Generalizations of Pleijel’s nodal domain theorem

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Improvements and conjectures
We study eigenvalue problems in a domain $\Omega$, that is to say an open, bounded and connected subset of $\mathbb{R}^n$, or more generally of an $n$-dimensional Riemannian manifold. The eigenvalue equation is

$$-\Delta u = \lambda u \text{ in } \Omega$$

with a boundary condition on $\partial \Omega$: $u = 0$ (Dirichlet), $\frac{\partial u}{\partial n} = 0$ (Neumann) or $\frac{\partial u}{\partial n} + hu = 0$ (Robin).

For an eigenfunction $u$, we define the nodal set as

$$\mathcal{N}(u) := \overline{u^{-1}(\{0\})}$$

and the nodal domains as the connected components of $\Omega \setminus \mathcal{N}(u)$. 
Some classical results

Theorem (J. C. F. Sturm [1836])
For a regular Sturm-Liouville problem on a segment \((a, b)\), with eigenvalues \((\lambda_k)_{k \geq 1}\), an eigenfunction associated with \(\lambda_k\) has \(k - 1\) zeros in \((a, b)\).

Remark
The theorem of Sturm and Liouville states that a \(k\)-th eigenfunction divides the segment into \(k\) nodal domains.

Theorem (R. Courant [1923])
Let \(\Omega\) be an open, bounded and connected set in \(\mathbb{R}^n\), with a sufficiently regular boundary. Let \((\lambda_k)_{k \geq 1}\) be the eigenvalues of the Laplacian, with Dirichlet, Neumann or Robin boundary condition. Then, an eigenfunction associated with \(\lambda_k\) has at most \(k\) nodal domains.
Pleijel’s result

Let $\Omega$ be an open, bounded and connected set in $\mathbb{R}^2$, and let $(\lambda_k)_{k \geq 1}$ denote the eigenvalue of the Laplacian with a Dirichlet boundary condition.

**Theorem (Å. Pleijel [1956])**
There is only a finite number of indices $k$ for which $\lambda_k$ has an eigenfunction with $k$ nodal domains.

Let $\nu_k$ denote the maximal number of nodal domains for an eigenfunction associated with $\lambda_k$.

**Proposition (Å. Pleijel [1956])**
We have the asymptotic upper bound

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^2}{\lambda_1(\Omega) |\Omega|^2} = \frac{4}{j^2} < 1.$$
The n-dimensional case

The result can be generalized to $\mathbb{R}^n$. We keep the same notation.

**Theorem**

If $\Omega \subset \mathbb{R}^n$,

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)^\frac{n}{2} |\mathbb{B}^n|^2} < 1.$$ 

We write

$$\gamma(n) := \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)^\frac{n}{2} \omega_n^2}.$$ 

We have the explicit expression

$$\gamma(n) = \frac{2^{n-2} n^2 \Gamma \left( \frac{n}{2} \right)^2}{j_{\frac{n}{2}-1,1}^n},$$

where $j_{\frac{n}{2}-1,1}$ is the smallest positive zero of the Bessel function of the first kind $J_{\frac{n}{2}-1}$.

**Theorem (B. Helffer, M. Persson Sundqvist [2016])**

The sequence $n \mapsto \gamma(n)$ is decreasing and goes to 0 exponentially fast ($\gamma(n + 1)/\gamma(n) \to 2/e$).
Outline of the proof

Let $u$ be an eigenfunction associated with $\lambda_k$ with $\nu_k$ nodal domains. Let us denote by $D_1, \ldots, D_{\nu_k}$ the nodal domains of $u$. We apply the Faber-Krahn inequality to a domain $D_i$:

$$\lambda_k^\frac{n}{2} |D_i| = \lambda_1(D_i)^\frac{n}{2} |D_i| \geq \lambda_1(\mathbb{B}^n)^\frac{n}{2} \omega_n.$$ 

Summing over $i \in \{1, \ldots, \nu_k\}$, we obtain

$$\lambda_k^\frac{n}{2} |\Omega| \geq \nu_k \lambda_1(\mathbb{B}^n)^\frac{n}{2} \omega_n$$

and therefore

$$\nu_k \leq \frac{\lambda_k^\frac{n}{2} |\Omega|}{\lambda_1(\mathbb{B}^n)^\frac{n}{2} \omega_n}.$$ 

According to Weyl’s law

$$\lambda_k^\frac{n}{2} |\Omega| \sim \frac{(2\pi)^n k}{\omega_n}.$$ 

We obtain

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)^\frac{n}{2} \omega_n^2}.$$
Some extensions

The asymptotic upper bound

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(n)$$

holds for the eigenfunctions of the following operators.

