Overdetermined PDE’s in Riemannian Geometry
II, Constant curvature spaces

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Serrin and other overdetermined problems on manifolds

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Recall that the metric $g$ defines a Laplace-Beltrami operator:

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which of course depends on the metric.
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If \(\nabla\) is the Levi-Civita connection associated to the metric, then

\[
\text{div} X = -\sum_{j=1}^{n} \langle \nabla_{e_j} X, e_j \rangle = -\sum_{j=1}^{n} \nabla X(e_j, e_j)
\]

so that

\[
\Delta f = -\text{tr}\nabla^2 f
\]

and \(\nabla^2 f = \nabla\nabla f\) is the Hessian of \(f\). Of course, the Green formula continues to hold.
The Dirichlet problem defining the mean exit time:

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\begin{align*}
\Delta u &= 1 \quad \text{on} \quad \Omega \\
\frac{\partial u}{\partial N} &= c \quad \text{on} \quad \partial \Omega
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If we ask that \( u \) has constant normal derivative we obtain \textit{Serrin problem on the manifold} \( M \):

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A \textit{harmonic domain} is a domain which supports a solution to the Serrin problem.
Here is a number of questions.

Existence: if $M$ is an arbitrary Riemannian manifold, do we always have harmonic domains there? Given a positive number $\alpha<|M|$, can we always find a harmonic domain of volume $\alpha$ inside $M$?

Rigidity: do we have geometric restrictions for the existence of harmonic domains? Can we describe them?

Classification: can we actually classify harmonic domains in the general Riemannian framework? We have seen that if $M$ is Euclidean space then the answer to the first question is positive (any ball is a harmonic domain) and we have a strong rigidity result: the only harmonic domains are balls. Naturally, the next step is to examine these questions in the other (simply connected) manifolds of constant curvature: hyperbolic space and the sphere.
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Serrin problem in hyperbolic space

It is not restrictive to study the case of the unique simply connected manifold of constant curvature $-1$, which is denoted $H^n$. 

A model of $H^n$ is given by the Poincare ball, that is, the unit ball in $\mathbb{R}^n$ with metric $g = 4(1-|x|^2)^2 \cdot g_E$ where $g_E$ is the Euclidean metric.

Existence. Geodesic balls are harmonic domains: in fact, the mean exit time function is radial. Here is a formula for the explicit expression valid in all space forms ($r$ is the distance function to a fixed point).

$$u(r) = \int_{\mathbb{R}} r \cdot \theta(s) \int_{0}^{s} \theta(t) \, dt \, ds$$

where

$$\theta(r) = \begin{cases} \sin^{n-1} r & \text{if } M = S^n, \\ r^{n-1} & \text{if } M = \mathbb{R}^n, \\ \sinh^{n-1} r & \text{if } M = H^n \end{cases}$$
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In the ball model of $\mathbb{H}^n$, if one fixes a point $p$ and a unit vector $\nu$, there exists a totally geodesic hypersurface $T_{p,\nu}$ orthogonal to $\nu$ at $p$: 
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- the reflection with respect to $T_{p,\nu}$ is in fact an isometry of $\mathbb{H}^n$. 
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- $T_{p,\nu}$ is a totally geodesic hypersurface, and
- the reflection with respect to $T_{p,\nu}$ is in fact an isometry of $H^n$.

Thus, if one fixes a unit vector $\nu$, there exists a one-parameter family of totally geodesic hypersurfaces $T_{p,\nu}$. 
This family plays the role of hyperplanes in the classical proof by Serrin for $\mathbb{R}^n$. Then, as $\Omega$ is compact, we can start the procedure by taking a hyperplane not intersecting $\Omega$, move it in the direction of $\Omega$ and play with reflections ... the final outcome is that the maximum principle will show that $\Omega$ must be symmetric with respect to $T_\nu$. As this is true for all hyperplanes $T_\nu$, $\Omega$ has to be a ball.

