Overdetermined PDE's in Riemannian Geometry
Part III : general manifolds

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Serrin problem on manifolds : existence

We now consider compact domains in a general Riemannian manifold and study the Serrin problem:

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The answer is expected to be, in general, negative, unless the metric is very "symmetric" (space-forms and few other situations).

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Perhaps one of the first results, which eventually triggered a good deal of research, is due to Ye, which proved that any non degenerate critical point of the scalar curvature admits a neighborhood which can be foliated by mean curvature hypersurfaces (basically, these are the boundaries of perturbed geodesic balls).

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We have seen that, in $\mathbf{S}^n$, any domain bounded by an isoparametric hypersurface is an isoparametric tube.

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If we write ρ for the distance function to P:

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then the function f on Ω is radial if and only if it can be expressed

$$f = \psi \circ \rho,$$

for a smooth function $\psi : [0, R] \to \mathbf{R}$.

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Averaging operator (radialization). We now define an operator

$$\mathcal{A} : C^\infty(\Omega) \to C^\infty(\Omega)$$

which will take a function to its *radialization*.

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Given $f \in C^\infty(\Omega)$ and $x \in \Omega \setminus P$ we define $\mathcal{A}f(x)$ as the average of f on the equidistant through x :

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where of course we use in both cases the measure induced by the Riemannian metric on Σ_x and P, respectively.

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All the above facts are not difficult to prove (only a little technical at times).

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Then, u and $\hat{u}$ are two solutions of the boundary value problem

$$\begin{cases} \Delta u = 1 \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

By uniqueness, they must coincide, hence $\mathcal{A}u = u$, as asserted.

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If Σ is a minimal hypersurface of $\mathbf{R}^n$, and $B(x_0, R)$ is a Euclidean ball, the manifold with boundary (or, a connected component of it)

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is called an *extrinsic ball*. Note that Ω has dimension $n-1$ and that the boundary $\partial\Omega$ is contained in the sphere $\partial B(x_0, R)$.

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An interesting case is when, at each point of the boundary, the tangent space to Ω is orthogonal to the sphere: we then say that Ω meets ∂B orthogonally.

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Using the above fact one shows that then:

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Clearly, $|x|^2 = 1$ on $\partial\Omega$ hence $1 - |x|^2$ vanishes on $\partial\Omega$.

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It remains to show that the normal derivative of u is constant on $\partial\Omega$. This follows because, on $\partial\Omega$, one has ∇u collinear with x, and that $N = -x$ because Ω meets ∂B orthogonally. Then $\dfrac{\partial u}{\partial N} = \dfrac{1}{n-1}$.

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So, the classification problem of harmonic domains in Riemannian manifolds is an open and interesting problem, because it will imply, in particular, a classification of minimal free boundary immersions.

Now, consider any minimal free boundary surface in B^3 with at least 3 boundary components, as in Fraser and Schoen.

This is a harmonic domain but cannot be an isoparametric tube, for topological reasons:

any tube over a connected submanifold can have at most two boundary components (the boundary of a solid tube around P is homeomorphic to the unit normal bundle of P).

So, the classification problem of harmonic domains in Riemannian manifolds is an open and interesting problem, because it will imply, in particular, a classification of minimal free boundary immersions.

In the next section we will consider another overdetermined problem, which is stronger than the Serrin problem, and for which a classification in the Riemannian case is actually possible.

The constant flow property

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If we write $u_t(x) \doteq u(t, x)$ this means that

$$\begin{cases} \Delta u_t + \dfrac{\partial u_t}{\partial t} = 0 \quad \text{on} \quad \Omega \\ u_0 = 1 \\ u_t = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

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The physical interpretation is the following: $u(t, x)$ is the temperature at time $t > 0$, at the point $x \in \Omega$, assuming the initial temperature is uniformly constant, equal to 1, and that the boundary is subject to absolute refrigeration.

Note $u(t, x) = \int_\Omega k(t, x, y)\, dy$, where $k(t, x, y)$ is the heat kernel of Ω.

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$$H'(t) = \frac{d}{dt}\int_\Omega u_t = \int_\Omega \frac{\partial u_t}{\partial t} = -\int_\Omega \Delta u_t = -\int_{\partial\Omega} \frac{\partial u_t}{\partial N}.$$

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It is a smooth, positive function defined on $\partial\Omega$ (hence, in particular $H'(t) < 0$ for all t).

