On the first Laplace eigenvalue of a homogeneous sphere

Emilio Lauret
CONICET and Universidad Nacional del Sur

Webinar in
Spectral geometry in the clouds
April 27, 2020.

This talk is based in the following articles:

- *The smallest Laplace eigenvalue of homogeneous 3-spheres*. Bull. Lond. Math. Soc. **51** (2019), 49-69. arXiv:1801.04259.
- *The first eigenvalue of a homogeneous CROSS*. With Renato Bettiol and Paolo Piccione. Preprint, January 2020. arXiv:2001.08471.

Laplacian

(M, g) a compact connected Riemannian manifold.

Δ the Laplace–Beltrami operator.

Spec(M, g):

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \cdots$$

and $\lambda_k(M, g) \to +\infty$ as $k \to \infty$.

Laplacian

(M, g) a compact connected Riemannian manifold.

Δ the Laplace–Beltrami operator.

Spec(M, g):

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \cdots$$

and $\lambda_k(M, g) \to +\infty$ as $k \to \infty$.

Shing-Tung Yau about the the fundamental tone $\lambda_1(M, g)$:

> *While this constant has analytic importance, it also gives strong insight in the geometry of the manifold.*

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

G a compact Lie group acting isometrically and transitively on M

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

G a compact Lie group acting isometrically and transitively on M,
$K :=$ isotropy subgroup of G at a point $p \in M$.

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

G a compact Lie group acting isometrically and transitively on M,
$K :=$ isotropy subgroup of G at a point $p \in M$.

We have the following identifications:

- $G/K \equiv M, \quad aK \mapsto a \cdot p$.

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

G a compact Lie group acting isometrically and transitively on M,
$K :=$ isotropy subgroup of G at a point $p \in M$.

We have the following identifications:

- $G/K \equiv M, \quad aK \mapsto a \cdot p.$
- $\mathfrak{p} \equiv T_pM, \quad$ where $\mathfrak{g} = \mathfrak{k} \oplus^{\perp} \mathfrak{p}.$

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

G a compact Lie group acting isometrically and transitively on M,
$K :=$ isotropy subgroup of G at a point $p \in M$.

We have the following identifications:

- $G/K \equiv M, \quad aK \mapsto a \cdot p.$
- $\mathfrak{p} \equiv T_pM, \quad \text{where } \mathfrak{g} = \mathfrak{k} \oplus^{\perp} \mathfrak{p}.$
- $\{G\text{-inv. met. on } M\} \longleftrightarrow \{\mathrm{Ad}(K)\text{-inv. inner products on } \mathfrak{p}\},$
  $$g \longmapsto \langle \cdot, \cdot \rangle := g_p.$$

Homogeneous Riemannian manifolds

(M, g) is called homogeneous if $\mathrm{Iso}(M, g)$ acts transitively on it.

G a compact Lie group acting isometrically and transitively on M,
$K :=$ isotropy subgroup of G at a point $p \in M$.

We have the following identifications:

- $G/K \equiv M, \quad aK \mapsto a \cdot p.$
- $\mathfrak{p} \equiv T_pM, \quad$ where $\mathfrak{g} = \mathfrak{k} \oplus^{\perp} \mathfrak{p}$.
- $\{G\text{-inv. met. on } M\} \longleftrightarrow \{\mathrm{Ad}(K)\text{-inv. inner products on } \mathfrak{p}\},$
  $$g \longmapsto \langle \cdot, \cdot \rangle := g_p.$$

To find all homogeneous Riemannian metrics on M, one needs to classify the compact Lie groups acting smoothly and transitively on M.

Homogeneous spheres

- $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

Homogeneous spheres

- $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.
- $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.

Homogeneous spheres

- $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

- $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.

Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).
Geometrically: canonical variation of the Hopf fibration

$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).
Geometrically: canonical variation of the Hopf fibration

$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

• $S^{15} \simeq \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$: two parameters.

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics.

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
Geometrically: canonical variation of the Hopf bundle
$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).
Geometrically: canonical variation of the Hopf fibration
$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

• $S^{15} \simeq \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$: two parameters.
Geometrically: canonical variation of the Hopf fibration
$$S^7 \longrightarrow S^{15} \longrightarrow S^8.$$

Homogeneous spheres

- $S^n \simeq SO(n+1)/SO(n)$: only round metrics. λ_1 ✓

- $S^{2n+1} \simeq SU(n+1)/SU(n)$: two parameters.

Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

- $S^3 \simeq SU(2)$: three parameters (up to isometry).

- $S^{4n+3} \simeq Sp(n+1)/Sp(n)$: four parameters (up to isometry).

