Dirichlet-to-Neumann operators on differential forms

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Introduction: classical Steklov problem

Laplace-Beltrami operator

- Let (M, g) be a smooth oriented connected Riemannian manifold of dimension n with smooth boundary.
- Let us consider the Laplace-Beltrami operator

$$\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right)$$

  and the following eigenvalue problem on M

$$\begin{aligned} \Delta f &= 0 \quad \text{on } M \\ \partial_n f &= \sigma f \quad \text{on } \partial M, \end{aligned} \tag{1}$$

  where n is the outward unit normal to ∂M.
- Values of σ for which a non-zero solution of (1) exists are called Steklov eigenvalues.

Dirichlet-to-Neumann map

- Equivalently, one can construct those numbers as eigenvalues of the Dirichlet-to-Neumann operator defined on $C^\infty(\partial M)$ in the following way.
- For any $u \in C^\infty(\partial M)$ there exists a unique harmonic extension $\hat{u} \in C^\infty(M)$, i.e. $\Delta \hat{u} = 0$ and $\hat{u}|_{\partial M} = u$.
- Then DtN operator L sends u to $\partial_n \hat{u}$.
- L is an elliptic self-adjoint ΨDO of order 1. Therefore, Steklov eigenvalues form an increasing sequence

$$0 = \sigma_1(M,g) < \sigma_2(M,g) \leqslant \sigma_3(M,g) \leqslant \sigma_4(M,g) \leqslant \dots$$

Note that enumeration starts with 1.

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Upper bounds

The most known general upper bound is due to Hersch, Payne and Schiffer. It states that for a manifold homeomorphic to a disk one has

$$\sigma_{p+1}\sigma_{q+1}L^2(\partial M) \leqslant \begin{cases} \pi^2(p+q-1)^2 & \text{if } p+q \text{ is odd} \\ \pi^2(p+q)^2 & \text{if } p+q \text{ is even} \end{cases}$$

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Upper bounds

For an orientable surface of genus γ with k boundary components one has the following inequality due to A. Girouard and I. Polterovich

$$\sigma_{p+1}\sigma_{q+1}L^2(\partial M) \leqslant \begin{cases} \pi^2(p+q-1)^2(\gamma+k)^2 & \text{if } p+q \text{ is odd} \\ \pi^2(p+q)^2(\gamma+k)^2 & \text{if } p+q \text{ is even.} \end{cases}$$

There are several other results for surfaces due to Fraser-Schoen, Hassannezhad, Kokarev and others.

New upper bound

Theorem (K., 2015)

For any orientable Riemannian surface of genus γ with k boundary components one has

$$\sigma_{p+1}\sigma_{q+1}L^2(\partial M) \leqslant \begin{cases} \pi^2(p+q+2\gamma+2k-3)^2 & \text{if } p+q \text{ is odd} \\ \pi^2(p+q+2\gamma+2k-2)^2 & \text{if } p+q \text{ is even} \end{cases}$$

Differential forms: prequel

Theorem (Yang, Yu, 2015)

Let M be an oriented Riemannian n-dimensional manifold with boundary. Then one has the inequality

$$\mu^{(n-2)}_{q+b_{n-2}(M)}\sigma_{p+1} \leqslant \lambda_{p+q+b_{n-1}(M)}(\partial M).$$

DtN map on differential forms

Boundary data of a differential form

Let $i\colon \partial M \to M$ be the inclusion map.

- $\omega|_{\partial M} = i^*\omega + dn \wedge i_n\omega.$
- Thus there are four components of boundary data associated with ω:

$$i^*\omega, \quad i^*\delta\omega, \quad i_n\omega, \quad i_n d\omega.$$

- Which are Dirichlet? Which are Neumann?

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Separation

Dirichlet

$i^*\omega$

Neumann

$i_n d\omega$

Separation

Dirichlet

$i^*\omega$

Neumann

$i_n d\omega$

Harmonic extension is **not unique**!

Separation

Dirichlet

$i^*\omega$, $i^*\delta\omega$

Neumann

$i_n\omega$, $i_n d\omega$

$i_n\omega$ $i^*\delta\omega$

Separation

Dirichlet

$i^*\omega$, $i^*\delta\omega$

Neumann

$i_n\omega$, $i_n d\omega$

Relative

$i^*\omega$, $i_n\omega$

Absolute

$i^*\delta\omega$, $i_n\omega$

$i_n\omega$ $i^*\delta\omega$

Definitions of DtN maps

Dirichlet-to-Neumann D

$$\begin{cases} \Delta\omega_D = 0 & x \in M \\ i^*\omega_D = \phi, \ \boxed{i_n\omega_D = 0} & x \in \partial M \end{cases}$$

$$D\phi = i_n d\omega_D$$

Relative-to-Absolute R

$$\begin{cases} \Delta\omega_R = 0 \\ i^*\omega_R = \phi, \ \boxed{i^*\delta\omega_R = 0} \end{cases}$$

$$R\phi = i_n d\omega_R$$

Properties of D

- D was introduced by Raulot and Savo
- D is a self-adjoint elliptic ΨDO of order 1 and thus has discrete spectrum on each of the spaces $\Omega^p(\partial M)$

$$0 \leqslant \mu_1^{(p)} \leqslant \mu_2^{(p)} \leqslant \mu_3^{(p)} \leqslant \dots$$

- Rayleigh quotient

$$R_D[\phi] = \frac{\int_M |d\omega_D|^2 + |\delta\omega_D|^2}{\int_{\partial M} |\phi|^2}$$

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Disclaimer

Operator similar to R (specifically $*R$) was introduced by Belishev and Sharafutdinov in the context of inverse problem for Maxwell equation. However, its spectral properties were not investigated.

