Inequalities for the lowest magnetic Neumann eigenvalue in a planar domain (after Fournais-Helffer).
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Abstract

We study the ground state energy of the Neumann magnetic Laplacian in planar domains. For a constant magnetic field we consider the question whether, under an assumption of fixed area, the disc maximizes in the simply connected case this eigenvalue. More generally, we discuss old and new bounds obtained on this problem in the case of a variable magnetic field and also consider the non simply connected case. Our talk will be completed with the discussion of recent results together with M. Persson Sunqvist on the Pauli operator.

The setup

We consider an open set Ω that is smooth, bounded and connected. We denote by $A(\Omega)$ the area of Ω, and define R_Ω to be the radius of the disc with the same area as Ω, i.e.

$$\pi R_\Omega^2 = A(\Omega). \tag{1}$$

Let $\lambda_1^N(B,\Omega)$ be the ground state energy for the magnetic Neumann Laplacian on Ω with constant magnetic field of intensity $B \geq 0$, i.e.

$$H^N(\mathbf{A}, B, \Omega) := (-i\nabla + B\mathbf{A})^2$$

where $\mathbf{A} = \frac{1}{2}(-x_2, x_1)$ (in particular $\nabla \times \mathbf{A} = 1$), and where we have imposed (magnetic) Neumann boundary conditions:

$$\nu \cdot \nabla_A u = 0 \text{ on } \partial\Omega\,,$$

with

$$\nabla_A u = \nabla u - i\mathbf{A}u\,.$$

We are not obliged to consider above a constant magnetic field. More general magnetic potentials can be considered for some of the questions. We can consider the magnetic Laplacian

$$-\Delta_A(B) := (D_{x_1} - BA_1)^2 + (D_{x_2} - BA_2)^2 \,.$$

Here $D_{x_j} = -i\partial_{x_j}$ for $j = 1, 2$.
The vector potential $\mathbf{A} = (A_1, A_2)$ satisfies

$$\beta(x) = \partial_{x_1} A_2(x) - \partial_{x_2} A_1(x) \,. \tag{2}$$

If Ω is not simply connected, it is better to write $\lambda^N(\mathbf{A}, B, \Omega)$ to mention the dependence with respect to the magnetic potential.

Similarly, $\lambda_1^D(\mathbf{A}, B, \Omega)$ will denote the ground state energy in the case where we impose the Dirichlet boundary condition. We are interested in upper and lower bounds on these eigenvalues, universal or asymptotic in the two regimes $B \to 0$ or $B \to +\infty$. When considering lower bounds, we first mention the following result obtained by L. Erdös for constant magnetic fields.

Theorem (Erdös)

For any planar domain Ω and $B > 0$, we have:

$$\lambda_1^D(B, \Omega) \geq \lambda_1^D(B, D(0, R)) \,, \tag{3}$$

Moreover the equality in (3) occurs if and only if $\Omega = D(0, R_\Omega)$.

We would like to analyze the same question for the Neumann magnetic Laplacian.

Quest. 1: For which $B > 0$ is $\lambda_1^N(B, \Omega) \leq \lambda_1^N(B, D(0, R_\Omega))$ true?

When Ω is assumed to be simply connected, our choice of $\mathbf{A}$ such that $\mathrm{curl}\, \mathbf{A} = 1$ is not important because, by gauge invariance, this spectral question depends only on the magnetic field.
To analyze Question 1, we first look at the two asymptotic regimes $B \to 0$ and $B \to +\infty$.

Weak magnetic field asymptotics

By rather standard perturbation theory (see the book of Fournais-Helffer), we have the following weak field asymptotics.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be smooth, bounded and connected. There exists a constant $C_\Omega > 0$ such that for all $B > 0$

$$A(\Omega)^{-1} B^2 \int_\Omega (\mathbf{A}')^2 \, dx - C_\Omega B^3 \leq \lambda_1^N(B, \Omega) \leq A(\Omega)^{-1} B^2 \int_\Omega (\mathbf{A}')^2 \, dx \,, \tag{4}$$

where the magnetic potential $\mathbf{A}'$ is the solution of

$$\nabla \times \mathbf{A}' = 1 \,, \; \nabla \cdot \mathbf{A}' = 0 \text{ and } \mathbf{A}' \cdot \nu = 0 \text{ on } \partial\Omega \,, \tag{5}$$

with (when Ω is not simply connected) the additional condition that $\mathbf{A} - \mathbf{A}'$ is exact.