Geometrically: canonical variation of the Hopf fibration

$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

- $S^{15} \simeq Spin(9)/Spin(7)$: two parameters.

Geometrically: canonical variation of the Hopf fibration

$$S^7 \longrightarrow S^{15} \longrightarrow S^8.$$

Homogeneous spheres

- $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$: only round metrics. λ_1 ✓

- $S^{2n+1} \simeq \mathrm{SU}(n+1)/\mathrm{SU}(n)$: two parameters.

Geometrically: canonical variation of the Hopf bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow P^n(\mathbb{C}).$$

- $S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).

Urakawa (1979) determined λ_1 for Berger 3-spheres, showing that $\lambda_1(M,g)/\mathrm{vol}(M,g)^{\frac{2}{\dim M}}$ is not bounded.

- $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$: four parameters (up to isometry).

Geometrically: canonical variation of the Hopf fibration

$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

- $S^{15} \simeq \mathrm{Spin}(9)/\mathrm{Spin}(7)$: two parameters.

Geometrically: canonical variation of the Hopf fibration

$$S^7 \longrightarrow S^{15} \longrightarrow S^8.$$

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics. λ_1 ✓

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
λ_1 ✓ Bettiol and Piccione [Calc. Var. PDE **47** (2013)] by using the method by Bérard Bergery and Bourguignon.

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).
Urakawa (1979) determined λ_1 for Berger 3-spheres, showing that $\lambda_1(M,g)/\mathrm{vol}(M,g)^{\frac{2}{\dim M}}$ is not bounded.

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).
Geometrically: canonical variation of the Hopf fibration

$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

• $S^{15} \simeq \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$: two parameters.
Geometrically: canonical variation of the Hopf fibration

$$S^7 \longrightarrow S^{15} \longrightarrow S^8.$$

Homogeneous spheres

• $S^n \simeq SO(n+1)/SO(n)$: only round metrics. λ_1 ✓

• $S^{2n+1} \simeq SU(n+1)/SU(n)$: two parameters.
λ_1 ✓ Bettiol and Piccione [Calc. Var. PDE **47** (2013)] by using the method by Bérard Bergery and Bourguignon.

•$S^3 \simeq SU(2)$: three parameters (up to isometry).
Urakawa (1979) determined λ_1 for Berger 3-spheres, showing that $\lambda_1(M,g)/\mathrm{vol}(M,g)^{\frac{2}{\dim M}}$ is not bounded.

• $S^{4n+3} \simeq Sp(n+1)/Sp(n)$: four parameters (up to isometry).
Geometrically: canonical variation of the Hopf fibration

$$S^3 \longrightarrow S^{4n+3} \longrightarrow P^n(\mathbb{H}).$$

• $S^{15} \simeq \mathrm{Spin}(9)/\mathrm{Spin}(7)$: two parameters. λ_1 ✓ Bettiol-Piccione.

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics. λ_1 ✓

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
λ_1 ✓ Bettiol and Piccione [Calc. Var. PDE **47** (2013)] by using the method by Bérard Bergery and Bourguignon.

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).
Urakawa (1979) determined λ_1 for Berger 3-spheres, showing that $\lambda_1(M,g)/\mathrm{vol}(M,g)^{\frac{2}{\dim M}}$ is not bounded.

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).
Bettiol and Piccione: variation of the round metric on S^3 (two parameters).

• $S^{15} \simeq \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$: two parameters. λ_1 ✓ Bettiol-Piccione.

Homogeneous spheres

• $S^n \simeq \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$: only round metrics. λ_1 ✓

• $S^{2n+1} \simeq \mathrm{SU}(n+1)/\,\mathrm{SU}(n)$: two parameters.
λ_1 ✓ Bettiol and Piccione [Calc. Var. PDE **47** (2013)] by using the method by Bérard Bergery and Bourguignon.

•$S^3 \simeq \mathrm{SU}(2)$: three parameters (up to isometry).
Urakawa (1979) determined λ_1 for Berger 3-spheres, showing that $\lambda_1(M,g)/\mathrm{vol}(M,g)^{\frac{2}{\dim M}}$ is not bounded.

• $S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n)$: four parameters (up to isometry).
Bettiol and Piccione: variation of the round metric on S^3 (two parameters).

• $S^{15} \simeq \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$: two parameters. λ_1 ✓ Bettiol-Piccione.

We next complete the list by filling the blue cases.

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$.

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ has $\mathbb{R}$-basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ has $\mathbb{R}$-basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any inner product on $\mathfrak{su}(2)$ induces a left-invariant metric on S^3.