- The Laplace-Beltrami operator in $\Omega$, with a Dirichlet boundary condition on $\partial \Omega$, where $\Omega$ is an open and connected set compactly included in a 2-dimensional Riemannian manifold $M$ ($n = 2$), with $M$ homeomorphic to a disk (J. Petree [1957]).

- The Laplace-Beltrami operator in $M$, a compact $n$-dimensional Riemannian manifold with or without boundary, with a Dirichlet boundary condition on $\partial M$ if it is not empty (P. Bérard and D. Meyer [1982]).

- The Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^n$, for several choices of $V$, including the (possibly anisotropic) harmonic potential and the Coulomb potential (P. Charron [2015] and P. Charron, B. Helffer and T. Hoffmann-Ostenhof [2016]).
Neumann boundary condition: an example

Given $\Omega \subset \mathbb{R}^2$ an open, bounded and connected set with a sufficiently regular boundary, we consider the Laplacian in $\Omega$ with Neumann boundary condition, and denote by $(\mu_k(\Omega))_{k \geq 1}$ its eigenvalues.

Proposition (Å. Pleijel [1956])
If $\Omega = Q := (0, \pi)^2$, 
\[
\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{4}{j^2}.
\]

Proof.
Given an eigenfunction $u$, divide its nodal domains into
- the interior domains $D^0_{1}, \ldots, D^0_{\nu_k}$ not touching $\partial Q$;
- the boundary domains $D^1_{1}, \ldots, D^1_{\nu_k}$ adjacent to $\partial Q$.

Using the fact that the eigenfunctions, restricted to one of the four sides of $Q$, are trigonometric polynomials of degree at most $\sqrt{\mu_k(Q)}$, we get $\nu^1_k \leq C \sqrt{\mu_k(Q)}$ for some constant $C$, so that $\lim_{k \to +\infty} \frac{\nu^1_k}{k} = 0$.

On the other hand, we can bound the number of interior domains as in the Dirichlet case. \qed
Neumann boundary condition: generalization

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set with a piecewise analytic boundary. We again consider the Laplacian in $\Omega$ with Neumann boundary condition, whose eigenvalues we denote by $(\mu_k)_{k \geq 1}$.

**Theorem (J.A. Toth, S. Zelditch [2009])**

For $k \geq 1$, we denote by $r_k$ the greatest possible number of zeros of $u$ on $\partial \Omega$, where $u$ is an eigenfunction associated with $\mu_k$. There exists a constant $C_\Omega$ such that

$$r_k \leq C_\Omega \sqrt{\mu_k}.$$

**Theorem (I. Polterovich [2009])**

Under the above hypotheses for $\Omega$, for the Neumann-Laplacian eigenfunctions,

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{4}{j^2}.$$
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Improvements and conjectures
Statement of the result

Let $\Omega$ be an open, bounded and connected open set in $\mathbb{R}^n$ with a boundary $\partial \Omega$ of class $C^{1,1}$. For $h \in \text{Lip}(\overline{\Omega})$ such that $h \geq 0$ on $\partial \Omega$, we consider the eigenvalue problem with Robin boundary condition

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega; \\ \frac{\partial u}{\partial n} + hu = 0 & \text{on } \partial \Omega. \end{cases}$$

We denote by $(\mu_k)_{k \geq 1}$ the associated sequence of eigenvalues, arranged in non-decreasing order and counted with multiplicities. Using standard regularity result for elliptic boundary value problems, we can show that any eigenfunction of the above problem is of class $C^1(\Omega)$.

As before, we denote by $\nu_k$ the maximal number of nodal domain for an eigenfunction associated with $\mu_k$.

