Shape optimisation for the Robin Laplacian: an overview

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The Robin eigenvalue problem

Consider

$$
\begin{aligned}
-\Delta u &= \lambda u && \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

where

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with Lipschitz bdry $\partial\Omega$,
- $\frac{\partial u}{\partial \nu}(x)$ is the outer normal derivative to $\partial\Omega$ at $x \in \partial\Omega$,
- $\alpha \in \mathbb{R}$ is a constant.

For each fixed Ω and α there is a sequence of eigenvalues

$$
\lambda_1(\Omega, \alpha) \leq \lambda_2(\Omega, \alpha) \leq \ldots \to \infty
$$

which depend (more or less) smoothly on α.

$$
\boxed{\alpha = 0 \implies \text{Neumann problem,}}
$$

$$
\boxed{\alpha = \pm\infty \implies \text{(formally) Dirichlet.}}
$$

The Robin eigenvalue problem

These eigenvalues interpolate between their Dirichlet and Neumann counterparts:

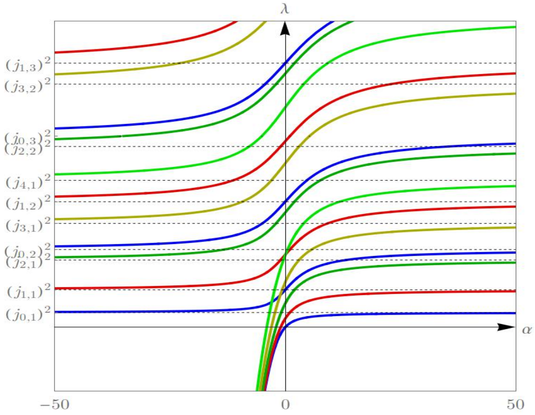


Figure: The first few eigenvalues of the unit disk as a function of α. Picture courtesy of Pedro Freitas

The Robin eigenvalue problem

Weak form

$$\int_\Omega \nabla\psi \cdot \nabla u \, \mathrm{d}x + \int_{\partial\Omega} \alpha\psi u \, \mathrm{d}\sigma = \lambda \int_\Omega \psi u \, \mathrm{d}x, \qquad u \in H^1(\Omega).$$

Variational characterisation

$$\lambda_1(\Omega, \alpha) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_{\partial\Omega} \alpha u^2 \, \mathrm{d}\sigma}{\int_\Omega u^2 \, \mathrm{d}x} : 0 \neq u \in H^1(\Omega) \right\}$$

with corresponding min-max and max-min principles for the higher eigenvalues.

Question

Let $\alpha \in \mathbb{R}$ and $k \geq 1$ be fixed. Which Ω optimises the eigenvalue $\lambda_k(\Omega, \alpha)$ among all domains of given volume?

The case $\alpha > 0$

The Bossel–Daners inequality

Conjecture (attributed to Krahn)

The ball minimises $\lambda_1(\Omega, \alpha)$ among all (Lipschitz) domains of given volume, for any fixed $\alpha > 0$.

- Proved by Bossel (2D, 1986) Daners (general case, 2006)
- Generalised to the p-Laplacian by Dai and Fu (2011) and Bucur and Daners (2010), i.e. for $p \in (1, \infty)$

$$\lambda_1^p(\Omega, \alpha) = \inf \left\{ \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_{\partial\Omega} \alpha |u|^p \, \mathrm{d}\sigma}{\int_\Omega |u|^p \, \mathrm{d}x} : 0 \neq u \in W^{1,p}(\Omega) \right\}$$

- Also shown to hold on general open sets of finite measure by Bucur and Giacomini (2010, 2015)
- The inequality is sharp: equality $\lambda_1(\Omega, \alpha) = \lambda_1(B, \alpha)$ implies Ω is a ball up to a negligible set (Daners and K., 2007; Bucur and Daners, 2010; Bucur and Giacomini, 2010), quantitative version (Bucur, Ferone, Nitsch and Trombetti, preprint)

The Bossel–Daners inequality: idea of the proof

Denote by ψ the first eigenfunction on Ω, chosen positive, then for all $u \in H^1(\Omega)$,

$$\int_\Omega \nabla\psi \cdot \nabla u + \int_{\partial\Omega} \alpha\psi u = \lambda_1 \int_\Omega \psi u.$$

Choose $u := 1/\psi$ as a test function, then $\nabla u = -\nabla\psi/\psi^2$, and

$$-\int_\Omega \frac{|\nabla\psi|^2}{\psi^2} + \int_{\partial\Omega} \alpha = \lambda_1 \int_\Omega 1,$$

that is,

$$\lambda_1 = \frac{1}{|\Omega|}\left(-\int_\Omega \frac{|\nabla\psi|^2}{\psi^2} + \int_{\partial\Omega} \alpha\right).$$