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ has $\mathbb{R}$-basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any inner product on $\mathfrak{su}(2)$ induces a left-invariant metric on S^3.

$g_{(a,b,c)} :=$ the inner prod. with orthonormal basis $\{aX_1, bX_2, cX_3\}$.

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ has $\mathbb{R}$-basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any inner product on $\mathfrak{su}(2)$ induces a left-invariant metric on S^3.

$g_{(a,b,c)} :=$ the inner prod. with orthonormal basis $\{aX_1, bX_2, cX_3\}$.

Up to isometries: $g_{(a,b,c)}$ for $a \geq b \geq c > 0$.

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ has $\mathbb{R}$-basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any inner product on $\mathfrak{su}(2)$ induces a left-invariant metric on S^3.

$g_{(a,b,c)} :=$ the inner prod. with orthonormal basis $\{aX_1, bX_2, cX_3\}$.

Up to isometries: $g_{(a,b,c)}$ for $a \geq b \geq c > 0$.

$g_{(1,1,1)}$ is the unit sphere.

Expression in dimension 3

$S^3 \simeq \mathrm{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ has $\mathbb{R}$-basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any inner product on $\mathfrak{su}(2)$ induces a left-invariant metric on S^3.

$g_{(a,b,c)} :=$ the inner prod. with orthonormal basis $\{aX_1, bX_2, cX_3\}$.

Up to isometries: $g_{(a,b,c)}$ for $a \geq b \geq c > 0$.

$g_{(1,1,1)}$ is the unit sphere.

Theorem (L. 2019)

$\lambda_1(S^3, g_{(a,b,c)}) = \min\{a^2 + b^2 + c^2,\ 4(b^2 + c^2)\}$,

with multiplicity $\begin{cases} \textit{four if} & a^2 + b^2 + c^2 < 4(b^2 + c^2), \\ \textit{seven if} & a^2 + b^2 + c^2 = 4(b^2 + c^2), \\ \textit{three if} & a^2 + b^2 + c^2 > 4(b^2 + c^2). \end{cases}$

Dimension $4n + 3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.

Dimension $4n + 3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

The isotropy representation decomposes as $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

The isotropy representation decomposes as $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where

$\mathfrak{p}_1 \equiv \mathbb{H}^n$, the standard representation

Dimension $4n + 3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

The isotropy representation decomposes as $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where

$\mathfrak{p}_1 \equiv \mathbb{H}^n$, the standard representation,

$$\mathfrak{p}_0 = \mathrm{Span}_{\mathbb{R}} \left\{ X_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \right\}.$$

The action is trivial on $\mathfrak{p}_0$.

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

The isotropy representation decomposes as $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where

$\mathfrak{p}_1 \equiv \mathbb{H}^n$, the standard representation,

$$\mathfrak{p}_0 = \mathrm{Span}_{\mathbb{R}} \left\{ X_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \right\}.$$

The action is trivial on $\mathfrak{p}_0$.

$\mathfrak{p}_0 = \mathrm{Im}(\mathbb{H}) \simeq \mathfrak{sp}(1) \simeq \mathfrak{su}(2)$

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\,\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

The isotropy representation decomposes as $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where

$\mathfrak{p}_1 \equiv \mathbb{H}^n$, the standard representation,

$$\mathfrak{p}_0 = \mathrm{Span}_{\mathbb{R}}\left\{X_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix}\right\}.$$

The action is trivial on $\mathfrak{p}_0$.

$\mathfrak{p}_0 = \mathrm{Im}(\mathbb{H}) \simeq \mathfrak{sp}(1) \simeq \mathfrak{su}(2) \ \rightsquigarrow\ H := \mathrm{Sp}(1) \simeq \mathrm{SU}(2) \simeq S^3$.

Dimension $4n+3$: isotropy representation

$S^{4n+3} \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(n) =: G/K$ for $n \geq 1$.
$G = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^*A = \mathrm{Id}\}$.
$\mathfrak{g} = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : X^* + X = 0\}$.

$$\left(\begin{array}{c|c} \mathfrak{k} & \mathfrak{p}_1 \\ \hline \mathfrak{p}_1 & \mathfrak{p}_0 \end{array}\right)$$

The isotropy representation decomposes as $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where

$\mathfrak{p}_1 \equiv \mathbb{H}^n$, the standard representation,

$$\mathfrak{p}_0 = \mathrm{Span}_{\mathbb{R}} \left\{ X_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \right\}.$$

The action is trivial on $\mathfrak{p}_0$.