**Theorem**

We have the asymptotic upper bound

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(n).$$
Preliminaries: inequalities in a nodal domain

Let \((\mu, u)\) is an eigenpair and \(D\) a nodal domain of \(u\). We have

\[
\int_D |\nabla u|^2 \leq \mu \int_D u^2.
\]

Indeed,

\[
\int_D |\nabla u|^2 = \int_D (-\Delta u) u + \int_{\partial D} \frac{\partial u}{\partial n} u = \mu \int_D u^2 - \int_{\partial D} h u^2 = \\
\mu \int_D u^2 - \int_{\partial D \cap \Omega} h u^2 - \int_{\partial D \cap \partial \Omega} h u^2 \leq \mu \int_D u^2.
\]

Furthermore, the Faber-Krahn inequality gives a lower bound for the Rayleigh quotients. More precisely, for each \(v \in H^1_0(D)\), we define

\[
R(v, D) = \frac{\int_D |\nabla v|^2}{\int_D v^2},
\]

and we have

\[
\lambda_1 (B^n)^{\frac{n}{2}} \omega_n \leq R(v, D)^{\frac{n}{2}} |D|.
\]

The function \(v\) does not need to be a groundstate of \(-\Delta\) in \(D\).
Given a nodal domain \( D \), the main idea is to distinguish between two cases:

- most of the mass of the eigenfunction is inside \( \Omega \);
- there is a non-negligible amount of mass near \( \partial \Omega \).

This is a natural distinction when we try to apply the previous form of the Faber-Krahn inequality.

Given \( r > 0 \), we consider

\[
\begin{align*}
\partial \Omega_r & := \{ x \in \mathbb{R}^n ; \text{dist}(x, \partial \Omega) < r \} ; \\
\partial \Omega_r^+ & := \partial \Omega_r \cap \Omega .
\end{align*}
\]

For \( \delta > 0 \) small enough, we construct non-negative smooth functions \( \varphi_0 \) and \( \varphi_1 \) such that

- \( \varphi_0^2 + \varphi_1^2 = 1 \) in \( \Omega \),
- \( \text{supp}(\varphi_0) \subset \Omega \setminus \overline{\partial \Omega_a^+} \) and \( \text{supp}(\varphi_1) \subset \partial \Omega_{A \delta}^+ \),
- \( \| \nabla \varphi_i \|_{L^\infty} \leq C \delta^{-1} \) for \( i \in \{ 0, 1 \} \),

with \( 0 < a < A \) and \( C \) independent of \( \delta \).
Let us consider an eigenpair \((\mu, u)\). We define \(u_0 := \varphi_0 u\) and \(u_1 := \varphi_1 u\).

By construction of \((\varphi_0, \varphi_1)\), we have, for each nodal domain \(D\) of \(u\),

\[
\int_D u^2 = \int_D u_0^2 + \int_D u_1^2.
\]

We fix \(\varepsilon \in (0, 1)\). With respect to this choice, we say that \(D\) is

\begin{itemize}
  \item a bulk domain if \(\int_D u_0^2 \geq (1 - \varepsilon) \int_D u^2\);
  \item a boundary domain if \(\int_D u_1^2 > \varepsilon \int_D u^2\).
\end{itemize}

We write

\begin{itemize}
  \item \(\nu^0(u, \varepsilon)\) for the number of bulk domains;
  \item \(\nu^1(u, \varepsilon)\) for the number of boundary domains.
\end{itemize}

Ultimately, we want to show that \(\nu^1(u, \varepsilon) \ll \nu^0(u, \varepsilon)\) for \(\mu\) large. We therefore take \(\delta\) depending on \(\mu\), namely

\[\delta := \mu^{-\theta}\]

with \(\theta > 0\) to be determined.

Following the steps of Pleijel's proof, we obtain, for the number of bulk domains,

\[
\nu^0(u, \varepsilon) \leq \frac{|\Omega|}{\lambda_1(B^n)^{\frac{n}{2}} \omega_n} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \mu + \frac{1 + \frac{1}{\varepsilon}}{1 - \varepsilon} C^2 \mu^{2\theta} \right)^{\frac{n}{2}}.
\]
Given a boundary domain $D$, we define
$$
\tilde{D} := D \cap \{ u_1 \neq 0 \} \subset \partial \Omega^+_A,
$$
where $u_1^R$ and $\tilde{D}^R$ are the reflection of $u_1$ and $\tilde{D}$ through $\partial \Omega$. We have in particular
$$
\tilde{D}^R \subset \partial \Omega_A.
$$