Properties of R

Theorem (K. 2017)

R is identically zero on the space of exact forms $\mathcal{E}^p(\partial M)$. Restricted to the space of co-closed forms $c\mathcal{C}^p(\partial M)$ it is a positive self-adjoint operator with discrete spectrum

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The Rayleigh quotient is given by

$$R_R[\phi] = \frac{\int_M |d\omega_R|^2}{\int_{\partial M} |\phi|^2}$$

Further properties of R

Let us denote by $\tilde{\mu}_i^{(p)}$ and $\tilde{\sigma}_i^{(p)}$ the i-th non-zero eigenvalue of D and R respectively.

Theorem (K. 2017)

For all $i \geqslant 0$

$$\tilde{\mu}_i^{(p)} \leqslant \tilde{\sigma}_i^{(p)}.$$

Further properties of R

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Rayleigh quotient for $\tilde{\mu}$

$$\frac{\int_M |d\omega_D|^2 + |\delta\omega_D|^2}{\int_{\partial M} |\phi|^2}$$

Rayleigh quotient for $\tilde{\sigma}$

$$\frac{\int_M |d\omega_R|^2}{\int_{\partial M} |\phi|^2}$$

Reminder

Theorem (Yang, Yu, 2015)
Let M be an oriented Riemannian n-dimensional manifold with boundary. Then one has the inequality

$$\tilde{\mu}_q^{(n-2)}\tilde{\mu}_p^{(0)} \leqslant \lambda_{p+q+b_{n-1}(M)}(\partial M).$$

Generalisation

Theorem (K. 2017)

Let M be an oriented Riemannian n-dimensional manifold with boundary. Then for any $0 \leqslant p \leqslant n-2$ one has the inequality

$$\tilde{\sigma}_m^{(n-2-p)} \tilde{\sigma}_r^{(p)} \leqslant \lambda'^{(p)}_{I_p+m+r+b_{n-p-1}(M)-1},$$

where

- $\lambda_i'^{(p)}$ *is the* p*-th eigenvalue of Hodge Laplacian on* $c\mathcal{C}^p(\partial M)$
- $I_q = \dim\ker R^{(q)} = \dim\operatorname{im}\{i^*\colon H^q(M) \to H^q(\partial M)\}$

For $n = 2p+2$ and $r = m$

$$\left(\tilde{\sigma}_m^{(p)}\right)^2 \leqslant \lambda'^{(p)}_{I_p+2m+b_{p+1}(M)-1}$$

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Geometric optimisation problem

Classical Steklov problem

- The spectrum is invariant with respect to conformal changes of metric which are constant on the boundary.
- For a fixed surface M and a fixed metric on ∂M the eigenvalues $\sigma_i(M)$ are bounded by a constant independent of the metric inside of M.

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New optimisation problem

Eigenvalues $\sigma_i^{(p)}$ for $n = 2p + 2$ possess similar properties.

- Conformal invariance follows from the invariance of

$$\int_M |d\omega_R|^2 \, dv_g$$

- Boundedness follows from

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New optimisation problem

We propose the following higher dimensional geometric optimisation problem

- Fix M and metric h on ∂M
- For all i find the supremum of $\tilde{\sigma}_m^{(p)}$ among all metrics g with $i^*g = h$

Simplest case

Theorem

The metric on euclidean ball $\mathbb{B}^{2p+2}$ *maximises* $\tilde{\sigma}_m^{(p)}$ *for* $p \leqslant \frac{1}{2}\binom{2p+2}{p+1}$ *among all metrics restricting to a canonical metric on* $\partial\mathbb{B}^{2p+2} = \mathbb{S}^{2p+1}$*.*

Open Questions

- Prove that the inequality

$$\left(\tilde{\sigma}_m^{(p)}\right)^2 \leqslant \lambda'^{(p)}_{I_p+2m+b_{p+1}(M)-1}$$

is sharp for all m.
- Rigidity result for the previous theorem.
- Maximiser for $\mathbb{S}^{2p}$.

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Sequel

- Let M be a simply connected surface. Let u be a function on ∂M, orthogonal to a constant function.
- Extend it to a harmonic function $\hat{u}$ on M.
- Then the 1-form $*d\hat{u}$ is closed. Since M is simply connected, there exists a unique function $\hat{v}$, such that $d\hat{u} = *d\hat{v}$ and $v = \hat{v}|_{\partial M}$ is orthogonal to a constant function. The function $\hat{v}$ is called harmonic conjugate to $\hat{u}$.
- Linear map $u \mapsto v$ is often refered to as Hilbert transform.

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$$\int_M \langle d\hat{v}, d\hat{v} \rangle = \int_{\partial M} v \partial_n \hat{v} \leqslant \left(\int_{\partial M} v^2 \right)^{\frac{1}{2}} \left(\int_{\partial M} \langle i_n d\hat{v}, i_n d\hat{v} \rangle \right)^{\frac{1}{2}}$$

Since $*d\hat{v} = d\hat{u}$ the latter factor equals $\left(\int_{\partial M} \langle du, du \rangle \right)^{\frac{1}{2}}$

Putting everything together we arrive at

$$R_S[u] R_S[v] \leqslant R_{\partial M}[u]$$

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