$\mathfrak{p}_0 = \mathrm{Im}(\mathbb{H}) \simeq \mathfrak{sp}(1) \simeq \mathfrak{su}(2) \rightsquigarrow H := \mathrm{Sp}(1) \simeq \mathrm{SU}(2) \simeq S^3$.

$$\begin{array}{ccccc} (K \times H)/K \equiv H & \longrightarrow & G/K & \longrightarrow & G/(K \times H) \\ S^3 & \longrightarrow & S^{4n+3} & \longrightarrow & P^n(\mathbb{H}). \end{array}$$

Dimension $4n + 3$: expression for λ_1

$\langle X, Y \rangle_0 := -\frac{1}{2}\operatorname{Tr}(XY)$, bi-invariant inner prod. on $\mathfrak{g}$.

Dimension $4n+3$: expression for λ_1

$\langle X, Y\rangle_0 := -\frac{1}{2}\operatorname{Tr}(XY)$, bi-invariant inner prod. on $\mathfrak{g}$.

For $a, b, c, s > 0$, the $\mathrm{Ad}(K)$-invariant inner product

$$\langle\cdot,\cdot\rangle_{(a,b,c,s)} := \frac{1}{2}\langle\cdot,\cdot\rangle_{(a,b,c)}|_{\mathfrak{p}_0} + \frac{1}{s^2}\langle\cdot,\cdot\rangle_0|_{\mathfrak{p}_1}$$

induces a G-invariant metric $g_{(a,b,c,s)}$ on $G/K = S^{4n+3}$.

Dimension $4n+3$: expression for λ_1

$\langle X, Y\rangle_0 := -\frac{1}{2}\operatorname{Tr}(XY)$, bi-invariant inner prod. on $\mathfrak{g}$.

For $a, b, c, s > 0$, the $\operatorname{Ad}(K)$-invariant inner product

$$\langle\cdot,\cdot\rangle_{(a,b,c,s)} := \frac{1}{2}\langle\cdot,\cdot\rangle_{(a,b,c)}|_{\mathfrak{p}_0} + \frac{1}{s^2}\langle\cdot,\cdot\rangle_0|_{\mathfrak{p}_1}$$

induces a G-invariant metric $g_{(a,b,c,s)}$ on $G/K = S^{4n+3}$.

Up to isometries: $g_{(a,b,c,s)}$ for $a \geq b \geq c > 0$, $s > 0$.

Dimension $4n+3$: expression for λ_1

$\langle X, Y\rangle_0 := -\frac{1}{2}\operatorname{Tr}(XY)$, bi-invariant inner prod. on $\mathfrak{g}$.

For $a, b, c, s > 0$, the $\operatorname{Ad}(K)$-invariant inner product

$$\langle \cdot, \cdot \rangle_{(a,b,c,s)} := \frac{1}{2}\langle \cdot, \cdot \rangle_{(a,b,c)}|_{\mathfrak{p}_0} + \frac{1}{s^2}\langle \cdot, \cdot \rangle_0|_{\mathfrak{p}_1}$$

induces a G-invariant metric $g_{(a,b,c,s)}$ on $G/K = S^{4n+3}$.

Up to isometries: $g_{(a,b,c,s)}$ for $a \geq b \geq c > 0$, $s > 0$.

$(a, b, c, s) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$ gives the unit sphere.

Dimension $4n+3$: expression for λ_1

$\langle X, Y\rangle_0 := -\frac{1}{2}\,\mathrm{Tr}(XY)$, bi-invariant inner prod. on $\mathfrak{g}$.

For $a, b, c, s > 0$, the $\mathrm{Ad}(K)$-invariant inner product

$$\langle\cdot,\cdot\rangle_{(a,b,c,s)} := \tfrac{1}{2}\langle\cdot,\cdot\rangle_{(a,b,c)}\big|_{\mathfrak{p}_0} + \tfrac{1}{s^2}\langle\cdot,\cdot\rangle_0\big|_{\mathfrak{p}_1}$$

induces a G-invariant metric $g_{(a,b,c,s)}$ on $G/K = S^{4n+3}$.

Up to isometries: $g_{(a,b,c,s)}$ for $a \geq b \geq c > 0$, $s > 0$.

$(a, b, c, s) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$ gives the unit sphere.

Theorem (Bettiol-L.-Piccione 2020)

$\lambda_1(S^{4n+3}, g_{(a,b,c,s)})$ *is given by the minimum among*

$$4ns^2 + 2(a^2+b^2+c^2), \quad 8(ns^2+b^2+c^2), \quad 8(n+1)s^2$$

Dimension $4n+3$: expression for λ_1

$\langle X, Y\rangle_0 := -\frac{1}{2}\operatorname{Tr}(XY)$, bi-invariant inner prod. on $\mathfrak{g}$.