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This additional request gives rise to an overdetermined problem, which can then be written:

$$\begin{cases} \Delta u_t + \dfrac{\partial u_t}{\partial t} = 0 \\ u_0 = 1 \quad \text{on} \quad \Omega \\ u_t = 0, \ \dfrac{\partial u_t}{\partial N} = \psi(t) \quad \text{on} \quad \partial\Omega \quad \text{for all} \quad t > 0 \end{cases} \tag{1}$$

for a suitable smooth function ψ of the only variable $t \in (0, \infty)$.

Perfect heat diffusers

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Given a smooth function $\phi \in C^\infty(\partial\Omega)$, define

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That is, $\hat{\phi}_t$ is the unique solution of the problem:

$$\begin{cases} \Delta\hat{\phi}_t + \dfrac{\partial\hat{\phi}_t}{\partial t} = 0 \\ \hat{\phi}_0 = 0 \\ \hat{\phi}_t = \phi \quad \text{on} \quad \partial\Omega, \quad \text{for all } t > 0 \end{cases}$$

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Then CFP holds if and only if the *incoming heat flow* is perfectly balanced, at all times, by the *outgoing heat flow*.

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In fact it can be shown that, as $t \to \infty$

$$u(t,x) \sim ce^{-\lambda_1 t}\phi_1(x)$$

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If Ω has the CFP one sees that, differentiating in the normal direction, one has $\dfrac{\partial v}{\partial N} = c$ as well.

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For the proof, recall that the radialization $\mathcal{A}$ commutes with the Laplacian. To show that the temperature function u_t has constant normal derivative (for all t), it is enough to show that it is radial, or that:

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As the initial and boundary values data of $\mathcal{A}u_t$ are the same as those of u_t, we have by uniqueness $\mathcal{A}u_t = u_t$.

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This also gives an analytic characterization of the isoparametric condition.

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Theorem

Let Ω be a smooth (not necessarily analytic) Riemannian manifold. Assume that Ω has the CFP. Then:

$$\frac{\partial^k \eta}{\partial N^k} = c_k = \text{constant on } \partial\Omega$$

for all $k = 0, 1, 2, \dots$.

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Typically, this involves the study of the singularity of the normal exponential map, that is, the study of the *cut-locus* Cut_Ω.

The cut-locus

Given $x \in \partial\Omega$, consider the geodesic $\gamma_x : [0, L] \to \Omega$ which starts at x and goes in the unit normal direction:

$$\gamma_x(0) = x, \quad \gamma_x'(0) = N(x).$$

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Hence $c(x)$ is defined as follows:

$$d(\gamma_x(t), \partial\Omega) = t \quad \text{if and only if} \quad t \in [0, c(x)].$$

$c(x)$ is called the *cut-radius at* $x \in \partial\Omega$ and $\gamma_x(c(x))$ is the *cut point* at x.

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However, generally speaking, it is far from being a regular subset.

End of proof

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We now set

$$P = \mathrm{Cut}_\Omega.$$

The equidistants from P coincide with the equidistants to $\partial\Omega$; as these have constant mean curvature, we see that Ω is an isoparametric tube over P, as asserted.

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The heat flow at $y \in \partial\Omega$ admits an asymptotic expansion for $t \to 0$, of type:

$$\frac{\partial u_t}{\partial N}(y) \sim \sum_{k=1}^{\infty} B_k(y) \cdot t^{\frac{k}{2}-1}$$
$$\sim B_1(y) \cdot \frac{1}{\sqrt{t}} + B_2(y) + B_3(y)\sqrt{t} + \dots$$

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for certain heat flow invariants $B_k(y) \in C^\infty(\partial\Omega)$.

If Ω has the CFP, then every B_k is constant on $\partial\Omega$.