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One tries the method of moving planes as before; for each pair of antipodal points $p, \bar{p}$ in $\mathbb{S}^n$ we have a one parameter family of totally geodesic hypersurfaces.
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These are equators obtained by intersecting the sphere with a plane containing the line through $p$ and $\bar{p}$. Again, reflection around each of these is an isometry.
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Thus, we have all the ingredients to start the moving plane method and prove rigidity ... There is only one requirement: for the method to work, one needs to start from a hyperplane not intersecting $\Omega$. This means that, for the method to work, $\Omega$ has to be contained in a hemisphere.
With that restriction, we have rigidity:

Theorem (Molzon)
The only harmonic domains in $S^n$ (the hemisphere) are geodesic balls.

Of course, one could ask if the restriction to the hemisphere is an essential hypothesis, or is just assumed to make the method work. In other words: Is it true that the only harmonic domains in $S^n$ are geodesic balls? The answer is: no, as we shall see in the next section.
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Exotic harmonic domains in spheres

The first example was given by Berenstein (in the nineties).

Consider the 2-surface (Clifford torus) isometrically embedded in $S^3$:

$$\Sigma = S^1\left(\frac{1}{\sqrt{2}}\right) \times S^1\left(\frac{1}{\sqrt{2}}\right).$$

It is easy to show that $\Sigma$ is the common boundary of two domains $\Omega_1$ and $\Omega_2$. Berenstein shows that both these domains are harmonic. As the boundary of each is a torus, which is topologically different from a sphere, it is clear that $\Omega_1$ and $\Omega_2$ are not isometric to geodesic balls. This gives the desired counterexample. His proof is analytical; we provide here a simpler proof.
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Then the natural map $\phi : \Sigma_{a,b} \to S^3$ defined by
\[ \phi((x_1, x_2), (x_3, x_4)) = (x_1, x_2, x_3, x_4) \]
is an isometric embedding.
Consider the domain $\Omega \subseteq S^3$ defined by the inequalities:

$$\Omega : \begin{cases} x_1^2 + x_2^2 \leq a^2 \\ x_3^2 + x_4^2 \geq b^2 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \end{cases}$$

so that $\partial \Omega = \Sigma_{a,b}$. 
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The group $SO(2)$ acts on each circle by rotations, hence the group

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**Proposition**

*The domain $\Omega$ is harmonic.*
Again, we need to show that the solution of

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has constant normal derivative:

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\frac{\partial u}{\partial N} = c.
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Now \(G\) acts on \(C^\infty(\Omega)\) as follows: for \(g \in G\) define \(g \cdot u \in C^\infty(\Omega)\) by

\[
(g \cdot u)(x) = u(g^{-1}(x)).
\]

As the Laplacian commutes with isometries, one has

\[
\Delta(g \cdot u) = g \cdot \Delta u = g \cdot 1 = 1
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and clearly \(g \cdot u = 0\) on \(\partial \Omega\).

Hence \(g \cdot u\) is also a solution of (1). By uniqueness, \(g \cdot u = u\) for all \(g \in G\).
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The invariance of $u$ shows that then

$$\frac{\partial u}{\partial N}(p) = \frac{\partial u}{\partial N}(q).$$
Hence $u$ is $G$-invariant, that is, it is constant on the orbits.

An isometry is in particular a conformal map, hence the action of the group preserves the unit normal vector field $N$; as the action of $G$ is transitive on $\partial \Omega$, given $p, q \in \partial \Omega$ we can always find $g \in G$ such that $g \cdot p = q$.

The invariance of $u$ shows that then

$$\frac{\partial u}{\partial N}(p) = \frac{\partial u}{\partial N}(q).$$

As $p$ and $q$ are arbitrary, this implies that $\frac{\partial u}{\partial N}$ is constant on $\partial \Omega$, as asserted.
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Each of the harmonic domains $\Omega$ above is foliated by a one parameter family of parallel Clifford tori:

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\begin{align*}
  x_1^2 + x_2^2 &= t^2 \\
  x_3^2 + x_4^2 &= 1 - t^2
\end{align*}
\]

where $t \in (0, a]$, collapsing to the great circle (minimal submanifold):

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Finally, we remark that the above example generalizes to get the following class of examples:

