R is

$$N(\Lambda, R) := \#\{(i_1, i_2, \dots, i_m) \in \mathbb{N}^m \cap E(\Lambda, R)\}.$$

Observe that

$$N(\Lambda, R) \leq \sum_{i_1,\dots,i_{m-1} \in \mathbb{N}} \left(\left(\frac{a_m^2 \Lambda}{\pi^2} - \sum_{j=1}^{m-2} \frac{a_m^2 i_j^2}{a_j^2} \right) - i_{m-1}^2 \right)_+^{1/2},$$

Proofs of the 2D and 3D cases use that the function $(y - i^2)^{n/2}$ is concave on $[0, y^{1/2}]$ for $n = 1, 2$.

Idea : approximate $\sum_{i=1}^{\lfloor \sqrt{y} \rfloor} (y - i^2)^{n/2}$ by the area under the curve $(y - x^2)^{n/2}$ and use concavity to deal with the error.

Are minimising cuboids uniformly bounded?

Dirichlet counting function on R is

$$N(\Lambda, R) := \#\{(i_1, i_2, \dots, i_m) \in \mathbb{N}^m \cap E(\Lambda, R)\}.$$

Observe that

$$N(\Lambda, R) \leq \sum_{i_1, \dots, i_{m-1} \in \mathbb{N}} \left(\left(\frac{a_m^2 \Lambda}{\pi^2} - \sum_{j=1}^{m-2} \frac{a_m^2 i_j^2}{a_j^2} \right) - i_{m-1}^2 \right)_+^{1/2},$$

Proofs of the 2D and 3D cases use that the function $(y - i^2)^{n/2}$ is concave on $[0, y^{1/2}]$ for $n = 1, 2$.

Idea : approximate $\sum_{i=1}^{\lfloor \sqrt{y} \rfloor} (y - i^2)^{n/2}$ by the area under the curve $(y - x^2)^{n/2}$ and use concavity to deal with the error.

For $m \geq 4$, we are concerned with a function of the form $(y - i^2)^{n/2}$ where $n \geq 3$. But $(y - i^2)^{n/2}$ is not concave on $[0, y^{1/2}]$ for $n \geq 3$.

Riesz mean of Dirichlet eigenvalues

Riesz mean of Dirichlet eigenvalues for $\Lambda \geq 0$ and $\gamma \geq 0$:

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = \sum_{k=1}^{\infty} (\Lambda - \lambda_k(\Omega))_+^\gamma.$$

Riesz mean of Dirichlet eigenvalues

Riesz mean of Dirichlet eigenvalues for $\Lambda \geq 0$ and $\gamma \geq 0$:

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = \sum_{k=1}^{\infty} (\Lambda - \lambda_k(\Omega))_+^\gamma.$$

Observe that

$$N(\Lambda, R) = \sum_{k:\lambda_k(R) \leq \Lambda} (\Lambda - \lambda_k(R))^0 = \mathrm{Tr}(-\Delta_\Omega - \lambda)_-^0.$$

Upper bound for counting function

Let $R' = (0, a_2) \times \cdots \times (0, a_m)$, then

$$\begin{aligned} N(\Lambda, R) &= \sum_{k:\lambda_k(R)\leq\Lambda} (\Lambda - \lambda_k(R))^0 \\ &= \sum_{l:\lambda_l((0,a_1))\leq\Lambda} \ \sum_{k:\lambda_k(R')\leq\Lambda-\lambda_l((0,a_1))} ((\Lambda - \lambda_l((0, a_1))) - \lambda_k(R'))^0 \\ &= \sum_{l:\lambda_l((0,a_1))\leq\Lambda} N((\Lambda - \lambda_l((0, a_1)))_+, R'), \end{aligned}$$

Upper bound for counting function

Let $R' = (0, a_2) \times \cdots \times (0, a_m)$, then

$$\begin{aligned} N(\Lambda, R) &= \sum_{k:\lambda_k(R)\le\Lambda} (\Lambda - \lambda_k(R))^0 \\ &= \sum_{l:\lambda_l((0,a_1))\le\Lambda} \ \sum_{k:\lambda_k(R')\le\Lambda-\lambda_l((0,a_1))} ((\Lambda - \lambda_l((0,a_1))) - \lambda_k(R'))^0 \\ &= \sum_{l:\lambda_l((0,a_1))\le\Lambda} N((\Lambda - \lambda_l((0,a_1)))_+, R'), \end{aligned}$$

