Geometric bounds for Steklov eigenvalues

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Joint work with Joachim Stubbe

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The Steklov problem

We consider the Steklov eigenvalue problem on $\Omega \subset \mathbb{R}^N$

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u, & \text{on } \partial\Omega. \end{cases}$$

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A. Girouard, I. Polterovich, *Spectral geometry of the Steklov problem*. Journal of Spectral Theory, 7 (2017), 321–359.

Basic properties

Variational characterization of Steklov eigenvalues:

$$\sigma_j = \min_{\substack{V \leq H^1(\Omega), \\ \dim V = j+1}} \max_{0 \neq u \in V} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\sigma}$$

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Weyl's asymptotic law (if Ω is piecewise C^1):

$$\sigma_j \sim 2\pi \omega_{N-1}^{-\frac{1}{N-1}} \left(\frac{j}{|\partial\Omega|} \right)^{\frac{1}{N-1}} \text{ as } j \to +\infty,$$

where ω_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$.

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Remark: analogous inequalities hold for Dirichlet eigenvalues (Li-Yau, lower bounds), Neumann eigenvalues (Kröger, upper bounds) and eigenvalues of the Laplacian on Riemannian manifolds (Buser, Cheng-Yang, Colbois-Maerten).

Geometric inequalities

The problem is completely solved for simply connected $\Omega \subset \mathbb{R}^2$:

$$\sigma_1 \leq \frac{2\pi}{|\partial\Omega|}$$

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where

$$I(\Omega) = \frac{|\partial\Omega|}{|\Omega|^{\frac{N-1}{N}}}$$

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We also have isodiametric control

$$\sigma_j \leq C_N \frac{j^{\frac{2}{N}+1}}{\mathrm{diam}(\Omega)}$$

B. Bogosel, D. Bucur, A. Giacomini, *Optimal Shapes Maximizing the Steklov Eigenvalues*. SIAM J. Math. Anal., 49(2) (2017), 1645–1680.

The Laplace-Beltrami operator

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with the variational characterization

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The eigenvalues satisfy the Weyl's asymptotic law

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and Weyl-type bounds of the form

$$\lambda_j \leq \frac{(N-2)\kappa^2}{4} + C_N \left(\frac{j}{|\partial\Omega|}\right)^{\frac{2}{N-1}}.$$

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Asymptotic formulas suggest that for large j

$$\sigma_j \approx \sqrt{\lambda_j}$$

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Theorem (P. - Stubbe, 2017)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with connected boundary $\partial\Omega$ of class C^2. Then there exists a constant c_Ω such that for all $j \in \mathbb{N}$

$$\lambda_j \leq \sigma_j^2 + 2c_\Omega\sigma_j, \quad \sigma_j \leq c_\Omega + \sqrt{c_\Omega^2 + \lambda_j}\,.$$

In particular,

$$\left|\sigma_j - \sqrt{\lambda_j}\right| \leq 2c_\Omega,$$

the constant c_Ω depending on the maximal possible size of a tubular neighborhood about $\partial\Omega$ and on the mean of the maximal curvatures of $\partial\Omega$.

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L. Provenzano, J. Stubbe, *Weyl-type bounds for Steklov eigenvalues*. *arXiv:1611.00929* (2017). To appear on the *Journal of Spectral Theory*.

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The constant c_Ω is given by

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- h is the maximal possible size of a tubular neighborhood about $\partial\Omega$;
- $$\bar{H}_\infty = \max_{x\in\partial\Omega}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}|\kappa_i(x)|\right)$$

  and $\kappa_i(x)$, $i = 1, ..., N-1$ are the principal curvatures of $\partial\Omega$ at x.

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Remark

One side of the estimate can be refined so that

$$\sigma_j - \sqrt{\lambda_j} \leq \frac{1}{h} + (N-1)H_\infty$$

where $H_\infty := \max_{x\in\partial\Omega} |H(x)|$ *with* $H(x)$ *the mean curvature of* $\partial\Omega$ *at* x.

Consequences

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Corollary (Weyl-type upper bounds for Steklov eigenvalues)

Let Ω be a bounded domain of class C^2 in $\mathbb{R}^N$ with connected boundary. Then for all $j \in \mathbb{N}$ it holds

$$\sigma_j \leq A_\Omega + C_N \left(\frac{j}{|\partial\Omega|} \right)^{\frac{1}{N-1}},$$

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The corollary follows from the main result and from upper bounds for Laplacian eigenvalues on $\partial\Omega$.