For $a, b, c, s > 0$, the $\operatorname{Ad}(K)$-invariant inner product

$$\langle\cdot,\cdot\rangle_{(a,b,c,s)} := \tfrac{1}{2}\langle\cdot,\cdot\rangle_{(a,b,c)}|_{\mathfrak{p}_0} + \tfrac{1}{s^2}\langle\cdot,\cdot\rangle_0|_{\mathfrak{p}_1}$$

induces a G-invariant metric $g_{(a,b,c,s)}$ on $G/K = S^{4n+3}$.

Up to isometries: $g_{(a,b,c,s)}$ for $a \geq b \geq c > 0$, $s > 0$.

$(a, b, c, s) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$ gives the unit sphere.

Theorem (Bettiol-L.-Piccione 2020)

$\lambda_1(S^{4n+3}, g_{(a,b,c,s)})$ *is given by the minimum among*

$$4ns^2 + 2(a^2+b^2+c^2), \quad 8(ns^2+b^2+c^2), \quad 8(n+1)s^2$$

mult: $\quad 4(n+1) \quad (n+1)(2n+3) \quad n(2n+3)$

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G,

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot, \cdot)$.

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot, \cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x \exp(tX_i)).$$

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\Big|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, i.e. $\pi : G \to \mathrm{GL}(V_\pi)$ cont. morphism, V_π does not have any non-trivial $\pi(G)$-invariant subspace. ($\dim V_\pi < \infty$).

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \dots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\Big|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \dots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\Big|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi \in \widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x\exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi\in\widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x)$.

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \dots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\Big|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi\in\widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x)$.

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\Big|_{t=0} f_{v\otimes\varphi}(x \exp(tX_i))$$

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot, \cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x) \cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi \in \widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa) \cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x).$$

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} \varphi(\pi(x \exp(tX_i)) \cdot v)$$

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi \in \widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x)$.

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = -\sum_{i=1}^{m} \varphi(\pi(x)\pi(X_i)^2 \cdot v) \qquad \text{[notation } \pi(X)\text{]}$$

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi \in \widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x).$$

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = \varphi\left(\pi(x)\,\pi(-\textstyle\sum_i {X_i}^2)\cdot v\right)$$

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x\exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi\in\widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x).$$

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = \varphi\left(\pi(x)\,\pi(-C_g)\cdot v\right)$$

where $C_g = \sum_i X_i^2$ lies in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \ldots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x \exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi\in\widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x).$$

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = f_{(\pi(-C_g)\cdot v)\otimes\varphi}(x)$$

where $C_g = \sum_i X_i^2$ lies in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

General procedure (Lie group case)

G a compact semisimple Lie group, g left-invariant metric on G, $\{X_1, \dots, X_m\}$ on.b. of $\mathfrak{g}$ w.r.t. $g_e(\cdot,\cdot)$. Then

$$(\Delta_g \cdot f)(x) = -\sum_{i=1}^{m} \frac{d^2}{dt^2}\bigg|_{t=0} f(x\exp(tX_i)).$$

$(\pi, V_\pi) \in \widehat{G}$, $V_\pi \otimes V_\pi^* \hookrightarrow C^\infty(G)$, $v \otimes \varphi \mapsto f_{v\otimes\varphi}(x) = \varphi(\pi(x)\cdot v)$.

Peter-Weyl Theorem: $L^2(G) \simeq \bigoplus_{\pi\in\widehat{G}} V_\pi \otimes V_\pi^*$ as G-modules, i.e.

$(R_a \cdot f_{v\otimes\varphi})(x) = f_{v\otimes\varphi}(xa) = \varphi(\pi(xa)\cdot v) = f_{(\pi(a)\cdot v)\otimes\varphi}(x)$.

$$(\Delta_g \cdot f_{v\otimes\varphi})(x) = f_{(\pi(-C_g)\cdot v)\otimes\varphi}(x)$$

where $C_g = \sum_i X_i^2$ lies in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

When $g_e = -$Killing form, C_g is the Casimir element which lies in the center of $\mathcal{U}(\mathfrak{g})$ (still true for g normal).

Implicit description of the spectrum

For $(\pi, V_\pi) \in \widehat{G}$, $v \in V_\pi$, $\varphi \in V_\pi^*$,

$$\Delta_g \cdot f_{v \otimes \varphi} = f_{(\pi(-C_g) \cdot v) \otimes \varphi},$$

with $\pi(-C_g) : V_\pi \to V_\pi$ is a linear map.