Now define for $t > 0$

$U_t := \{x \in \Omega : \psi(x) > t\}$	upper level set
$\partial_e U_t := \partial U_t \cap \partial\Omega$	exterior part of ∂U_t
$\partial_i U_t := \partial U_t \cap \Omega$	interior part of ∂U_t

The Bossel–Daners inequality: idea of the proof

One can show (formally take $u = (1/\psi) \cdot 1_{U_t}$ as a test function):

$$\lambda_1 = \frac{1}{|U_t|} \left(-\int_{U_t} \frac{|\nabla \psi|^2}{\psi^2} + \int_{\partial_i U_t} \frac{|\nabla \psi|}{\psi} + \int_{\partial_e U_t} \alpha \right).$$

Hence define

$$H_\Omega(U_t, \varphi) := \frac{1}{|U_t|} \left(-\int_{U_t} \varphi^2 + \int_{\partial_i U_t} \varphi + \int_{\partial_e U_t} \alpha \right).$$

Lemma

For a broad class of φ, assuming $\varphi \neq |\nabla \psi|/\psi$, there exists a set of $t \in (\min \psi, \max \psi)$ of positive measure such that $\lambda_1 > H_\Omega(U_t, \varphi)$.

Now take the function $|\nabla \psi^*|/\psi^*$ on B, rearrange it onto a test function φ on Ω and use the isoperimetric inequality:

$$\lambda_1(\Omega, \alpha) \geq H_\Omega(U_t, \varphi) \geq H_B(B_{r(t)}, |\nabla \psi^*|/\psi^*) = \lambda_1(B, \alpha).$$

Further reading: Bucur and Daners, Calc. Var. 37 (2010), 75–86.
Bucur, Ferone, Nitsch, Trombetti, arXiv:1611.06704

The higher eigenvalues

- For λ_2, for any fixed $\alpha > 0$, the minimiser is the disjoint union of two equal balls (K., 2009)
- For the higher eigenvalues, the situation becomes more interesting (K., 2010; Antunes, Freitas and K., 2013): The minimisers will depend on α
- For any given Ω, as $\alpha \to 0$,

$$\lambda_k(\Omega, \alpha) \to \lambda_k(\Omega, 0) := \mu_k(\Omega),$$

the kth Neumann eigenvalue: $\mu_k(\Omega) = 0$ if Ω has (at least) k connected components

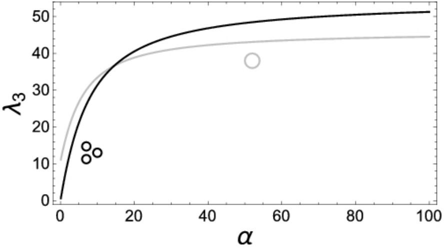
Proposition (K., 2010)

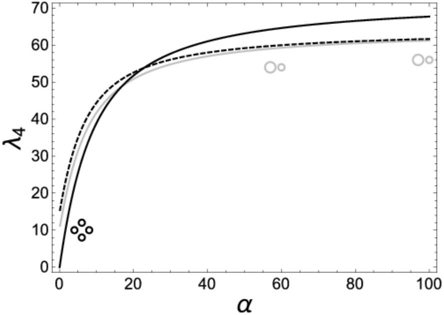
For any Ω and any k there exists $\alpha^*(k, \Omega)$ such that

$$\lambda_k(\Omega, \alpha) \geq \lambda_k(B_k, \alpha) \qquad \text{for all } \alpha \in (0, \alpha^*],$$

where B_k is the disjoint union of k equal balls, each of vol. $|\Omega|/k$.

Some numerics (thanks to Pedro Antunes)





