Overdetermined PDE in Riemannian Geometry
I, Euclidean space

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CUSO Mini-Course
Workshop on Geometric Spectral Theory
Neuchatel, June 19-20, 2017

Pompeiu problem

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$$\Delta u = -\sum_j \frac{\partial^2 u}{\partial x_j^2}$$

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Does this imply that $f = 0$ *?*

When Ω is a ball of radius R, the question is the following: suppose that there exists a continuous function on the whole $\mathbf{R}^n$ such that

$$\int_{B(x_0,R)} f = 0$$

for all $x_0 \in \mathbf{R}^n$.
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We will discuss the proof below.

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Note: A.G. Ramm published a paper (*Solution to the Pompeiu problem and the related symmetry problem* Applied Mathematics Letters 63 (2017) 28-33) in which he proves Pompeiu problem in dimension 3. However, he does not assume explicitly that $\partial\Omega$ is connected (which is a necessary assumption).

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Proposition

Let $\Omega = B^n(R)$ and consider any eigenfunction of the Laplacian which is defined on the whole $\mathbf{R}^n$, for example

$$f(x_1, \dots, x_n) = \sin(\sqrt{\lambda} x_1), \quad \lambda > 0$$

Take any $\lambda > 0$ so that $\sqrt{\lambda}R$ is a positive zero of the Bessel function $J_{\frac{n}{2}}$. Then

$$\int_{B(x_0,R)} f = 0$$

for all $x_0 \in \mathbf{R}^n$.

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In fact, consider $z_2 > z_1 > 0$ and let $r < R$ be such that

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Then $f(x_1, \dots, x_n) = \sin(\sqrt{\lambda} x_1)$ as above integrates to zero on all balls of radius r, and also on all balls of radius R.

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Then, it will integrate to zero also on any domain obtained removing a ball of radius r from a ball of radius R. That is why in the above conjecture we need to assume that Ω is homeomorphic to a ball.

Proof of Proposition

Let then $B(x_0, r)$ be the ball of center x_0 (an arbitrary point) and radius R, and let f be any C^2-function globally defined on $\mathbf{R}^n$. Let ρ be the distance function to the center x_0. Introduce the function $F = [0, \infty) \to \mathbf{R}$:

$$F(r) = \int_{B(x_0,r)} f.$$

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and (Green formula and some easy work):

$$F''(r) = \int_{\partial B(x_0,r)} \langle \nabla f, \nabla \rho \rangle - f \Delta \rho.$$

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It follows that

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Assume that f satisfies $\Delta f = \lambda f$. Then F satisfies the ODE

$$F''(r) - \frac{n-1}{r} F'(r) + \lambda F(r) = 0.$$

(the ODE is independent on x_0).

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$$h(x) = x^k g(x).$$

One verifies that $g(x)$ satisfies

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Choose $k = \frac{n}{2}$. The equation becomes:

$$g'' + \frac{1}{x}g' + \Big(1 - \frac{n^2}{4}\cdot\frac{1}{x^2}\Big)g = 0.$$

which is of Bessel type:

$$g'' + \frac{1}{x}g' + \Big(1 - \frac{\nu^2}{x^2}\Big)g = 0.$$

with $\nu = n/2$.

The space of *bounded* solutions to the previous ODE is one-dimensional, and is spanned by the Bessel function J_ν, defined by the power series:

$$J_\nu = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}.$$

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In conclusion, we get that if $\Delta f = \lambda f$ and $F(r) = \int_{B(x_0,r)} f$, then F satisfies

$$F''(r) - \frac{n-1}{r} F'(r) + \lambda F(r) = 0,$$

and there is a constant c such that

$$F(r) = c r^{\frac{n}{2}} J_{\frac{n}{2}}(\sqrt{\lambda} r).$$

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Hence, if $\sqrt{\lambda} R = z_j$, we see indeed that $F(R) = \int_{B(x_0,R)} f = 0$: note that this is true for all choices of x_0. The Proposition follows.

A spectral theoretic formulation: Schiffer conjecture

There are several equivalent formulations of Pompeiu problem. Many of them involve the Fourier transform of the characteristic function of the domain:

$$\tilde{\Omega}(\xi) = \int_\Omega e^{i\langle x,\xi\rangle}\, dx.$$

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Neumann eigenvalue problem. Given any bounded domain (in any Riemannian manifold) the problem

$$\begin{cases} \Delta f = \lambda f \quad \text{on} \quad \Omega \\ \dfrac{\partial f}{\partial N} = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

is known as the *Neumann eigenvalue problem*.

It admits a discrete sequence of eigenvalues (each repeated according to its multiplicity):

$$\lambda_1^N(\Omega) \le \lambda_2^N(\Omega) \le \dots$$

with associated eigenfunctions $f_1, f_2, \dots$, and $\lambda_k^N \to \infty$ as $k \to \infty$. (Note that $\lambda_1^N = 0$ because it is associated to the constant eigenfunction $f_1 = 1$).