Notice that in the case of the disc, we have $\mathbf{A}' = \mathbf{A}$ with $\mathbf{A}$ given above. A weak version of Question 1 above would consequently be:

$$\text{Quest. 2:}\quad \int_\Omega (\mathbf{A}')^2\, dx \leq \frac{1}{4}\int_{D(0,1/\sqrt{\pi})} r^2\, dx \text{ if } A(\Omega) = 1\,?$$

We can give an affirmative answer to this question.

Strong magnetic field asymptotics

For a smooth domain Ω and a point $P \in \partial\Omega$ we denote by $\kappa(P)$ the curvature of the boundary at P. We denote by $\kappa_{\max}(\Omega)$ the maximum value of $\kappa(P)$, $P \in \partial\Omega$.
In the limit when $B \to +\infty$, we have the following theorem (Bernoff-Sternberg , Helffer-Morame, Lu-Pan).

Theorem

Let $\Omega \subset \mathbb{R}^2$ be smooth and bounded. There exist $c_\Omega, B_\Omega > 0$ such that

$$\left|\lambda_1^N(\mathbf{A}, B, \Omega) - \left(\Theta_0 B - C_1 \kappa_{\max}(\Omega) B^{1/2}\right)\right| \leq c_\Omega B^{1/3}, \tag{6}$$

for all $B \geq B_\Omega$. Here $\Theta_0, C_1 > 0$ are universal constants, in particular, independent of B and Ω.

The asymptotics for strong magnetic fields leads us to the next question

Quest. 3: Is the maximal boundary curvature minimized by the disc ?

Reverse Faber-Krahn inequality for magnetic fields

The analysis of Question 2 and 3, i.e. the study of the limits of large and small magnetic field strength, suggest that, when Ω is simply connected,

$$\lambda_1^N(B,\Omega) \leq \lambda_1^N(B, D(0,R_\Omega)), \tag{7}$$

for all B.
This would correspond to a reverse Faber-Krahn inequality for magnetic fields. Notice though, that we do not prove such an inequality. Also notice that this inequality is not true in general non-simply connected domains.

Around maximal curvature

Under the assumption that Ω is simply connected, we have

Theorem

For a given area, the maximal curvature is minimized by the disc.

This is actually an old theorem which was rediscovered by Pankrashkin (and proved in the star-shaped case). The general (simply connected but not star-shaped) case seemed open until recently. However, it has been settled by Pankrashkin on the basis of a result due to Pestov-Ionin:

Proposition (Pestov-Ionin)

For a smooth closed Jordan curve, the interior of the curve contains a disk of radius $\frac{1}{\kappa_{\max}(\Omega)}$.

An easier proof can be given for star-like open sets.

Finally, we mention that, as observed by Pankrashkin, this implies through the semi-classical analysis recalled previously that in the large magnetic field strength limit we have

$$\lambda_1^N(B,\Omega) \leq \lambda_1^N(B, D(0,R_\Omega)) + \mathcal{O}(B^{1/3})$$

From this we deduce:

Proposition

Let $\Omega \subset \mathbb{R}^2$ be smooth and simply connected. There exists $B_1(\Omega) > 0$ such that, for all $B \geq B_1(\Omega)$,

$$\lambda_1^N(B,\Omega) \leq \lambda_1^N(B, D(0,R_\Omega))\,. \tag{8}$$

Furthermore, the inequality (8) is strict unless $\Omega = D(0,R_\Omega)$.

Torsional rigidity

For $\mathbf{A}$ as in the introduction, we define

$$\widehat{S}_\Omega := \int_\Omega (\mathbf{A}')^2 \, dx$$

where the magnetic potential $\mathbf{A}'$ is the solution of

$$\nabla \times \mathbf{A}' = 1\,,\ \nabla \cdot \mathbf{A}' = 0 \text{ and } \mathbf{A}' \cdot \nu = 0 \text{ on } \partial\Omega\,, \tag{9}$$

such that $\mathbf{A} - \mathbf{A}'$ is exact.
As observed in Fournais-Helffer we have the identity

$$\widehat{S}_\Omega = \inf_\phi \int_\Omega |\nabla\phi + \mathbf{A}|^2 \, dx.$$

We do not have to assume here that the magnetic field is constant. Note that with this gauge the magnetic Neumann condition is the standard Neumann condition.