By Pólya's inequality $N(\Lambda, R') \le L^{\mathrm{cl}}_{0,m-1}|R'|\Lambda^{(m-1)/2}$,

$$\begin{aligned} N(\Lambda, R) &\le \sum_{l:\lambda_l((0,a_1))\le\Lambda} L^{\mathrm{cl}}_{0,m-1}|R'|(\Lambda - \lambda_l((0,a_1)))_+^{(m-1)/2} \\ &= L^{\mathrm{cl}}_{0,m-1}|R'| \operatorname{Tr}(-\Delta_{(0,a_1)} - \Lambda)_-^{(m-1)/2}. \end{aligned}$$

Minimising cuboids are uniformly bounded

Theorem (G - Larson, 2017)

Let $R = \prod_{i=1}^{m}(0, a_i)$*,* $m \geq 4$*, be such that* $0 < a_1 \leq a_2 \leq \cdots \leq a_m$ *and* $|R| = \prod_{i=1}^{m} a_i = 1$*. Then the bound*

$$N(\Lambda, R) \leq L^{cl}_{0,m}\Lambda^{m/2} - \frac{3bL^{cl}_{0,m-1}}{8a_1}\Lambda^{(m-1)/2} + \frac{b^2 L^{cl}_{0,m-2}}{8\pi a_1^2}\Lambda^{(m-2)/2},$$

holds for all $\Lambda > 0$ *and all* $b \in [0, b_0]$*, with* $b_0 := \pi - \sqrt{\frac{6-8\pi+3\pi^2}{3}}$*.*

Minimising cuboids are uniformly bounded

Theorem (G - Larson, 2017)

Let $R = \prod_{i=1}^{m}(0, a_i)$, $m \geq 4$, *be such that* $0 < a_1 \leq a_2 \leq \cdots \leq a_m$ *and* $|R| = \prod_{i=1}^{m} a_i = 1$. *Then the bound*

$$N(\Lambda, R) \leq L^{cl}_{0,m}\Lambda^{m/2} - \frac{3bL^{cl}_{0,m-1}}{8a_1}\Lambda^{(m-1)/2} + \frac{b^2 L^{cl}_{0,m-2}}{8\pi a_1^2}\Lambda^{(m-2)/2},$$

holds for all $\Lambda > 0$ *and all* $b \in \left[0, b_0\right]$, *with* $b_0 := \pi - \sqrt{\frac{6-8\pi+3\pi^2}{3}}$.

For any $\epsilon > 0$, $N(\lambda_k^* - \epsilon, Q) \leq N(\lambda_k(Q) - \epsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$.

Minimising cuboids are uniformly bounded

Theorem (G - Larson, 2017)

Let $R = \prod_{i=1}^{m}(0, a_i)$, $m \geq 4$, *be such that* $0 < a_1 \leq a_2 \leq \cdots \leq a_m$ *and* $|R| = \prod_{i=1}^{m} a_i = 1$. *Then the bound*

$$N(\Lambda, R) \leq L^{cl}_{0,m}\Lambda^{m/2} - \frac{3bL^{cl}_{0,m-1}}{8a_1}\Lambda^{(m-1)/2} + \frac{b^2 L^{cl}_{0,m-2}}{8\pi a_1^2}\Lambda^{(m-2)/2},$$

holds for all $\Lambda > 0$ *and all* $b \in \left[0, b_0\right]$, *with* $b_0 := \pi - \sqrt{\frac{6-8\pi+3\pi^2}{3}}$.

For any $\epsilon > 0$, $N(\lambda_k^* - \epsilon, Q) \leq N(\lambda_k(Q) - \epsilon, Q) < k \leq N(\lambda_k^*, R_k^*)$. We can deduce that

$$a^*_{m,k} \leq \left(\frac{32m\Gamma\left(\frac{m+1}{2}\right)^2}{3\pi^{3/2}\Gamma\left(\frac{m}{2}\right)}\right)^{m-1} + o(1).$$

Convergence of eigenvalues

Theorem (G - Larson, 2017)

