Overdetermined PDE's in Riemannian Geometry II, Constant curvature spaces

Alessandro Savo
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If ∇ is the Levi-Civita connection associated to the metric, then

$$\mathrm{div}X = -\sum_{j=1}^{n}\langle\nabla_{e_j}X, e_j\rangle = -\sum_{j=1}^{n}\nabla X(e_j, e_j)$$

so that

$$\Delta f = -\mathrm{tr}\nabla^2 f$$

and $\nabla^2 f = \nabla\nabla f$ is the Hessian of f. Of course, the Green formula continues to hold.

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Naturally, the next step is to examine these questions in the other (simply connected) manifolds of constant curvature : hyperbolic space and the sphere.

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Here is a formula for the explicit expression valid in all space forms (r is the distance function to a fixed point).

$$u(r) = \int_r^R \frac{1}{\theta(s)} \int_0^s \theta(t)\, dtds$$

where

$$\theta(r) = \begin{cases} \sin^{n-1} r & \text{if} \quad M = \mathbf{S}^n \\ r^{n-1} & \text{if} \quad M = \mathbf{R}^n \\ \sinh^{n-1} r & \text{if} \quad M = \mathbf{H}^n \end{cases}$$

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- *the reflection with respect to $T_{p,\nu}$ is in fact an isometry of $\mathbf{H}^n$.*

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Thus, if one fixes a unit vector ν, there exists a one-parameter family of totally geodesic hypersurfaces $T_{p,\nu}$.

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- *Note: Weinberger proof does not work in hyperbolic space.*

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Thus, we have all the ingredients to start the moving plane method and prove rigidity ... There is only one requirement: for the method to work, one needs to start from a hyperplane not intersecting Ω. This means that, for the method to work, Ω *has to be contained in a hemisphere.*

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Is it true that the only harmonic domains in $\mathbf{S}^n$ *are geodesic balls ?*

The answer is : no, as we shall see in the next section.

Exotic harmonic domains in spheres

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His proof is analytical; we provide here a simpler proof.

More generally, for positive numbers a, b such that $a^2 + b^2 = 1$, consider the *Clifford torus*

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Then the natural map $\phi : \Sigma_{a,b} \to \mathbf{S}^3$ defined by

$$\phi((x_1, x_2), (x_3, x_4)) = (x_1, x_2, x_3, x_4)$$

is an isometric embedding.

Consider the domain $\Omega \subseteq \mathbf{S}^3$ defined by the inequalities:

$$\Omega : \begin{cases} x_1^2 + x_2^2 \leq a^2 \\ x_3^2 + x_4^2 \geq b^2 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \end{cases}$$

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Proposition

The domain Ω is harmonic.

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Hence $g \cdot u$ is also a solution of (1). By uniqueness,

$$g \cdot u = u$$

for all $g \in G$.

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As p and q are arbitrary, this implies that $\dfrac{\partial u}{\partial N}$ is constant on $\partial\Omega$, as asserted.

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Each of the harmonic domains Ω above is foliated by a one parameter family of parallel Clifford tori :

$$\begin{cases} x_1^2 + x_2^2 = t^2 \\ x_3^2 + x_4^2 = 1 - t^2 \end{cases}$$

where $t \in (0, a]$, collapsing to the great circle (minimal submanifold):

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Finally, we remark that the above example generalizes to get the following class of examples :

Let Ω be any compact Riemannian manifold with boundary on which G acts by isometries. Assume that the action of G restricts to a transitive action on $\partial\Omega$. Then, Ω is a harmonic domain.

Isoparametric hypersurfaces

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Let M^n be a Riemannian manifold and Σ an hypersurface of M. We can define, at least locally, a unit normal vector field N to Σ.

A way to see how curved is Σ in M is to examine the rate of change of the normal field N infinitesimally, along a curve γ in Σ.

At any point $p \in \Sigma$ and for any tangent vector $X \in T_p\Sigma$ we can define the covariant derivative of N along X, denoted $S_p(X)$:

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S_p is called the *shape operator* of Σ at p. Its eigenvalues $k_1, \ldots, k_{n-1}$ are called *principal curvatures* of Σ at p, and its trace, divided by $n-1$, is the *mean curvature* :

$$H(p) = \frac{1}{n-1}(k_1 + \cdots + k_{n-1}).$$

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We say that Σ is a *critical point of the area* if

$$\frac{d}{dt}|_{t=0}\mathrm{vol}(\Sigma_V(t)) = 0$$

for all normal vector fields V on Σ.

A calculation shows that

$$\frac{d}{dt}|_{t=0}\mathrm{vol}(\Sigma_V(t)) = c\int_\Sigma \phi(x)H(x)\,dx$$

where $H(x)$ is the mean curvature at x and $V(x) = \phi(x)N(x)$.