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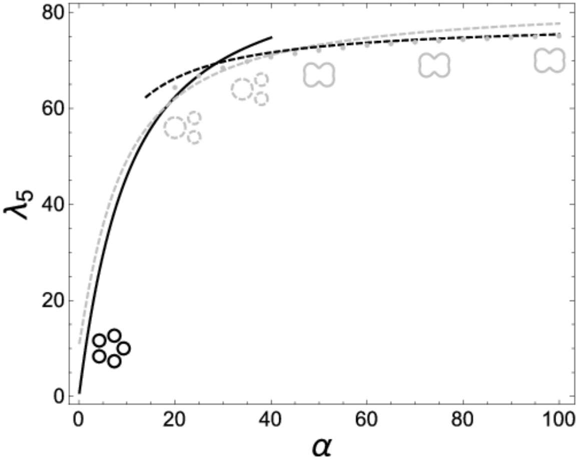
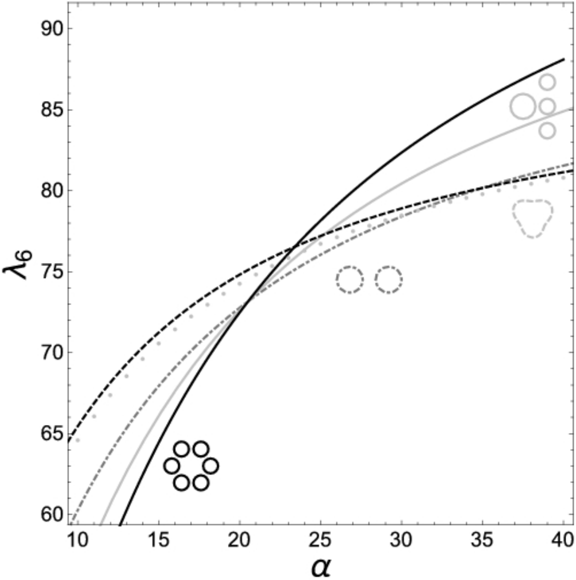


Some numerics (thanks to Pedro Antunes)



The higher eigenvalues II

Homothetic scalings $\Omega \mapsto t\Omega$:

$$|\Omega|^{2/d}\lambda_k(\Omega,\alpha) = |t\Omega|^{2/d}\lambda_k(t\Omega,\alpha/t)$$

Consequence (Antunes–Freitas–K., 2013)

Set $\lambda_k^*(V,\alpha) := \inf\{\lambda_k(\Omega,\alpha) : \Omega \textit{ smooth}, |\Omega| = V\}$. *Then*

$$\lambda_k^*(V,\alpha) \leq \lambda_k(B_k,\alpha) \leq d\alpha\left(k\omega_d V^{-1}\right)^{1/d}.$$

In particular, for fixed $\alpha > 0$, $\lambda_k^*(V,\alpha) = o(k^{1/d+\varepsilon})$ *as* $k \to \infty$, *in contrast to the Weyl asymptotics* $\lambda_k(\Omega,\alpha) = O(k^{2/d})$ *for fixed* Ω.

Consequence (Antunes–Freitas–K., 2013)

The dimensionally normalised optimal "spectral gap" tends to 0*:*

$$(\lambda_{k+1}^*)^{d/2} - (\lambda_k^*)^{d/2} \to 0 \qquad \text{as } k \to \infty \ (\alpha \text{ fixed}).$$

Open problems

What do we not have/not know?

- Universal inequalities, inequalities for ratios, sums etc.
- Existence (and regularity etc) of a domain minimising λ_k for fixed $|\Omega|$ and $\alpha > 0$
- There exists $\alpha^* = \alpha^*(k, d, |\Omega|)$ such that $\lambda_k(\Omega, \alpha)$ is minimised by d balls of equal volume whenever $\alpha \in (0, \alpha^*]$. Moreover, $\alpha^* \to \infty$ as $k \to \infty$
- Asymptotic optimality of B as $k \to \infty$, for fixed $\alpha > 0$

The case $\alpha < 0$

What do we expect for λ_1?

- $\lambda_1(\Omega, \alpha) < 0$. In fact, $\lambda_1(\Omega, \alpha) < \alpha \frac{|\partial\Omega|}{|\Omega|} = \frac{\int_\Omega |\nabla 1|^2 + \int_{\partial\Omega} \alpha \cdot 1^2}{\int_\Omega 1^2}$
- There is no *minimiser* among all domains of given volume: recalls the *Neumann* problem

Conjecture (Bareket, 1977)

For any $\alpha < 0$, $\lambda_1(\Omega, \alpha)$ is maximised among all domains of given volume by the ball.

The conjecture was reiterated in 2007 (Brock–Daners) after the proof of the Bossel–Daners inequality
Why this conjecture?

- Bareket showed the disk is a local maximiser in two dimensions generalised recently by Ferone, Nitsch and Trombetti (2015)
- The formula $\frac{d}{d\alpha}\Big|_{\alpha=0} \lambda_1(\Omega, \alpha) = \frac{|\partial\Omega|}{|\Omega|}$ implies the ball is the "asymptotic" maximiser as $\alpha \to 0$
- The solution of any isoperimetric problem is always a ball

What happens as $\alpha \to -\infty$?

Theorem (Lacey–Ockendon–Sabina, 1998; Lou-Zhu, 2004)

For any bounded domain $\Omega \subset \mathbb{R}^d$ with C^1 boundary,

$$\lambda_1(\Omega, \alpha) \sim -\alpha^2 + o(\alpha^2), \qquad \alpha \to -\infty.$$

Actually, $\lambda_1(\Omega, \alpha) \leq -\alpha^2$ for all Ω and $\alpha < 0$.
If Ω has "corners", the asymptotic behaviour is $-C\alpha^2$ for some $C \geq 1$ depending on the corner (Lacey *et al*; Levitin–Parnovski, 2008).