As $k \to \infty$,

$$|\lambda_k(Q) - \lambda_k^*| = O(k^{(\theta_m - (m-2))/m}) = \begin{cases} O(k^{(\theta_m - (m-2))/m}) & 2 \le m \le 4, \\ O(1) & m \ge 5, \end{cases}$$

where θ_m is any exponent such that for all $a_1, \dots, a_m \in \mathbb{R}_+$

$$\#\{(i_1, \dots, i_m) \in \mathbb{Z}^m : a_1^{-2} i_1^2 + \cdots + a_m^{-2} i_m^2 \le r^2\} - \omega_m r^m \prod_{i=1}^m a_i = O(r^{\theta_m}),$$

as $r \to \infty$ uniformly for a_i on compact subsets of $\mathbb{R}_+$.

Convergence of eigenvalues

Theorem (G - Larson, 2017)

As $k \to \infty$,

$$|\lambda_k(Q) - \lambda_k^*| = O(k^{(\theta_m-(m-2))/m}) = \begin{cases} O(k^{(\theta_m-(m-2))/m}) & 2 \leq m \leq 4, \\ O(1) & m \geq 5, \end{cases}$$

where θ_m is any exponent such that for all $a_1, \dots, a_m \in \mathbb{R}_+$

$$\#\{(i_1, \dots, i_m) \in \mathbb{Z}^m : a_1^{-2} i_1^2 + \cdots + a_m^{-2} i_m^2 \leq r^2\} - \omega_m r^m \prod_{i=1}^{m} a_i = O(r^{\theta_m}),$$

as $r \to \infty$ uniformly for a_i on compact subsets of $\mathbb{R}_+$.

Restrict the collection of cuboids to a sub-collection, then any sequence of minimising cuboids converges to the cuboid of smallest perimeter in this sub-collection.

Further results for Dirichlet Laplacian

- For $\gamma \geq 0$ and $\Lambda \geq 0$,

$$\max\left\{\mathrm{Tr}(-\Delta_R - \Lambda)_-^\gamma : R = R_{a_1,\ldots,a_m}, |R| = 1\right\}.$$

Any sequence of maximising cuboids $(R_\Lambda)_\Lambda$ converges to the unit cube as $\Lambda \to \infty$.

Further results for Dirichlet Laplacian

- For $\gamma \geq 0$ and $\Lambda \geq 0$,

$$\max\Big\{\mathrm{Tr}(-\Delta_R - \Lambda)_-^{\gamma} : R = R_{a_1,\dots,a_m}, |R| = 1\Big\}.$$

  Any sequence of maximising cuboids $(R_\Lambda)_\Lambda$ converges to the unit cube as $\Lambda \to \infty$.

- For $k \in \mathbb{N}$,

$$\min\Big\{\frac{1}{k}\sum_{i=1}^{k}\lambda_i(R) : R = R_{a_1,\dots,a_m}, |R| = 1\Big\}.$$

  Any sequence of minimising cuboids $(R_k^{**})_k$ converges to the unit cube as $k \to \infty$.

Neumann eigenvalues

Let $\Omega \subset \mathbb{R}^m$ be a bounded, open set with Lipschitz boundary.
Neumann eigenvalues of the Laplacian on Ω, $\mu_k(\Omega) \in \mathbb{R}$, satisfy

$$\begin{cases} -\Delta u_k(x) = \mu_k(\Omega) u_k(x) & x \in \Omega, \\ \frac{\partial u_k(x)}{\partial \vec{n}} = 0 & x \in \partial\Omega, \end{cases}$$

where $\vec{n}$ is the outward pointing unit normal vector to $\partial\Omega$.

Neumann eigenvalues

Let $\Omega \subset \mathbb{R}^m$ be a bounded, open set with Lipschitz boundary.
Neumann eigenvalues of the Laplacian on Ω, $\mu_k(\Omega) \in \mathbb{R}$, satisfy

$$\begin{cases} -\Delta u_k(x) = \mu_k(\Omega) u_k(x) & x \in \Omega, \\ \frac{\partial u_k(x)}{\partial \vec{n}} = 0 & x \in \partial\Omega, \end{cases}$$

where $\vec{n}$ is the outward pointing unit normal vector to $\partial\Omega$.