Implicit description of the spectrum

For $(\pi, V_\pi) \in \widehat{G}$, $v \in V_\pi$, $\varphi \in V_\pi^*$,

$$\Delta_g \cdot f_{v\otimes\varphi} = f_{(\pi(-C_g)\cdot v)\otimes\varphi},$$

with $\pi(-C_g) : V_\pi \to V_\pi$ is a linear map.

If $\pi(-C_g) \cdot v = \lambda v$

Implicit description of the spectrum

For $(\pi, V_\pi) \in \widehat{G}$, $v \in V_\pi$, $\varphi \in V_\pi^*$,

$$\Delta_g \cdot f_{v\otimes\varphi} = f_{(\pi(-C_g)\cdot v)\otimes\varphi},$$

with $\pi(-C_g) : V_\pi \to V_\pi$ is a linear map.

If $\pi(-C_g) \cdot v = \lambda v$, then $\Delta_g \cdot f_{v\otimes\varphi} = f_{\lambda v\otimes\varphi}$

Implicit description of the spectrum

For $(\pi, V_\pi) \in \widehat{G}$, $v \in V_\pi$, $\varphi \in V_\pi^*$,

$$\Delta_g \cdot f_{v\otimes\varphi} = f_{(\pi(-C_g)\cdot v)\otimes\varphi},$$

with $\pi(-C_g) : V_\pi \to V_\pi$ is a linear map.

If $\pi(-C_g) \cdot v = \lambda v$, then $\Delta_g \cdot f_{v\otimes\varphi} = f_{\lambda v\otimes\varphi} = \lambda f_{v\otimes\varphi}$.

Implicit description of the spectrum

For $(\pi, V_\pi) \in \widehat{G}$, $v \in V_\pi$, $\varphi \in V_\pi^*$,

$$\Delta_g \cdot f_{v\otimes\varphi} = f_{(\pi(-C_g)\cdot v)\otimes\varphi},$$

with $\pi(-C_g) : V_\pi \to V_\pi$ is a linear map.

If $\pi(-C_g) \cdot v = \lambda v$, then $\Delta_g \cdot f_{v\otimes\varphi} = f_{\lambda v\otimes\varphi} = \lambda f_{v\otimes\varphi}$.

Peter-Weyl Theorem gives

$$\mathrm{Spec}(\Delta_g) = \bigcup_{\pi\in\widehat{G}} \mathrm{Spec}(\pi(-C_g))^{\dim V_\pi},$$

Implicit description of the spectrum

For $(\pi, V_\pi) \in \widehat{G}$, $v \in V_\pi$, $\varphi \in V_\pi^*$,

$$\Delta_g \cdot f_{v\otimes\varphi} = f_{(\pi(-C_g)\cdot v)\otimes\varphi},$$

with $\pi(-C_g) : V_\pi \to V_\pi$ is a linear map.

If $\pi(-C_g)\cdot v = \lambda v$, then $\Delta_g \cdot f_{v\otimes\varphi} = f_{\lambda v\otimes\varphi} = \lambda f_{v\otimes\varphi}$.

Peter-Weyl Theorem gives

$$\mathrm{Spec}(\Delta_g) = \bigcup_{\pi\in\widehat{G}} \mathrm{Spec}(\pi(-C_g))^{\dim V_\pi},$$

$$\lambda_1(\Delta_g) = \min_{\pi\in\widehat{G}\smallsetminus\{1_G\}} \lambda_{\min}(\pi(-C_g)).$$

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$$G = \mathrm{SU}(2), \quad X_1 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), X_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), X_3 = \left(\begin{smallmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right),$$

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$$G = \mathrm{SU}(2), \qquad X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

$g_{(a,b,c)}$ has on.b. $\{aX_1, bX_2, cX_3\}$, thus $C_g = a^2X_1^2 + b^2X_2^2 + c^2X_3^2$.

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$G = \mathrm{SU}(2), \quad X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$

$g_{(a,b,c)}$ has on.b. $\{aX_1, bX_2, cX_3\}$, thus $C_g = a^2X_1^2 + b^2X_2^2 + c^2X_3^2$.

$\widehat{G} = \{(\pi_k, V_k) : k \geq 0\}$ with $\dim V_k = k + 1$.

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$$G = \mathrm{SU}(2), \qquad X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

$g_{(a,b,c)}$ has on.b. $\{aX_1, bX_2, cX_3\}$, thus $C_g = a^2X_1^2 + b^2X_2^2 + c^2X_3^2$.