Example

If $\Omega = (0, \infty)$, then $\psi(x) := e^{\alpha x}$ is an eigenfunction with eigenvalue $-\alpha^2$:

$$\psi''(x) = \alpha^2 e^{\alpha x}, \qquad \frac{\partial \psi}{\partial \nu}\Big|_{x=0} = -\psi'(0) = -\alpha\psi(0).$$

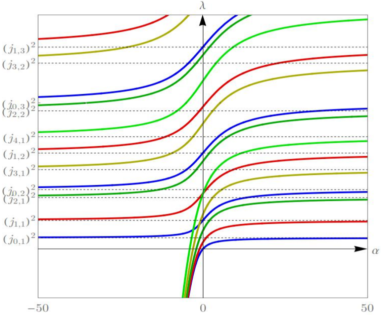
What happens as $\alpha \to -\infty$?

Theorem (Daners–K., 2010)

For any bounded domain $\Omega \subset \mathbb{R}^d$ with C^1 boundary, $d \geq 2$, and any $k \geq 1$,

$$\lambda_k(\Omega, \alpha) \sim -\alpha^2 + o(\alpha^2), \qquad \alpha \to -\infty.$$

However, another subset of the eigenvalues converges to the Dirichlet spectrum from above as $\alpha \to -\infty$.



But does this help us with the isoperimetric problem?

Theorem (Exner-Minakov-Parnovski, '14; Pankrashkin-Popoff, '15)

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be bounded with smooth (C^3) boundary. Then for any $k \geq 1$,

$$\lambda_k(\Omega, \alpha) = -\alpha^2 + (d-1)\gamma_{\max}\alpha + O(|\alpha|^{2/3}), \qquad \alpha \to -\infty,$$

where $\gamma_{\max}$ is the maximum mean curvature of $\partial\Omega$.

This was obtained independently by Freitas and Krejčiřík for the ball and spherical shells and used to negate Bareket's conjecture:

Theorem (Freitas–Krejčiřík, 2015)

For any ball B, there exist $r_1, r_2 > 0$ and $\alpha_0 < 0$ such that the spherical shell $S_{r_1,r_2} = \{x \in \mathbb{R}^d : r_1 < |x| < r_2\}$ has the same volume as B and

$$\lambda_1(B, \alpha) \leq \lambda_1(S_{r_1,r_2}, \alpha) \qquad \text{for all } \alpha \leq \alpha_0.$$

Bareket's conjecture is true for sufficiently small $\alpha < 0$

Theorem (Freitas–Krejčiřík, 2015)

There exists an $\alpha_0 < 0$ depending only on the area of the domains such that for all $\alpha \in [\alpha_0, 0]$ and all bounded $\Omega \subset \mathbb{R}^2$ of that given area with C^2 boundary

$$\lambda_1(\Omega, \alpha) \leq \lambda_1(B, \alpha)$$

where B is any disk of the same area.

If we fix the *perimeter* rather than the volume, then there is an inequality for all $\alpha < 0$:

Theorem (Antunes–Freitas–Krejčiřík, preprint 2016)

For all bounded $\Omega \subset \mathbb{R}^2$ with C^2 boundary and all $\alpha < 0$

$$\lambda_1(\Omega, \alpha) \leq \lambda_1(B, \alpha)$$

where B is any disk with $|\partial B| = |\partial \Omega|$.

Further problems; the other eigenvalues

- The proofs use *parallel coordinates* inspired by Payne–Weinberger:
- Construct a test function whose level lines run parallel to (the outermost part of) $\partial\Omega$ to compare $\lambda_1(\Omega, \alpha)$ from above with the first eigenvalue of an annulus with Robin-Neumann conditions
- This Robin-Neumann eigenvalue is smaller than $\lambda_1(B, \alpha)$ if α is small enough (how small depending on $|\Omega|$), or always if $|\partial\Omega|$ and not $|\Omega|$ is fixed
- Open problem: generalise to higher dimensions
- Higher eigenvalues: shells "asymptotically" optimal for $\lambda_k(\Omega, \alpha)$ as $\alpha \to -\infty$ for any fixed k. Existence?
- Numerical work by Antunes, Freitas and Krejčiřík (2016) suggests the existence of $\alpha_*(d, k, |\Omega|)$ such that $\lambda_k(\Omega, \alpha)$ is maximised by a shell for all $\alpha \leq \alpha_*$
- Is the maximiser of $\lambda_2(\Omega, \alpha)$ a ball for α close to zero?

Thank you for your attention!