Generating function and application

Define now $\psi = \psi_\Omega$ to be the solution of

$$\Delta\psi = 1\,,\ \psi\big|_{\partial\Omega} = 0\,. \tag{10}$$

Then we have in the simply connected case

$$\mathbf{A}' = \nabla^\perp \psi\,,$$

where $\nabla^\perp\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$.
Hence, we get:

$$\int_\Omega (\mathbf{A}')^2\,dx = \int_\Omega |\nabla\psi|^2\,dx\,.$$

The quantity

$$S_\Omega := \int_\Omega |\nabla \psi|^2 \, dx \tag{11}$$

with ψ solution of (10), is a well known quantity in Mechanics, which is called (up to a factor 2) the torsional rigidity of Ω.
By an integration by parts, we get in the simply connected case

$$S_\Omega = -\int_\Omega \psi \, dx \,. \tag{12}$$

If $\psi = \psi_\Omega$ is the solution of (10) then, by the maximum principle, $\psi < 0$ in Ω and attains its infimum $\psi_{min}(\Omega)$ in Ω. In Helffer–Persson-Sundqvist it was observed, using a theorem of Erdös and the asymptotics for B large, that:

$$0 > \psi_{min}(\Omega) \geq \psi_{min}(D(0, R_\Omega)) \tag{13}$$

where $D(0, R_\Omega)$ is the disk of same area as Ω.

To address Question 2, we compare S_Ω for different domains. This is actually the particular case (already mentioned in Polya-Szegö) of a general result communicated to us by D. Bucur (2017)

Proposition

Suppose that Ω is simply connected, then

$$S_\Omega \leq S_{D(0,R)} \,. \tag{14}$$

The proof is based on the formula

$$S_\Omega = -\left(\int_\Omega |\nabla \psi_\Omega|^2 \, dx + 2 \int_\Omega \psi_\Omega \, dx \right) \tag{15}$$

One can then follow the standard proof of the Faber-Krahn inequality using the Schwarz symmetrization procedure.

As a corollary, we obtain

Proposition

Suppose that Ω is smooth bounded and simply connected. There exists $B_0 > 0$ such that, for all $B \in (0, B_0)$,

$$\lambda_1^N(B, \Omega) \leq \lambda_1^N(B, D(0, R_\Omega)) . \qquad (16)$$

Using recent results by Brasco-De Philippis-Velechnikov, it is possible to show that we can take, assuming $A(\Omega) = 1$,

$$B_0(\Omega) = C\mathcal{A}(\Omega) ,$$

where $C > 0$ is a universal constant and $\mathcal{A}(\Omega)$ is the Fraenkel assymmetry

$$\mathcal{A}(\Omega) := \inf_{D\,,\ \text{unit disk}} A(\Omega \Delta D) ,$$

where the symbol Δ stands for the symmetric difference between sets.

Extensions and open questions.

Of course the main question is: *Can we prove the reverse Faber-Krahn inequality* (7) *for any B ?*

Let us also mention the following connected questions

1. What can we say when B is no more constant ?
2. What can we say in three dimensions ?
3. What can we say in the non-simply connected case ?
4. What about Pauli operators ? (Helffer–Persson-Sundqvist) See two papers in ArXiv.
5. Specific questions when the magnetic field is not of constant sign.(Helffer–Kowarik–Persson-Sundqvist (work in progress)

The case of a non constant magnetic field.

Let us assume that the magnetic potential $B\mathbf{A}$ has as magnetic field $B\beta(x)$, with β not necessarily constant. Define,

$$\widehat{S}_\Omega^{\mathbf{A}} := \int_\Omega |\mathbf{A}'|^2\, dx\,,$$

where $\mathbf{A}'$ is the unique magnetic potential, such that $\mathbf{A}' - \mathbf{A}$ is a gradient and satisfying

$$\text{curl } \mathbf{A}' = \beta\,,\ \mathrm{Div}\,\mathbf{A}' = 0 \text{ and } \mathbf{A}' \cdot \nu = 0 \text{ on } \partial\Omega\,.$$

We have the following easy perturbation proposition:

Proposition

If Ω is connected,

$$A(\Omega)^{-1} B^2 \widehat{S}_\Omega^{\mathbf{A}} - C_\Omega B^4 \leq \lambda_1^N(B\mathbf{A}, \Omega) \leq A(\Omega)^{-1} B^2 \widehat{S}_\Omega^{\mathbf{A}}\,. \qquad (17)$$

When Ω is simply connected, $\mathbf{A}' = \nabla^\perp \psi$ where ψ is the solution of

$$\Delta\psi = \beta\,,\ \psi = 0 \text{ on } \partial\Omega\,.$$

Hence in this case,

$$\widehat{S}_\Omega^{\mathbf{A}} = S_\Omega^\beta\,,$$

where

$$S_\Omega^\beta = \int_\Omega |\nabla\psi|^2 dx\,.$$

As in the constant case, we would like now to find an isoperimetric inequality for S_Ω^β.
If we asssume that the magnetic field β satisfies

$$\beta(x) > 0 \text{ on } \overline{\Omega}\,,$$

we get by Maximum principle that $\psi < 0$ in Ω and follow the different steps of the constant magnetic field case.