We have
$$
R \left( u_1^R, \tilde{D}^R \right) = R \left( u_1, \tilde{D} \right) = \frac{\int_D |\nabla u_1|^2}{\int_D u_1^2} \leq \frac{2}{\varepsilon} \left( \mu + C^2 \mu^2 \theta \right),
$$
and, Faber-Krahn inequality applied to $\tilde{D}^R$ gives us
$$
\lambda_1 \left( \mathbb{B}^n \right) \frac{n}{2} \omega_n \leq \left| R \left( u_1^R, \tilde{D}^R \right) \right| \frac{n}{2} \left| \tilde{D}^R \right| = 2 \left| R \left( u_1, \tilde{D} \right) \right| \frac{n}{2} \left| \tilde{D} \right|
$$

Summing over all boundary domains, we get
$$
\nu^1(u, \varepsilon) \leq C' \frac{|\partial \Omega_A|}{\lambda_1 \left( \mathbb{B}^n \right) \frac{n}{2} \omega_n} \left( \mu + C^2 \mu^2 \theta \right) \frac{n}{2},
$$
and therefore
$$
\nu^1(u, \varepsilon) \leq C'' \mu^{-\theta} \left( \mu + C^2 \mu^2 \theta \right) \frac{n}{2}.
$$
Let \((u_k)_{k \geq 1}\) be a sequence of eigenfunction associated with \((\mu_k)\) and having each the maximal number of nodal domain, \(\nu_k\).

Let us recall that we have

\[
\nu^0(u_k, \varepsilon) \leq \frac{|\Omega|}{\lambda_1(B^n)^{\frac{n}{2}} \omega_n} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \mu_k + \frac{1 + \frac{1}{\varepsilon}}{1 - \varepsilon} C^2 \mu_k^2 \theta \right)^{\frac{n}{2}}.
\]

and

\[
\nu^1(u_k, \varepsilon) \leq C'' \mu^{-\theta} \left( \mu_k + C^2 \mu_k^2 \theta \right)^{\frac{n}{2}}.
\]

We choose \(\theta \in (0, \frac{1}{2})\), for instance \(\theta = \frac{1}{4}\). Using Weyl’s law, we have

\[
\mu_k \leq \lambda_k \sim \frac{4\pi^2}{(\omega_n |\Omega|)^{\frac{2}{n}}} k^{\frac{2}{n}}.
\]

Therefore

\[
\lim_{k \to +\infty} \frac{\nu^1(u_k, \varepsilon)}{k} = 0 \quad \text{and} \quad \limsup_{k \to +\infty} \frac{\nu^0(u_k, \varepsilon)}{k} \leq \frac{(2\pi)^n}{\lambda(B^n)^{\frac{n}{2}} \omega_n^2} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{n}{2}}.
\]

Since \(\nu_k = \nu^0(u_k, \varepsilon) + \nu^1(u_k, \varepsilon)\), we get

\[
\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda(B^n)^{\frac{n}{2}} \omega_n^2} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{n}{2}},
\]

and the conclusion when \(\varepsilon \to 0\).
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Improvements and conjectures
Recent improvements: sharper upper bound

We now go back to eigenvalues of the Laplacian with a Dirichlet boundary condition.

\[
\Omega \subset \mathbb{R}^2 \quad \limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(2) - 3 \cdot 10^{-9} \quad \text{(J. Bourgain [2013])}
\]

\[
\Omega \subset \mathbb{R}^2 \quad \limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(2) - \varepsilon(2) \quad \text{(S. Steinerberger [2013])}
\]

\[
\Omega \subset M^n \quad \limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(n) - \varepsilon(n) \quad \text{(H. Donnelly [2014])}
\]

**Conjecture (I. Polterovich [2009])**

For any domain \( \Omega \subset \mathbb{R}^2 \),

\[
\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{2}{\pi}.
\]

The analysis of rectangles show that the conjecture upper bound is optimal.
Recent improvements: geometric control

General form of the results
(P. Bérard and B. Helffer [2016], M. van den Berg and K. Gittins [2016])

Given a "regular enough" domain $\Omega$, there are no more than $N(\Omega)$ eigenvalues satisfying equality in Courant Theorem, where $N(\Omega)$ depends on known geometric quantities associated with $\Omega$.

The proofs rely on explicit geometric estimates for the remainder in Weyl’s law.

**Theorem (M. van den Berg and K. Gittins [2016])**

Let $\Omega$ be a convex domain in $\mathbb{R}^n$, then there is an (explicit) constant $C(n)$ such that

$$N(\Omega) \leq C(n) \frac{\mathcal{H}^{n-1}(\partial \Omega)^n}{|\Omega|^{n-1}}$$

Can we prove similar results for Neumann/Robin eigenfunctions?