Consequences

Theorem (Asymptotically sharp upper bounds for Steklov Riesz-means)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with connected boundary $\partial\Omega$ of class C^2. Then for all $z \geq 0$

$$\sum_{j=0}^{\infty}(z-\sigma_j)_+^2 \leq \frac{2}{N(N+1)}(2\pi)^{-(N-1)}\omega_{N-1}|\partial\Omega|\left(z+c_\Omega\right)^{N+1}.$$

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Bounds are asymptotically sharp since

$$\lim_{z\to+\infty}\frac{1}{z^{N+1}}\sum_{j=0}^{\infty}(z-\sigma_j)_+^2 = \frac{2}{N(N+1)}(2\pi)^{-(N-1)}\omega_{N-1}|\partial\Omega|.$$

Consequences

Corollary (Sharp upper bounds for the trace of the Steklov heat kernel)

Let Ω be a bounded domain of class C^2 in $\mathbb{R}^N$ with connected boundary. Then

$$\sum_{j=0}^{\infty} e^{-\sigma_j t} \leq \frac{1}{N(N+1)} (2\pi)^{-(N-1)} \omega_{N-1} |\partial\Omega| t^{-N-1} e^{c_\Omega t} \Gamma(N+2, c_\Omega t)$$

for all $t > 0$, where $\Gamma(a,b) = \int_b^{\infty} t^{a-1} e^{-t}\, dt$.

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The estimate is sharp as $t \to 0^+$ since it implies

$$\limsup_{t \to 0_+} t^{N-1} \sum_{j=0}^{\infty} e^{-\sigma_j t} \leq (2\pi)^{-N-1} B_{N-1} \Gamma(N) |\partial\Omega|.$$

Corollary (Weyl-type lower bounds)

Let Ω be a bounded domain of class C^2 in $\mathbb{R}^N$ with connected boundary. Then for all $j \in \mathbb{N}$:

$$\sigma_j \geq r_N 2\pi \omega_{N-1}^{-\frac{1}{N-1}} \left(\frac{j+1}{|\partial\Omega|} \right)^{\frac{1}{N-1}} - c_\Omega$$

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This corollary is an immediate consequence of the bounds on the Steklov heat kernel.

Strategy of the proof

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$$i) \qquad \int_{\partial\Omega} |\nabla_{\partial\Omega} v|^2 d\sigma \leq \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma + 2c_\Omega \left(\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma\right)^{\frac{1}{2}}$$

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$$ii)\qquad \left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}} \leq c_{\Omega} + \sqrt{c_{\Omega}^2 + \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma}$$

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If we formally substitute $\lambda_j \sim \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2d\sigma$ and $\sigma_j \sim \left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2d\sigma\right)^{\frac{1}{2}}$, $i)$ and $ii)$ become

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In particular, for λ_j:

$$\lambda_j = \inf_{\substack{V \leq H^1(\partial\Omega) \\ \dim V = j+1}} \sup_{\substack{0 \neq v \in V \\ \int_{\partial\Omega} v^2 d\sigma = 1}} \int_{\partial\Omega} |\nabla_{\partial\Omega} v|^2 d\sigma$$

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where $V_S = \left\langle {u_0}_{|_{\partial\Omega}}, \dots, {u_j}_{|_{\partial\Omega}} \right\rangle$, u_i are the first $j+1$ Steklov eigenfunctions on $\partial\Omega$.

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where $V_{LB} = \langle \phi_0, ..., \phi_j \rangle$, ϕ_i are the harmonic extension in Ω of the first $j+1$ Laplace-Beltrami eigenfunctions on $\partial\Omega$.

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where $V_{LB} = \langle \phi_0, ..., \phi_j \rangle$, ϕ_i are the harmonic extension in Ω of the first $j+1$ Laplace-Beltrami eigenfunctions on $\partial\Omega$. Since $v \in V_{LB}$ are harmonic,

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It remains then to prove *i)* and *ii)*.