$\widehat{G} = \{(\pi_k, V_k) : k \geq 0\}$ with $\dim V_k = k + 1$.

$$[\pi_1(-C_g)] = \begin{pmatrix} a^2+b^2+c^2 & \\ & a^2+b^2+c^2 \end{pmatrix}.$$

$$[\pi_2(-C_g)] = \begin{pmatrix} 4a^2+2b^2+2c^2 & 2b^2-2c^2 & \\ 2b^2-2c^2 & 4a^2+2b^2+2c^2 & \\ & & 4b^2+4c^2 \end{pmatrix}.$$

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$$G = \mathrm{SU}(2), \qquad X_1 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right),\ X_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right),\ X_3 = \left(\begin{smallmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right),$$

$g_{(a,b,c)}$ has on.b. $\{aX_1, bX_2, cX_3\}$, thus $C_g = a^2X_1^2 + b^2X_2^2 + c^2X_3^2$.

$\widehat{G} = \{(\pi_k, V_k) : k \geq 0\}$ with $\dim V_k = k + 1$.

$$[\pi_1(-C_g)] = \begin{pmatrix} a^2+b^2+c^2 & \\ & a^2+b^2+c^2 \end{pmatrix}.$$

$$[\pi_2(-C_g)] = \begin{pmatrix} 4a^2+2b^2+2c^2 & 2b^2-2c^2 & \\ 2b^2-2c^2 & 4a^2+2b^2+2c^2 & \\ & & 4b^2+4c^2 \end{pmatrix}.$$

It turns out that $[\pi_k(-C_g)]$ is tridiagonal ($a_{ij} = 0$ if $|i - j| \geq 2$).

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$$G = \mathrm{SU}(2), \quad X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

$g_{(a,b,c)}$ has on.b. $\{aX_1, bX_2, cX_3\}$, thus $C_g = a^2X_1^2 + b^2X_2^2 + c^2X_3^2$.

$\widehat{G} = \{(\pi_k, V_k) : k \geq 0\}$ with $\dim V_k = k + 1$.

$$[\pi_1(-C_g)] = \begin{pmatrix} a^2 + b^2 + c^2 & \\ & a^2 + b^2 + c^2 \end{pmatrix}.$$

$$[\pi_2(-C_g)] = \begin{pmatrix} 4a^2 + 2b^2 + 2c^2 & 2b^2 - 2c^2 & \\ 2b^2 - 2c^2 & 4a^2 + 2b^2 + 2c^2 & \\ & & 4b^2 + 4c^2 \end{pmatrix}.$$

It turns out that $[\pi_k(-C_g)]$ is tridiagonal ($a_{ij} = 0$ if $|i - j| \geq 2$).

Gershgorin Circle Theorem gives $\lambda_{\min}(\pi_k(-C_g)) \geq 2kb^2 + k^2c^2$.

Procedure for $\lambda_1(S^3, g_{(a,b,c)})$

$G = \mathrm{SU}(2), \quad X_1 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), X_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), X_3 = \left(\begin{smallmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right),$

$g_{(a,b,c)}$ has on.b. $\{aX_1, bX_2, cX_3\}$, thus $C_g = a^2X_1^2 + b^2X_2^2 + c^2X_3^2$.

$\widehat{G} = \{(\pi_k, V_k) : k \geq 0\}$ with $\dim V_k = k + 1$.

$$[\pi_1(-C_g)] = \begin{pmatrix} a^2 + b^2 + c^2 & \\ & a^2 + b^2 + c^2 \end{pmatrix}.$$

$$[\pi_2(-C_g)] = \begin{pmatrix} 4a^2 + 2b^2 + 2c^2 & 2b^2 - 2c^2 & \\ 2b^2 - 2c^2 & 4a^2 + 2b^2 + 2c^2 & \\ & & 4b^2 + 4c^2 \end{pmatrix}.$$

It turns out that $[\pi_k(-C_g)]$ is tridiagonal ($a_{ij} = 0$ if $|i - j| \geq 2$).

Gershgorin Circle Theorem gives $\lambda_{\min}(\pi_k(-C_g)) \geq 2kb^2 + k^2c^2$.

Hence $\quad \lambda_1(S^3, g_{(a,b,c)}) = \min\{a^2 + b^2 + c^2,\ 4b^2 + 4c^2\}$.

First application: spectrum distinguishes homogeneous spheres

First application: spectrum distinguishes homogeneous spheres

Theorem (L. 2019 and Bettiol-L.-Piccione 2020)

Two isospectral homogeneous spheres are isometric.

First application: spectrum distinguishes homogeneous spheres

Theorem (L. 2019 and Bettiol-L.-Piccione 2020)

Two isospectral homogeneous spheres are isometric.