*Let $\Omega$ be any compact Riemannian manifold with boundary on which $G$ acts by isometries. Assume that the action of $G$ restricts to a transitive action on $\partial \Omega$. Then, $\Omega$ is a harmonic domain.*
Isoparametric hypersurfaces

The scope of this section is to further enlarge the class of harmonic domains inside the sphere, by observing that we have one such domain whenever the boundary is *isoparametric.*
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A way to see how curved is $\Sigma$ in $M$ is to examine the rate of change of the normal field $N$ infinitesimally, along a curve $\gamma$ in $\Sigma$. 
At any point \( p \in \Sigma \) and for any tangent vector \( X \in T_p \Sigma \) we can define the covariant derivative of \( N \) along \( X \), denoted \( S_p(X) : \)

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S_p(X) = -\nabla_X N.
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which is self-adjoint with respect to the inner product given by the metric.

\( S_p \) is called the \textit{shape operator} of \( \Sigma \) at \( p \). Its eigenvalues \( k_1, \ldots, k_{n-1} \) are called \textit{principal curvatures} of \( \Sigma \) at \( p \), and its trace, divided by \( n - 1 \), is the \textit{mean curvature}:

\[
H(p) = \frac{1}{n-1}(k_1 + \cdots + k_{n-1}).
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First variation of area

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$$\Sigma_V(t) = \{\exp_x(tV(x)) : x \in \Sigma\}$$ for $t \in (-\epsilon, \epsilon)$.

We can then consider the area of the perturbed surface $\Sigma_V(t)$, as a function of $t$:

$$t \mapsto \text{vol}(\Sigma_V(t))$$.

We say that $\Sigma$ is a critical point of the area if

$$\frac{d}{dt}\bigg|_{t=0} \text{vol}(\Sigma_V(t)) = 0$$

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To first order, these correspond to vector fields \( V \) for which
\[ \int_{\Sigma} \langle V, N \rangle = \int_{\Sigma} \phi = 0. \]
The above then gives:

\[ \Sigma \] is a critical point of the area (restricted to volume preserving deformations) if and only if it has constant mean curvature, that is \( H = c \) on \( \Sigma \).

In fact, a function on \( \Sigma \) is constant if and only if it is \( L_2 \)-orthogonal to the subspace of functions with zero mean.

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We have seen the geometric significance of the constancy of the mean curvature. But what is the meaning of the constancy of the principal curvatures?

Let $M$ be a constant curvature space form and $\Sigma$ a hypersurface of $M$. We say that $\Sigma$ is isoparametric if it has constant principal curvatures, that is, if the characteristic polynomial of $\Sigma$ is the same at all points of $\Sigma$. Obvious examples: geodesic spheres are isoparametric (all principal curvatures are the same).

Here is one of the first results in the theory (Cartan): In $\mathbb{R}^n$, $H^n$ and the hemisphere $S^n^+\text{the only compact isoparametric hypersurfaces are geodesic spheres. So, the theory is quite boring in those spaces: for some interesting facts we need to look at } S^n, \text{ and actually we will focus on that space from now on.}$
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It is not difficult to see that $\Sigma$ admits only two distinct principal curvatures, namely

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*In $S^3$ the only isoparametric surfaces are geodesic spheres and Clifford tori.*
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Consider the function $F : \mathbb{R}^5 \to \mathbb{R}$.

$$F(x_1, x_2, x_3, x_4, x_5) = x_5^3 + \frac{3}{2} x_5 (x_1^2 - 2x_2^2 + x_3^2 - 2x_4^2)$$

$$+ \frac{3\sqrt{3}}{2} x_4 (x_1^2 - x_3^2) - 3\sqrt{3} x_1 x_2 x_3.$$

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We see that there are more isoparametric hypersurface besides spheres and Clifford tori.
Summary

We were looking at the so-called Serrin problem

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\begin{aligned}
\Delta v &= 1 \quad \text{on} \quad \Omega \\
v &= 0, \quad \frac{\partial v}{\partial N} = c \quad \text{on} \quad \partial \Omega
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Any compact domain supporting a solution to this overdetermined problem is called a harmonic domain.
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An isoparametric hypersurface of \( S^n \) is a hypersurface \( \Sigma \) which has constant principal curvatures (that is, the characteristic polynomial of its shape operator is the same at all points). We will see another proof of this fact later, in a more general context.
Classification of IH ? Algebraic facts