Strategy of the proof

Inequalities *i*) and *ii*) follow by applying a Rellich-Pohozaev identity for harmonic functions v and Lipschitz vector fields F:

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} F \cdot \nabla v d\sigma - \frac{1}{2} \int_{\partial\Omega} |\nabla v|^2 F \cdot \nu d\sigma \\ + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \mathrm{div} F dx - \int_{\Omega} (DF \cdot \nabla v) \cdot \nabla v dx = 0,$$

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We need a very specific F in order to obtain *i*) and *ii*).

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Inequalities *i*) and *ii*) are consequence of the following choice

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$$\eta(x) := \frac{(h - \mathrm{dist}(x, \partial\Omega))^2}{2}.$$

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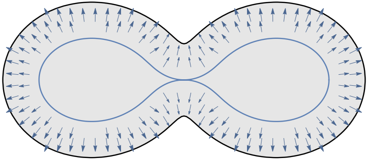

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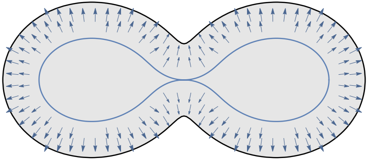

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Plugging F into the Rellich-Pohozaev identity we obtain that for harmonic functions v it holds

$$\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma - \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma = \frac{1}{h}\int_{\omega_h}\left(2(D^2\eta\cdot\nabla v)\cdot\nabla v - |\nabla v|^2\Delta\eta\right)dx.$$

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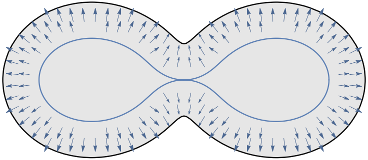

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$$\rho_i(x) = \begin{cases} \frac{(h - \mathrm{dist}(x,\partial\Omega))\kappa_i(x')}{1 - \mathrm{dist}(x,\partial\Omega)\kappa_i(x')}, & \text{if } i = 1, ..., N-1, \\ 1, & \text{if } i = N, \end{cases}$$

where x' is the nearest point to x on $\partial\Omega$.

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where x' is the nearest point to x on $\partial\Omega$.

$$\left| \int_{\omega_h} 2(D^2\eta \cdot \nabla v) \cdot \nabla v - |\nabla v|^2 \Delta\eta dx \right| \leq (1 + (N-1)\bar{H}_\infty h) \int_\Omega |\nabla v|^2 dx.$$

This and suitable estimates imply *i)* and *ii)*.

Example 1: convex domains

If Ω is a bounded and convex domain of class C^2 in $\mathbb{R}^N$ then

$$\lambda_j \leq \sigma_j^2 + (N-1)K_\infty \sigma_j$$

and

$$\sigma_j \leq \frac{K_\infty}{2} + \sqrt{\frac{K_\infty^2}{4} + \lambda_j},$$

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$$\sigma_j \leq K_\infty + C_N \left(\frac{j}{|\partial\Omega|}\right)^{\frac{1}{N-1}}.$$

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It is known that if Ω is a ball of radius R in $\mathbb{R}^N$, then given a Steklov eigenvalue σ it is of the form

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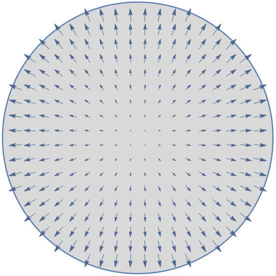
for some $l \in \mathbb{N}$. The corresponding eigenfunction are the restriction to Ω of the harmonic polynomials in $\mathbb{R}^N$. Given a Laplace-Beltrami eigenvalue λ, it is of the form

$$\lambda = \frac{l(l+N-2)}{R^2}$$

for some $l \in \mathbb{N}$. The corresponding eigenfunctions are the spherical harmonics on $\partial\Omega$.

Example 2: balls

If Ω is a ball, $\eta(x) = \frac{|x|^2}{2}$ so we can take $F(x) = x$ in the Rellich-Pohozaev identity



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We obtain that for a function v harmonic in Ω

$$\int_{\partial\Omega} |\nabla_{\partial\Omega} v|^2 d\sigma = \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma + \frac{N-2}{R} \int_{\Omega} |\nabla v|^2 d\sigma$$

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- Remove the hypothesis of connected boundary;
- Less regular domains (piecewise C^1, polygonal) with other choices of F in the Rellich-Pohozaev identity.

THANK YOU