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Consider now the following problem, called *Schiffer problem*:

$$\begin{cases} \Delta f = \lambda f \quad \text{on} \quad \Omega \\ \dfrac{\partial f}{\partial N} = 0, \quad f = c \neq 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

It is the Neumann problem, with the additional request that the eigenfunction is *constant on the boundary*.

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(2) is an example of *overdetermined problem*, because a generic domain will not support solutions: solutions exist (if they do) only in few "fortunate" cases, that is, for domains with a good amount of "symmetries", and for "special geometries".

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It is a remarkable fact that the Laplace operator preserves the class of radial function on the ball: if $f = f(r)$ is radial also Δf will be. In fact a calculation shows that

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Then, the eigenfunction equation $\Delta f = \lambda f$ for the Neumann problem, for radial functions, becomes the ODE on the interval $[0, R]$:

$$f''(r) + \frac{n-1}{r} f'(r) + \lambda f(r) = 0$$

with boundary conditions

$$f'(0) = f'(R) = 0$$

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If $f(r)$ is a radial eigenfunction with Neumann boundary conditions then automatically f solves (2).

Standard procedure shows that there are, in fact, infinitely many radial solutions (classical Sturm-Liouville theory). More precisely, proceeding as before we see that the above ODE has solutions

$$f(r) = cr^{-\nu} J_\nu(\sqrt{\lambda}), \quad \text{with } \nu = \frac{n-2}{2}.$$

The condition $f'(0) = 0$ is automatically satisfied. The condition $f'(R) = 0$ forces $\sqrt{\lambda}R$ to be a zero of the function $\psi(x) = xJ'_\nu(x) - \nu J_\nu(x)$.

We conclude

Proposition

Let $\{z_1, z_2, \dots\}$ *be the set of zeroes of the function* $\psi(x) = xJ'_\nu(x) - \nu J_\nu(x)$*, with* $\nu = \frac{n-2}{2}$*. Then, for each* $k = 1, 2, \dots$ *the function*

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is a solution to the Schiffer overdetermined problem (2) *on the ball* $B(0, R)$*.*

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Having established that balls supports solutions, it is natural to ask whether there are other domains with that property. At present, no new such domains are known, and here is a conjecture. (In dimension 3 the conjecture has recently been confirmed by A.G. Ramm.)

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The following is a remarkable connection between this conjecture and the Pompeiu problem. It was discovered by Williams.

We conclude

Proposition

Let $\{z_1, z_2, \dots\}$ *be the set of zeroes of the function* $\psi(x) = xJ'_\nu(x) - \nu J_\nu(x)$*, with* $\nu = \frac{n-2}{2}$*. Then, for each* $k = 1, 2, \dots$ *the function*

$$f_k(r) = r^{-\nu} J_\nu(\frac{z_k r}{R})$$

is a solution to the Schiffer overdetermined problem (2) *on the ball* $B(0, R)$*.*

Having established that balls supports solutions, it is natural to ask whether there are other domains with that property. At present, no new such domains are known, and here is a conjecture. (In dimension 3 the conjecture has recently been confirmed by A.G. Ramm.)

Schiffer conjecture. *Assume that* $\Omega \subseteq \mathbf{R}^n$ *supports a function* f *solving the overdetermined problem* (2)*. Then* Ω *is a ball.*

The following is a remarkable connection between this conjecture and the Pompeiu problem. It was discovered by Williams.

The Schiffer conjecture is equivalent to the Pompeiu conjecture.

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We now focus on the most famous case: $F \equiv 1$ and $c_1 = 0$, for which we do have a rigidity result.

Serrin problem

Consider the *mean-exit time function* $v = v(x)$, unique solution of the boundary value problem:

$$\begin{cases} \Delta v = 1 \quad \text{on} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

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If we make this extra assumption we then get an overdetermined problem, often called *Serrin problem*:

$$\begin{cases} \Delta v = 1 \quad \text{on} \quad \Omega, \\ v = 0, \, \dfrac{\partial v}{\partial N} = c \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3}$$

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A domain Ω *supports a solution to* (3) *if and only if the mean-value of any harmonic function* h *on* Ω *equals its mean-value on* $\partial\Omega$. *That is*:

$$\frac{1}{|\Omega|}\int_{\Omega} h = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} h$$

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It can be shown that harmonic domains are critical points of the torsional rigidity

Serrin rigidity result

First, we remark that any ball is a harmonic domain: mean exit time is a radial function $v = v(r)$ and in fact, for the ball of radius R centered at the origin in $\mathbf{R}^n$:

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Theorem

(Serrin) Assume that $\Omega \in \mathbf{R}^{n+1}$ admits a solution to (3)*. Then Ω is a ball and v is radially symmetric. That is, harmonic Euclidean domains are balls.*

He proved more generally that if there is a *positive* solution u of the problem

$$\begin{cases} \Delta u = F(u) \\ u = 0, \ \dfrac{\partial u}{\partial N} = c \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

then Ω is a ball.