It turns out that every isoparametric hypersurface is a regular level set of the restriction to $S^n$ of a polynomial in $\mathbb{R}^{n+1}$. Hence isoparametric hypersurfaces are smooth algebraic varieties.
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The following is mainly the work of Münzner. Let $\Sigma$ be isoparametric in $S^n$. Then $\Sigma = f^{-1}(t)$ where $f$ is the restriction to $S^n$ of a homogeneous polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree $g$, where $g$ is the number of distinct principal curvatures of $\Sigma$. $F$ satisfies:

\[
\begin{align*}
|\nabla F|^2 &= g^2 |x|^{2g-2} \\
\Delta F &= c |x|^{g-2}
\end{align*}
\]

for a suitable rational number $c$. The polynomial $F$ is called the Cartan-Münzner polynomial.
Geometric properties of isoparametric hypersurfaces

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Therefore, all equidistants sufficiently close to a IH \( \Sigma \) are isoparametric as well, and we obtain a whole one-parameter family of such.
For example:

\[ S^p(t) \times S^q(\sqrt{1 - t^2}) \]

defines one such family (Clifford tori) in \( S^{p+q+1} \), when \( t \in (0, 1) \).
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Note that when \( t \to 0 \) this family collapses to the lower dimensional submanifold \( \Sigma_+ \equiv S^q(1) \) and when \( t \to 1 \) it collapses to \( \Sigma_- \equiv S^p(1) \), both being totally geodesic (in particular, minimal) submanifolds of \( S^{p+q+1} \).
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This is a general fact.

**Proposition**

Let \( \Sigma \) be a isoparametric hypersurface. Then there are two regular, connected submanifolds \( \Sigma_+, \Sigma_- \) of \( S^n \) such that \( \Sigma \) is the tube of radius \( r_+ \) (resp. of radius \( r_- \)) around \( \Sigma_+ \) (resp. around \( \Sigma_- \)). These submanifolds are called the focal submanifolds of \( \Sigma \), and are minimal in \( S^n \).
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When $\Sigma$ is a geodesic sphere, the focal submanifolds are just two opposite points.
Isoparametric tubes

Any connected (two-sided) hypersurface $\Sigma$ of the sphere is the common boundary of two spherical domains $\Omega^+, \Omega^-$. If $\Sigma$ is isoparametric, and if $\Omega^+$ is the side containing the focal submanifold $\Sigma^+$, then $\Omega^+$ has the following properties:

a) $\Omega^+$ is the solid tube of fixed radius around $\Sigma^+$, that is, $\Omega^+ = \{ x \in S^n : d(x, \Sigma^+) \leq R \}$ for some $R > 0$.

b) Each equidistant from $\Sigma^+$, that is, each set $\Sigma^t = \{ x \in \Omega : d(x, \Sigma) = t \}$ is a regular hypersurface having constant mean curvature, for all $t < R$.

Domains with the properties a) and b) stated above are called isoparametric tubes. Thus, isoparametric tubes are those domains bounded by a (connected) isoparametric hypersurface.
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Isoparametric tubes are harmonic

We finally get to the largest class known so far of harmonic domains in $S^n$. 

Theorem

Any isoparametric tube is a harmonic domain, that is, it supports a solution to the Serrin problem:

$$\begin{cases}
\Delta u = 1 \\
u = 0, \quad \frac{\partial u}{\partial N} = c
\end{cases}$$

on $\Omega$.

The proof will be given later, for a more general situation.

It will turn out that any isoparametric tube will support a solution to the Schiffer problem (D), for infinitely many eigenvalues $\lambda$:

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We focused on the Serrin problem in the space forms $\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n$:

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Domains which support a solution to this problem are called harmonic. We have seen that in $\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n$, the only harmonic domains are geodesic balls. In the whole sphere, we have many more harmonic domains, namely, isoparametric tubes (domains bounded by isoparametric hypersurfaces).

Question: are there others? This seems to be an open question.

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