If Ω is simply-connected,

$$S_\Omega^\beta = -\int_\Omega \beta(x)\psi(x)\,. \tag{18}$$

We rewrite S_Ω^β in the form

$$S_\Omega^\beta = -\left(\int_\Omega |\nabla\psi(x)|^2\,dx + 2\int_\Omega \beta(x)\psi(x)\,dx\right)\,.$$

Hence we get

Proposition

If Ω is simply connected and $\beta \geq 0$ then

$$S_\Omega^\beta \leq S_{D(0,R_\Omega)}^{\beta^*}\,, \tag{19}$$

where β^* is the Schwarz symmetrization of β.

Remarks

In our particular case, we recover under the additional assumption that Ω is simply connected, the following variant[1] of Colbois-El Soufi–Ilias-Savo's result

$$\lambda_1^N(B\beta,\Omega) \leq B^2 \frac{1}{A(\Omega)\lambda^D(\Omega)} \left(\int_\Omega \beta^2 dx\right) . \tag{20}$$

We observe indeed, using (18), that, if Ω is simply connected,

$$S_\Omega^\beta \leq \|\beta\| \, \|\psi\| \leq \frac{\|\beta\|}{\sqrt{\lambda_1^D(\Omega)}} \left(S_\Omega^\beta\right)^{\frac{1}{2}} ,$$

which gives, by the standard isoperimetric inequality for λ_1^D

$$S_\Omega^\beta \leq \|\beta\|^2 \lambda_1^D(\Omega)^{-1} \leq \|\beta\|^2 \lambda_1^D(D(0,R_\Omega))^{-1} . \tag{21}$$

[1] The authors use another lowest eigenvalue corresponding to a Laplacian on 2-forms satisfying specific boundary conditions but work in any dimension. Here we are in dimension 2 and identify 2-forms and functions.

As in the constant magnetic field case, the previous estimates are only good for B small. When B is large, we refer to the semi-classical analysis of N. Raymond or to the universal estimates of Erdös or Colbois-Savo.

The 3D-case.

There is no hope to have in 3D the inequality

$$\lambda^N(\Omega, B) \leq \lambda^N(B(R_\Omega), B) \text{ if } |\Omega| = |B(0, R_\Omega)| = \frac{4}{3}\pi R_\Omega^3\,, \qquad (22)$$

for a constant magnetic field of intensity B.
Take indeed $\Omega_L = \omega \times [0, L]$ and a magnetic field $\beta = B(0, 0, 1)$.
We have

$$|\Omega_L| = A(\omega)\, L = \frac{4\pi R_{\Omega_L}^3}{3}\,,$$

and (by separation of variables)

$$\lambda^N(\Omega^L, B) = \lambda^N(\omega, B)\,.$$

But, using the constant function $|\Omega_L|^{-\frac{1}{2}}$ as trial state, we get a contradiction as $L \to 0$. This is actually not surprising because the magnetic field introduces a privileged direction. The "optimal domain" should have the same property.

The nonsimply connected case

We mention a recent preprint of B. Colbois and A. Savo, partially developed in collaboration with A. ElSoufi and S. Ilias for the upper bounds, devoted to the Neumann problem and two papers by Helffer-Persson Sundqvist initially motivated by Ekholm-Kowarik-Portman.
Here we denote by $\lambda^N(\mathbf{A},\Omega)$ the first eigenvalue of the Neumann problem. We observe that if Ω has k holes D_j ($j = 1, \cdots, k$),

$$\Omega := \widetilde{\Omega} \setminus \cup_j D_j \,,$$

the generating function is now solution of

$$\Delta\psi = \beta \text{ in } \Omega\,, \text{ with } \psi|_{\partial\widetilde{\Omega}} = 0 \text{ and } \psi|_{\partial D_j} = C_j\,, \qquad (23)$$

for some real constants C_j.