Theorem (van den Berg - Bucur - G, 2016)

Among rectangles of unit area, any sequence of maximising rectangles $(R_k^*)_k$ for μ_k converges to the unit square as $k \to \infty$. Moreover,

$$a_{2,k}^* = 1 + O(k^{(\theta_2 - 1)/4}), k \to \infty,$$

where θ_2 is any exponent such that for all $a_1, a_2 \in \mathbb{R}_+$

$$\#\{(i_1, i_2) \in \mathbb{Z}^2 : a_1^{-2} i_1^2 + a_2^{-2} i_2^2 \le r^2\} - \pi a_1 a_2 r^2 = O(r^{\theta_2}), r \to \infty.$$

Neumann eigenvalues

Theorem (G - Larson, 2017)

Let $m \geq 3$. *For* $k \in \mathbb{N}$, *let* $R_k^* = R_{a_{1,k}^*, a_{2,k}^*, \ldots, a_{m,k}^*}$ *denote an m-dimensional cuboid which maximises* μ_k *where* $0 < a_{1,k}^* \leq a_{2,k}^* \leq \cdots \leq a_{m,k}^*$, $\prod_{i=1}^m a_{i,k}^* = 1$. *Then, as* $k \to \infty$,

$$a_{m,k}^* = 1 + O(k^{(\theta_m - (m-1))/2m}),$$

where θ_m *is any exponent such that*

$$\#\{(i_1, \ldots, i_m) \in \mathbb{Z}^m : a_1^{-2} i_1^2 + \cdots + a_m^{-2} i_m^2 \leq r^2\} - \omega_m r^m \prod_{i=1}^m a_i = O(r^{\theta_m})$$

as $r \to \infty$ *uniformly for* a_i *on compact subsets of* $\mathbb{R}_+$.
Any sequence of maximising m-dimensional cuboids $(R_k^*)_k$ *of unit measure for* μ_k *converges to the m-dimensional unit cube as* $k \to \infty$.

References

- P. R. S. Antunes, P. Freitas, Optimal spectral rectangles and lattice ellipses. *Proc. R. Soc. A* 469 (2013), 20120492.
- M. van den Berg, D. Bucur, K. Gittins, Maximising Neumann eigenvalues on rectangles. *Bull. London Math. Soc.* 48 (2016) 877–894.
- M. van den Berg, K. Gittins, Minimising Dirichlet eigenvalues on cuboids of unit measure, *Mathematika* 63 (2017) 469–482.
- K. Gittins, S. Larson, Asymptotic behaviour of cuboids optimising Laplacian eigenvalues, arXiv:1703.10249 [math.SP] (2017).
- S. Larson Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains, arXiv:1611.05680 (2016).

Flat tori

Let $T_{a,b}$ denote the flat torus obtained from the parallelogram in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, (a,b), $(a+1,b)$ by identifying parallel edges.
Let $\lambda_k(a,b)$ denote the eigenvalues of Laplace-Beltrami operator on $T_{a,b}$.

For $k \in \mathbb{N}$, there is a maximising flat torus $T_{a_k^*, b_k^*}$ which realises the supremum

$$\sup\{b \cdot \lambda_k(a,b) : (a,b) \in \mathbb{R}^2\}.$$

Conjecture (Kao, Lai and Osting, 2016)

For $k \in \mathbb{N}$, the maximising flat torus $T_{a_k^*, b_k^*}$ has

$$(a_k^*, b_k^*) = \left(\frac{1}{2}, \sqrt{\left\lceil \frac{k}{2} \right\rceil^2 - \frac{1}{4}}\right).$$

Optimal cylinders

Cylinder $C(r,\ell) = S_r \times [0,\ell]$, where S_r is a circle of radius r.

For $b > 0$, set $\ell = b$ and $r = b^{-1}$ so $|C(b^{-1}, b)| = 2\pi$.

Dirichlet eigenvalues on $C(b^{-1}, b)$:

$$\frac{\pi^2 i^2}{b^2} + j^2 b^2,\ i \in \mathbb{N},\ j \in \mathbb{Z}.$$

$\inf\{\lambda_k(C(b^{-1}, b)) : b > 0\} = 0$ via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to \infty$ as $\ell \to \infty$.

$\sup\{\lambda_k(C(b^{-1}, b)) : b > 0\} = \infty$ via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to 0$ as $\ell \to \infty$.

Optimal cylinders

Cylinder $C(r,\ell) = S_r \times [0,\ell]$, where S_r is a circle of radius r.