This was already known for S^3 by Schmidt and Sutton.

First application: spectrum distinguishes homogeneous spheres

Theorem (L. 2019 and Bettiol-L.-Piccione 2020)

Two isospectral homogeneous spheres are isometric.

This was already known for S^3 by Schmidt and Sutton.

Dimension 3: $\mathrm{vol}(S^3, g_{(a,b,c)})$, $\mathrm{Scal}(S^3, g_{(a,b,c)})$, $\lambda_1(S^3, g_{(a,b,c)})$ with its multiplicity determine a, b, c.

First application: spectrum distinguishes homogeneous spheres

Theorem (L. 2019 and Bettiol-L.-Piccione 2020)

Two isospectral homogeneous spheres are isometric.

This was already known for S^3 by Schmidt and Sutton.

Dimension 3: $\mathrm{vol}(S^3, g_{(a,b,c)})$, $\mathrm{Scal}(S^3, g_{(a,b,c)})$, $\lambda_1(S^3, g_{(a,b,c)})$ with its multiplicity determine a, b, c.

Dimension $4n+3$, $n \geq 1$: quite involved.

First application: spectrum distinguishes homogeneous spheres

Theorem (L. 2019 and Bettiol-L.-Piccione 2020)

Two isospectral homogeneous spheres are isometric.

This was already known for S^3 by Schmidt and Sutton.

Dimension 3: $\mathrm{vol}(S^3, g_{(a,b,c)})$, $\mathrm{Scal}(S^3, g_{(a,b,c)})$, $\lambda_1(S^3, g_{(a,b,c)})$ with its multiplicity determine a, b, c.

Dimension $4n+3$, $n \geq 1$: quite involved.

Remaining dimensions: not difficult.

Second application: Yamabe problem

Second application: Yamabe problem

Bettiol and Piccione (2013) proved that a non-round homogeneous metric g on S^d satisfying

$$\frac{\mathrm{Scal}(S^d, g)}{d-1} < \lambda_1(S^d, g)$$

is a locally rigid solution of the Yamabe problem.

Second application: Yamabe problem

Bettiol and Piccione (2013) proved that a non-round homogeneous metric g on S^d satisfying

$$\frac{\mathrm{Scal}(S^d, g)}{d-1} < \lambda_1(S^d, g)$$

is a locally rigid solution of the Yamabe problem.

Every non-round homogeneous metric on S^3 is locally rigid.

Second application: Yamabe problem

Bettiol and Piccione (2013) proved that a non-round homogeneous metric g on S^d satisfying

$$\frac{\mathrm{Scal}(S^d, g)}{d-1} < \lambda_1(S^d, g)$$

is a locally rigid solution of the Yamabe problem.

Every non-round homogeneous metric on S^3 is locally rigid.

In higher dimensional spheres, $\lambda_1(S^{4n+3}, g_{(a,b,c,s)})$ is $\min\{4ns^2 + 2(a^2 + b^2 + c^2), 8(ns^2 + b^2 + c^2), 8(n+1)s^2\}$.

Second application: Yamabe problem

Bettiol and Piccione (2013) proved that a non-round homogeneous metric g on S^d satisfying

$$\frac{\mathrm{Scal}(S^d, g)}{d-1} < \lambda_1(S^d, g)$$

is a locally rigid solution of the Yamabe problem.

Every non-round homogeneous metric on S^3 is locally rigid.

In higher dimensional spheres, $\lambda_1(S^{4n+3}, g_{(a,b,c,s)})$ is $\min\{4ns^2 + 2(a^2 + b^2 + c^2), 8(ns^2 + b^2 + c^2), 8(n+1)s^2\}$.

$\mathrm{Scal} = (4n+2)\,\lambda_1$ is only possible when $\lambda_1 = 8(n+1)s^2$.

Second application: Yamabe problem

Bettiol and Piccione (2013) proved that a non-round homogeneous metric g on S^d satisfying

$$\frac{\mathrm{Scal}(S^d, g)}{d-1} < \lambda_1(S^d, g)$$

is a locally rigid solution of the Yamabe problem.

Every non-round homogeneous metric on S^3 is locally rigid.

In higher dimensional spheres, $\lambda_1(S^{4n+3}, g_{(a,b,c,s)})$ is $\min\{4ns^2 + 2(a^2+b^2+c^2), 8(ns^2+b^2+c^2), 8(n+1)s^2\}$.

$\mathrm{Scal} = (4n+2)\,\lambda_1$ is only possible when $\lambda_1 = 8(n+1)s^2$.