We can then write

$$\psi = \psi^0 + \sum_{j=1}^{k} C_j \, \theta^j$$

where ψ^0 is the solution of

$$\Delta\psi^0 = \beta \, , \; \psi^0|_{\partial\Omega} = 0 \, ,$$

and θ^j is the solution of

$$\Delta\theta^j = 0 \, , \; \theta^j|_{\partial D_i} = \delta_{ij} \, , \; \theta^j|_{\partial\widetilde{\Omega}} = 0 \, .$$

$\widehat{S}_\Omega^{\mathbf{A}}$ is given by

$$\widehat{S}_\Omega^{\mathbf{A}} = -\int_\Omega \beta\psi^0 + \sum_j C_j C_j \int_\Omega \nabla\theta^i \nabla\theta^j \, dx \, .$$

Hence we obtain

$$\widehat{S}_{\Omega}^{\mathbf{A}} \leq \|\beta\|^2 \lambda_D(\Omega)^{-1} + \sum_j M_{ij} C_j C_j \,,$$

with

$$M_{ij} = \int_\Omega \nabla\theta^i \nabla\theta^j = -\int_{\partial D_i} \partial_\nu \theta^j \,.$$

For the upper bound to $\lambda_1^N(\beta B, \Omega)$, we implement the gauge invariance in order to minimize the C_j. We introduce

$$\mathbf{A}^0 = \nabla^\perp \psi^0 \,,$$

and the circulations of $\mathbf{A}^0$ and $\mathbf{A}$ along ∂D_i

$$\Phi_i^0 = \int_{\partial D_i} A_0 \, ds \,,\; \Phi_i = \int_{\partial D_i} \mathbf{A} \, ds \text{ for } i = 1, \ldots, k \,.$$

We note that

$$\Phi_i = \Phi_i^0 - \sum_j M_{ij} C_j \,,$$

M is positive definite and we get

$$\widehat{S}_\Omega^{\mathbf{A}} = S_\Omega^\beta + |M^{-\frac{1}{2}}(\Phi - \Phi^0)|^2 \,.$$

Coming back to the upper bound for $\lambda^N(\mathbf{A})$, we can use the gauge invariance of the problem to get:

$$\lambda^N(\mathbf{A}) \leq \frac{1}{A(\Omega)} \left(\|\beta\|^2 \lambda_1^D(\Omega)^{-1} + \inf_{\gamma \in \mathbb{Z}^k} \left(|M^{-\frac{1}{2}}(\Phi - \Phi^0 - 2\pi\gamma)|^2 \right) \right) \tag{24}$$

Using the isoperimetric inequality for λ_1^D, we get

$$\lambda^N(\mathbf{A}) \leq \frac{1}{A(\Omega)} \left(\|\beta\|^2 \lambda_1^D(D(0, R_\Omega))^{-1} + \inf_{\gamma \in \mathbb{Z}^k} \left(|M^{-\frac{1}{2}}(\Phi - \Phi^0 - 2\pi\gamma)|^2 \right) \right. \tag{25}$$

Pauli operator

Let Ω be a connected, regular domain in $\mathbb{R}^2$, $B = B(x)$ (denoted by β in the previous slides) be a magnetic field in $C^\infty(\bar{\Omega})$, and $h > 0$ a semiclassical parameter.
We are interested in the analysis of the ground state energy $\lambda^D_{P_-}(h, \mathbf{A}, B, \Omega)$ of the Dirichlet realization of the Pauli operator

$$P_\pm(h, \mathbf{A}, B, \Omega) := (hD_{x_1} - A_1)^2 + (hD_{x_2} - A_2)^2 \pm hB(x)\,.$$

Here $D_{x_j} = -i\partial_{x_j}$ for $j = 1, 2$.
The vector potential $\mathbf{A} = (A_1, A_2)$ satisfies

$$B(x) = \partial_{x_1} A_2(x) - \partial_{x_2} A_1(x)\,. \tag{26}$$

The Pauli operator is non-negative on $C_0^\infty(\Omega)$.
This follows by an integration by parts or think also of the square of the Dirac operator

$$D_A := \sum_j \sigma_j (hD_{x_j} - \mathbf{A}_j)\,,$$

where the σ_j ($j = 1, 2$) are the 2×2-Pauli matrices.
We have, on $C_0^\infty(\Omega; \mathbb{C}^2)$