For $b > 0$, set $\ell = b$ and $r = b^{-1}$ so $|C(b^{-1}, b)| = 2\pi$.

Neumann eigenvalues on $C(b^{-1}, b)$:

$$\frac{\pi^2 i^2}{b^2} + j^2 b^2,\ i \in \mathbb{N} \cup \{0\},\ j \in \mathbb{Z}.$$

$\inf\{\mu_k(C(b^{-1}, b)) : b > 0\} = 0$ via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to \infty$ as $\ell \to \infty$, and via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to 0$ as $\ell \to \infty$.

Maximising cylinders for μ_k exist.

Q: What are they? What is the asymptotic behaviour of a sequence of maximising cylinders as $k \to \infty$?

Optimisation of Dirichlet eigenvalues: fixed perimeter

For $\ell \in \mathbb{R}$, $\ell > 0$,

$$\lambda_k(\Omega_k^*) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ open}, |\Omega| < \infty, \text{Per}(\Omega) = \ell\}.$$

De Philippis and Velichkov: a minimiser exists, is bounded and connected.

Bucur and Freitas: any sequence of minimisers $\Omega_k^* \subset \mathbb{R}^2$ of λ_k with perimeter ℓ converges to the disc of perimeter ℓ as $k \to \infty$.

van den Berg: for each $k \in \mathbb{N}$, there exists $\Omega_k^* \subset \mathbb{R}^m$, $m \geq 2$, such that

$$\lambda_k(\Omega_k^*) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ open}, \text{convex}, |\Omega| < \infty, \text{Per}(\Omega) = \ell\}.$$

For any sequence of minimisers, there exists a sequence of isometries of these minimisers which converges to a ball of perimeter ℓ as $k \to \infty$.

Antunes and Freitas: any sequence of m-dimensional minimising cuboids of perimeter ℓ converges to the cube of perimeter ℓ as $k \to \infty$.

Optimisation of Neumann eigenvalues: fixed perimeter

Let

$$R_{a,c} = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < c, 0 < x_2 < a\}.$$

Theorem (van den Berg, Bucur, Gittins, 2016)

(i) *If* $k = 1$, *then* $\inf\{\mu_k(R_{a,c}) : \mathrm{Per}(R_{a,c}) = 2(a + c) = 4\}$ *does not have a minimiser, and the infimum equals* $\frac{\pi^2}{4}$.

(ii) *Any sequence of minimising rectangles* $(R_{a_k^*,c_k^*})_k$, $k \geq 2$, *converges to the unit square as* $k \to \infty$.

(iii) *For* $k \in \mathbb{N}$, *there is a unique maximising rectangle* $R_{a_k^*,c_k^*}$ *such that* $\mu_k(R_{a_k^*,c_k^*}) = \sup\{\mu_k(R_{a,c}) : \mathrm{Per}(R_{a,c}) = 2(a + c) = 4\}$ *with* $a_k^* = \frac{2}{k+1} \in (0, 1]$ *and* $c_k^* = 2 - a_k^*$. *The sequence of maximising rectangles collapses as* $k \to \infty$.

Optimisation of Neumann eigenvalues: fixed perimeter

For $k \in \mathbb{N}$ and $m \geq 3$,

$$\sup\{\mu_k(R) : R \text{ is a cuboid in } \mathbb{R}^m, \mathrm{Per}(R) = 4\}.$$

Either

- there is a non-degenerate maximising sequence for μ_k, or
- there is a maximising sequence $\left(R_{a_1^{(n)},\dots,a_m^{(n)}}\right)_n$ for μ_k with one vanishing side-length $a_1^{(n)} \to 0$ and, for all $i \in \{1,\dots,m\}$, $a_i^{(n)} \to a_i$ as $n \to \infty$. The perimeter constraint becomes $a_2 a_3 \dots a_m = 2$.

The eigenvalues of $R_{a_1,\dots,a_m}$ are the eigenvalues of the $(m-1)$-dimensional cuboid with edges of length $a_2, a_3, \dots, a_m$ and measure 2.

For a maximising cuboid, μ_k^* behaves like $k^{2/m}$ as $k \to \infty$.

On a degenerating sequence which collapses towards a fixed $(m-1)$-dimensional cuboid, μ_k^* behaves like $k^{2/(m-1)}$ as $k \to \infty$.