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Here they are.

$$B_1 = \frac{2}{\sqrt{\pi}}$$

$$B_2 = -\frac{1}{2}\eta$$

$$B_3 = -\frac{1}{6\sqrt{\pi}}\left(2\mathrm{tr}(R_N + S^2) - \eta^2\right)$$

$$B_4 = \frac{1}{16}\left(\eta\mathrm{tr}(R_N + S^2) - \mathrm{tr}(\nabla_N R_N + 2S \circ R_N + 2S^3) + \Delta^{\partial\Omega}\eta\right)$$

where R_N is the Jacobi operator $R_N(X) = R(N, X)N$ and S is the shape operator of $\partial\Omega$ relative to the inner unit normal N.

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where R_N is the Jacobi operator $R_N(X) = R(N, X)N$ and S is the shape operator of $\partial\Omega$ relative to the inner unit normal N.

Given that, we see that the CFP property immediately implies some valuable informations.

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Let us describe the outcome and the way the invariants B_k are presented.

Sketch of proof of Theorem 5

On a small tubular neighborhood U of $\partial\Omega$, the distance function ρ is smooth, and then we can define the first order operator:

$$N\phi = 2\langle \nabla\phi, \nabla\rho\rangle - \phi\Delta\rho,$$

for all $\phi \in C^\infty(U)$.

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Hence the operator $N\phi$ can also be written:

$$N\phi = 2\frac{\partial\phi}{\partial N} - \eta\phi.$$

In particular:

$$N1 = -\eta.$$

Let Δ be the Laplacian of the ambient manifold Ω. We let $\mathcal{A}$ be the algebra of operators acting on $C^\infty(U)$ and generated by the operators N and Δ. Here is the main calculation.

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Theorem

(S. 2004) *For all $k = 1, 2, \ldots$ there exists an operator $D_k \in \mathcal{A}$ (that is, a polynomial in N and Δ of homogeneous degree $k-1$) such that:*

$$B_k = D_k 1|_{\partial\Omega}.$$

The sequence of operators $\{D_k\}$ is explicitly computable by a recursive formula.

Here are the first few operators:

$$D_1 = \frac{2}{\sqrt{\pi}} \cdot I$$

$$D_2 = \frac{1}{2} N$$

$$D_3 = \frac{1}{6\sqrt{\pi}} (N^2 - 4\Delta)$$

$$D_4 = -\frac{1}{16} (\Delta N + 3N\Delta)$$

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hence we obtain the following presentation of the invariants $B_1, \ldots, B_5$: note in fact that $N1 = -\eta$ etc.

In what follows, $\Delta^T \nabla^T, \delta^T$ will denote the tangential operators (that is, the operators acting on the restriction to each equidistant).

$$\begin{aligned}
B_1 =& \frac{2}{\sqrt{\pi}} \\
B_2 =& -\frac{1}{2}\eta \\
B_3 =& \frac{1}{6\sqrt{\pi}}\Big(-2\frac{\partial \eta}{\partial N} + \eta^2\Big) \\
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& + 12\Delta^T(\frac{\partial \eta}{\partial N}) + 16|\nabla^T \eta|^2 + 4\delta^T(S\nabla^T \eta)\Big)
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It is now easy to show that, if $B_1, \ldots, B_5$ are constant, then η and $\dfrac{\partial^k \eta}{\partial N^k}$ are constant for $k \leq 3$.

Final iteration

Iterating the above argument and using the recursive formulae one proves that, for all $k \in \mathbf{N}$:

$$B_k = c_k \frac{\partial^{k-2}\eta}{\partial N^{k-2}} + \text{terms involving normal derivatives of lower order}$$

where c_k is a constant.

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This ends the proof.

Final remarks

What we said refers to *compact* manifolds with boundary. The case where Ω is not compact is very rich and interesting (see the works of Pacard, Ros and Sicbaldi, etc.) There are unbounded domains in $\mathbf{R}^n$, not isometric to isoparametric tubes, which supports solutions to the Serrin problem.

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There is a big variety of overdetermined problems. For example, people considered the following Steklov overdetermined problem:

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad \Omega \\ \dfrac{\partial u}{\partial N} = -\sigma u, \ |\nabla u| = 1 \quad \text{on} \quad \partial\Omega \end{cases}$$

Payne and Philippin proved that, if $\Omega \subseteq \mathbf{R}^2$ supports a solution when $\sigma = \sigma_2$ (first positive Steklov eigenvalue), then Ω must be a ball.

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When $\sigma > \sigma_2$ there are exotic plane domains supporting solutions (Alessandrini and Magnanini).