$$D_A^2 := (P_-(h, \mathbf{A}, B, \Omega), P_+(h, \mathbf{A}, B, \Omega))\,.$$

This implies that

$$\lambda^D_{P_\pm}(h, \mathbf{A}, B, \Omega) \geq 0\,.$$

When $\Omega = \mathbb{R}^2$ and $B > 0$ constant, we have

$$\lambda_{P_-}(h, \mathbf{A}, B, \mathbb{R}^2) = 0\,.$$

When $\Omega = \mathbb{R}^2$ under weak assumptions on $B(x)$ (see Helffer-Nourrigat-Wang (1989), Thaller (book 1992))

$$0 \in \sigma_{ess}(P_-(h, \mathbf{A}, B, \mathbb{R}^2)) \cup \sigma_{ess}(P_+(h, \mathbf{A}, B, \mathbb{R}^2))\,.$$

Is 0 in the kernel ? Aharonov-Casher's theorem (see Cycon-Froese-Kirsch-Simon (book 1986)).
What is going on when Ω is bounded ?

Two years ago, T. Ekholm, H. Kowarik and F. Portmann [8] give a lower bound which has a universal character

Theorem EKP

Let Ω be regular, bounded, simply connected in $\mathbb{R}^2$. If B does not vanish identically in Ω, $\exists\, \epsilon > 0$ s.t. $\forall h > 0$, $\forall \mathbf{A}$ s.t. $\mathrm{curl}\mathbf{A} = B$,

$$\lambda^D_{P_-}(h, \mathbf{A}, B, \Omega) \geq \lambda^D(\Omega)\, h^2 \exp\big(-\epsilon/h\big)\,. \tag{27}$$

where $\lambda^D(\Omega)$ denotes the ground state energy of the Laplacian on Ω.

Our goal is to determine (when $B > 0$) the optimal ϵ, to give exponentially small upper bounds [17] and to analyze the non simply connected case [18]. This will be done in the semi-classical limit: $h \to 0$.

The main theorem in [17] is

Theorem HPS1

If $B(x) > 0$, Ω is simply connected and if ψ_0 is the solution of

$$\Delta\psi_0 = B(x) \text{ in } \Omega\,,\ \psi_{0/\partial\Omega} = 0\,,$$

then, for any $h > 0$,

$$\lambda^D_{P\_}(h, B, \Omega) \geq \lambda^D(\Omega)\, h^2 \exp\big(2 \inf \psi_0 / h\big)\,. \tag{28}$$

and, in the semi-classical limit

$$\lim_{h\to 0} h \log \lambda^D_{P\_}(h, B, \Omega) \leq 2 \inf \psi_0\,.$$

In the non simply connected case, such formulation could be wrong. The result could depend on the circulations of the magnetic potential along the different components of the boundary. Below we show that in the semi-classical limit the circulation effects disappear in the rate of the exponential decay.

Theorem HPS2

If $B(x) > 0$, Ω is connected, and if ψ_0 is the solution of

$$\Delta\psi_0 = B(x) \text{ in } \Omega\,,\ \psi_{0/\partial\Omega} = 0\,,$$

then, for any $\mathbf{A}$ such that $\mathrm{curl}\,\mathbf{A} = B$,

$$\lim_{h\to 0} h \log \lambda^D_{P\_}(h, \mathbf{A}, B, \Omega) = 2 \inf \psi_0\,. \tag{29}$$

We observe that the lower bound in this generalization is no more universal and only true in the semi-classical limit. The proof uses strongly the gauge invariance of the problem.

Dirichlet forms and Witten Laplacians

The problem we study is quite close to the question of analyzing the smallest eigenvalue of the Dirichlet realization of the operator associated with the quadratic form:

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \, dx \,. \tag{30}$$

For this case, we can mention Theorem 7.4 in Freidlin-Wentcell, which says (in particular) that, if f has a unique non-degenerate local minimum x_{min}, then the lowest eigenvalue $\lambda_1(h)$ of the Dirichlet realization $\Delta_{f,h}^{(0)}$ in Ω satisfies:

$$\lim_{h\to 0} -h \log \lambda_1(h) = 2 \inf_{x\in\partial\Omega} (f(x) - f(x_{min})) \,. \tag{31}$$

More precise or general results (prefactors) are given in Bovier-Eckhoff-Gayrard-Klein. This is connected with the semi-classical analysis of Witten Laplacians (Witten, Helffer-Sjöstrand, Cycon-Froese-Kirsch-Simon, Simon, Helffer-Klein-Nier, Helffer-Nier, Lepeutrec, Michel,....)

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