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To first order, these correspond to vector fields V for which

$$\int_\Sigma \langle V, N\rangle = \int_\Sigma \phi = 0.$$

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In particular, solutions of the isoperimetric problem (that is, domains having minimal boundary area for a fixed volume) must have boundary of constant mean curvature.

These facts stress the importance of mean curvature in geometry.

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So, the theory is quite boring in those spaces : for some interesting facts we need to look at $\mathbf{S}^n$, and actually we will focus on that space from now on.

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In $\mathbf{S}^3$ the only isoparametric surfaces are geodesic spheres and Clifford tori.

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Consider the function $F : \mathbf{R}^5 \to \mathbf{R}$.

$$F(x_1, x_2, x_3, x_4, x_5) = x_5^3 + \frac{3}{2}x_5(x_1^2 - 2x_2^2 + x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}x_4(x_1^2 - x_3^2) - 3\sqrt{3}x_1x_2x_3.$$

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Fact: *Any regular level set of $\tilde{F}$ is an isoparametric hypersurface of $\mathbf{S}^4$.*

We see that there are more isoparametric hypersurface besides spheres and Clifford tori.

Summary

We were looking at the so-called *Serrin problem*

$$\begin{cases} \Delta v = 1 \quad \text{on} \quad \Omega \\ v = 0, \, \dfrac{\partial v}{\partial N} = c \quad \text{on} \quad \partial\Omega \end{cases}$$

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Things get more interesting in $\mathbf{S}^n$: it turns out that any domain bounded by an isoparametric hypersurface is harmonic. (Shklover).

An isoparametric hypersurface of $\mathbf{S}^n$ is a hypersurface Σ which has constant principal curvatures (that is, the characteristic polynomial of its shape operator is the same at all points). We will see another proof of this fact later, in a more general context.

Classification of IH ? Algebraic facts

It turns out that every isoparametric hypersurface is a regular level set of the restriction to $\mathbf{S}^n$ of a polynomial in $\mathbf{R}^{n+1}$. Hence isoparametric hypersurfaces are smooth algebraic varieties.

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The following is mainly the work of Münzner. Let Σ be isoparametric in $\mathbf{S}^n$. Then $\Sigma = f^{-1}(t)$ where f is the restriction to $\mathbf{S}^n$ of a homogeneous polynomial $F : \mathbf{R}^{n+1} \to \mathbf{R}$ of degree g, where g is the number of distinct principal curvatures of Σ. F satisfies:

$$\begin{cases} |\nabla F|^2 = g^2 |x|^{2g-2} \\ \Delta F = c|x|^{g-2} \end{cases}$$

for a suitable rational number c. The polynomial F is called the Cartan-Münzner polynomial.

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It is a classical fact that IH come in families. For $t \in \mathbf{R}$ define the equidistant

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Therefore, all equidistants sufficiently close to a IH Σ are isoparametric as well, and we obtain a whole one-parameter family of such.

For example:

$$\mathbf{S}^p(t) \times \mathbf{S}^q(\sqrt{1-t^2})$$

defines one such family (Clifford tori) in $\mathbf{S}^{p+q+1}$, when $t \in (0,1)$.

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Note that when $t \to 0$ this family collapses to the lower dimensional submanifold $\Sigma_+ \doteq \mathbf{S}^q(1)$ and when $t \to 1$ it collapses to $\Sigma_- \doteq \mathbf{S}^p(1)$, both being totally geodesic (in particular, minimal) submanifolds of $\mathbf{S}^{p+q+1}$.

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This is a general fact.

Proposition

Let Σ be a isoparametric hypersurface. Then there are two regular, connected submanifolds Σ_+, Σ_- of $\mathbf{S}^n$ such that Σ is the tube of radius r_+ (resp. of radius r_-) around Σ_+ (resp. around Σ_-). These submanifolds are called the focal submanifolds *of Σ, and are minimal in $\mathbf{S}^n$.*

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When Σ is a geodesic sphere, the focal submanifolds are just two opposite points.

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Thus, isoparametric tubes are those domains bounded by a (connected) isoparametric hypersurface.

Isoparametric tubes are harmonic

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The proof will be given later, for a more general situation.

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It willl turn out that any isoparametric tube will support a solution to the Schiffer problem (D), for infinitely many eigenvalues λ:

$$\begin{cases} \Delta u = \lambda u \quad \text{on} \quad \Omega \\ u = 0, \dfrac{\partial u}{\partial N} = c \quad \text{on} \quad \partial\Omega \end{cases}$$

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Conjecture. *Any harmonic spherical domain is an isoparametric tube.*