Maximum principle

We give two proofs of the theorem: one is the original Serrin's proof by the moving planes method, the other is due to Weinberger, and uses the Bochner formula and an integral identity (Pohozaev identity). Both use the maximum principle, some consequences of which are summarized below.

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Theorem

(Maximum principle) Let Ω be a bounded domain and let $u \in C^2(\Omega)$ be a superharmonic function: $\Delta u \geq 0$ on Ω. Then:

a) *u attains its minimum on the boundary of Ω.*

b) *Let $x_0 \in \partial\Omega$ be a point where u attains its minimum value, and assume that $\partial\Omega$ is C^1 at x_0. Then either $\dfrac{\partial u}{\partial N}(x_0) > 0$ or u is constant on Ω.*

c) *If u takes its minimum value in the interior of Ω then u is constant.*

In fact, a) is the weak maximum principle, b) is the Hopf boundary point lemma and c) is the strong maximum principle.

The proof by Serrin : moving planes method

We will show that, for any unit vector ν in $\mathbf{R}^n$, there is a hyperplane H_ν orthogonal to ν with respect to which Ω is symmetric. This forces Ω to be invariant under all reflections, hence invariant under the whole group of rotations. So, Ω is a ball.

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From that moment on, the resulting plane T will cut off from Ω a cup $\Sigma(T)$. That is:

$$\Sigma(T) = \Omega \cap T_+,$$

where T_+ is the half space containing the starting plane T_0.

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$$\Sigma'(T) = \text{ reflection of } \Sigma(T) \text{ with respect to } T.$$

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i) $\Sigma'(T)$ *becomes internally tangent to* $\partial\Omega$ *at some point* $p \notin T$*;*

ii) T *reaches a position where it is orthogonal to* $\partial\Omega$ *at some point* q*.*

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We now focus on this "critical" hyperplane $\bar{T}$ and show that Ω must be symmetric with respect to it. We denote $D = \Sigma'(\bar{T})$, for simplicity, and define a function

$$v : D \to \mathbf{R}$$

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Clearly $\Delta v = 1$. On D we can consider also the function

$$h = u - v.$$

Clearly h is harmonic; since $D \subseteq \Omega$ it is also clear that $h \geq 0$ on ∂D. Therefore

$$h \geq 0 \quad \text{on } D.$$

Case 1. Assume that i) holds.

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Note that at p one has $v(p) = 0$ hence $h(p) = 0$: thus p is an absolute minimum of h on D; by the Hopf boundary point lemma, either $h \equiv 0$ on D or $\dfrac{\partial h}{\partial N}(p) > 0$.

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But ∂D and $\partial\Omega$ are tangent at p, hence they have the same inner unit normal; as v is the reflection of u one has:

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We conclude that $h = 0$ and $u = v$. This happens only if Ω is symmetric about $\bar{T}$, and we are done.

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First, Serrin proves a boundary point lemma adapted to the situation, and concludes that either h must be constant on D, or one of the two cases:

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Then, one shows that h has a zero of order 2 at q, in the sense that all derivatives up and including order 2 are zero. Hence, the second case cannot occur, which means that $h = 0$ and, again, Ω is simmetric w.r.t. $\bar{T}$. The proof is complete.

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Lemma

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$$\langle \nabla \Delta u, \nabla u \rangle = |\nabla^2 u|^2 + \frac{1}{2}\Delta(|\nabla u|^2).$$

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Another useful formula is Pohozaev identity (sometimes also called Rellich identity) :

Lemma

Let $x = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$ *be the position vector. For any smooth function* u *on* Ω*:*

$$\int_\Omega 2\langle x, \nabla u \rangle \Delta u + (n-2)|\nabla u|^2 = \int_{\partial\Omega} 2\langle x, \nabla u \rangle \frac{\partial u}{\partial N} - |\nabla u|^2 \langle x, N \rangle.$$

Recall that we have to prove that, if Ω is a domain in $\mathbf{R}^n$ which supports a solution to the overdetermined problem

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That is, one has on Ω:

$$|\nabla u|^2 + \frac{2}{n} u \leq c^2.$$

We integrate this inequality over Ω. By the Green formula

$$\int_\Omega |\nabla u|^2 = \int_\Omega u \Delta u - \int_{\partial\Omega} u \frac{\partial u}{\partial N} = \int_\Omega u.$$

and we arrive at

$$\frac{n+2}{n} \int_\Omega u \leq c^2 |\Omega|. \tag{6}$$

It is important to remark that, if equality holds, then $\nabla^2 u$ must be a scalar matrix, and since $\mathrm{tr}\nabla^2 u = -1$ at all points of Ω one sees that

$$\nabla^2 u = -\frac{1}{n} I. \tag{7}$$

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Therefore, equating the two sides:

$$\frac{n+2}{n}\int_\Omega u = c^2|\Omega|.$$

That is, we have equality in (6) and thus we have also (7).

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Hence $\partial\Omega$ is an umbilical hypersurface of $\mathbf{R}^n$, hence, it must be a sphere (of the appropriate radius). This means that Ω is a ball. The proof is complete.