# Sharp Weyl laws on 3d manifolds with rough potentials


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# Various forms of Weyl's law

Let $(M, g)$ be a three-dimensional compact, $C^4$ Riemannian manifold without boundary.

**Notation**:

$$-\Delta_g = \text{(nonnegative) Laplace–Beltrami operator on } (M, g)$$

$$\mathbb{1}(-\Delta_g \leq \lambda) = \text{spectral projection corresponding to eigenvalues } \leq \lambda$$

$$\mathbb{1}(-\Delta_g \leq \lambda)(x, y) = \text{integral kernel of } \mathbb{1}(-\Delta_g \leq \lambda)$$

$$N(\lambda, -\Delta_g) = \operatorname{Tr} \mathbb{1}(-\Delta_g \leq \lambda) = \int_M \mathbb{1}(-\Delta_g \leq \lambda)(x, x)\, dv_g(x) = \#\, \text{ev's } \leq \lambda$$

We are interested in four different forms of Weyl's law concerning the limit $\lambda \to \infty$:

- integrated Weyl law: $N(\lambda, -\Delta_g) = \frac{\lambda^{3/2}}{6\pi^2} \operatorname{Vol}_g(M) + o(\lambda^{3/2})$
- pointwise Weyl law: $\mathbb{1}(-\Delta_g \leq \lambda)(x, x) = \frac{\lambda^{3/2}}{6\pi^2} + o(\lambda^{3/2})$ uniformly in $x \in M$
- sharp integrated Weyl law: $N(\lambda, -\Delta_g) = \frac{\lambda^{3/2}}{6\pi^2} \operatorname{Vol}_g(M) + \mathcal{O}(\lambda)$
- sharp pointwise Weyl law: $\mathbb{1}(-\Delta_g \leq \lambda)(x, x) = \frac{\lambda^{3/2}}{6\pi^2} + \mathcal{O}(\lambda)$ uniformly in $x \in M$

We won't discuss improved integrated/pointwise Weyl laws with $o(\lambda)$ for some $(M, g)$.

**History**: Weyl (1911), Carleman (1934), Minakshisundaram–Pleijel (1949), Levitan (1952), Avakumović (1952 & 1956)

# Adding a rough potential

Question: What happens with Weyl's law if we consider $-\Delta_g + V$ instead of $-\Delta_g$?

For 'nice' $V$, the answer is 'nothing': By standard semiclassics, with $\lambda = h^{-2}$,

$$\mathbb{1}(-\Delta_g + V \leq \lambda)(x,x) = \mathbb{1}(-h^2\Delta_g + h^2 V - 1 \leq 0)(x,x) = \frac{h^{-3}}{6\pi^2} + \mathcal{O}(h^{-2}) = \frac{\lambda^{3/2}}{6\pi^2} + \mathcal{O}(\lambda)$$

We are interested in rough $V$, namely from the Kato class

$$\lim_{\epsilon\to 0} \sup_{x\in M} \int_{d_g(x,y)<\epsilon} \frac{|V(y)|}{d_g(x,y)}\, dv_g(y) = 0\,.$$

Why rough $V$?

- Question raised by Blair–Sire–Sogge (2019), Huang–Sogge (2020)
- Singular potentials appear naturally in some problems in quantum physics
- Linear problems with rough data often appear as a step in nonlinear problems
- We'll find interesting (and, at least for us, unexpected) phenomena

Why Kato class $V$?

- The Kato class is scaling critical and almost optimal for selfadjointness (like $L^{3/2}$)
- Eigenfunctions of $-\Delta_g + V$ with Kato class $V$ are bounded (unlike $L^{3/2}$)

And: We really think Avakumović's method deserves to be more widely known!

# Vojislav G. Avakumović

March 12, 1910 in Zemun
– August 19, 1990 in Marburg

PhD student of Karamata

PhD advisor of Bojanić, Vucković, Stanković, Marić, Maravić, Eberhard, Neunzert, Gromes, Bautsch, Brüning, Brübach

**Relevant papers for this talk:**


V. G. Avakumović, *Bemerkung über einen Satz des Herrn T. Carleman*, Math. Z. **53** (1950), 53–58.

———, *Über die Eigenfunktionen der Schwingungsgleichung*, Acad. Serbe Sci. Publ. Inst. Math. **4** (1952), 95–96.

———, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z. **65** (1956), 327–344.


http://www.mi.sanu.ac.rs/novi_sajt/aboutus/history/distiguished_mathematicians/avakumovic.php

## Main results

**Theorem (Pointwise Weyl law for Kato class potentials)**

*Let $V : M \to \mathbb{R}$ be in the Kato class. Then, uniformly in $x \in M$, as $\lambda \to +\infty$,*

$$\mathbb{1}(-\Delta_g + V \leq \lambda)(x,x) = \frac{\lambda^{3/2}}{6\pi^2} + o(\lambda^{3/2})\,.$$

The following example shows that $o(\lambda^{3/2})$ cannot be replaced by $\mathcal{O}(\lambda^{3/2-\epsilon})$ for any $\epsilon > 0$.

**Example (Violation of the sharp pointwise Weyl law)**

Let $\eta \in (0,1)$, $x_0 \in M$, $\gamma \in \mathbb{R}$, $\chi$ a cut-off function which is $\equiv 1$ near zero and

$$V(x) = \gamma\,\frac{\chi(d_g(x,x_0))}{d_g(x,x_0)^{2-\eta}}\,.$$

Then, in geodesic normal coordinates around $x_0$,

$$\mathbb{1}(-\Delta_g + V \leq \lambda)(y/\sqrt{\lambda}, y/\sqrt{\lambda}) = \frac{\lambda^{3/2}}{6\pi^2} - \gamma\,\Xi_\eta(y)\;\lambda^{(3-\eta)/2} + o(\lambda^{(3-\eta)/2})\,,$$

where $\Xi_\eta(0) > 0$ and $\Xi_\eta(y) \sim (2\pi^2)^{-1}|y|^{-2+\eta}$ as $|y| \to \infty$.

## Main results, cont'd

**Theorem (Sharp pointwise Weyl law)**

*If for some* $\epsilon > 0$, $\sup_{x\in M} \int_{d_g(y,x)<\epsilon} d_g(y,x)^{-2}|V(y)|\, dv_g(y) < \infty$, *then, uniformly in* $x \in M$, *as* $\lambda \to +\infty$,

$$\mathbb{1}(-\Delta_g + V \leq \lambda)(x,x) = \frac{\lambda^{3/2}}{6\pi^2} + \mathcal{O}(\lambda)\,.$$

- The condition in this theorem is, in particular, satisfied if $V \in L^p(M)$ for some $p > 3$.
- Conversely, for any $p < 3$ there is an $\eta \in (0,1)$ such that the $V$ from the previous example belongs to $L^p$.

**Theorem (Sharp integrated Weyl law)**

*If* $V \in$ *Kato class* $+ L^{3/2}(M)$, *then, as* $\lambda \to +\infty$,

$$N(t, -\Delta_g + V) = \frac{\lambda^{3/2}}{6\pi^2} + \mathcal{O}(\lambda)\,.$$

- For Kato class $V$, this is due to Huang–Sogge (2020).
- Despite the failure of a sharp Weyl law in ptw form, it is valid in integrated form.

# Comparison of our / Avakumović's and Levitan's strategy of proof

**Two ingredients in the proof**:

- Parametrix estimate for $(-\Delta_g + V + \lambda)^{-2}$ for $\lambda$ large positive
- Tauberian theorem for a Stieltjes transform

Morally, the same should probably work for $e^{t(\Delta_g - V)}$ and the Laplace transform.

We do not work with wave propagators $\exp(-it\sqrt{-\Delta_g + V})$ (as Levitan, Hörmander, ...) or resolvents close to the spectrum $(-\Delta_g + V - \lambda + i\epsilon)^{-k}$ (as Agmon, Métivier,...)

Compared to these latter works, here

- the parametrix estimates are simpler. All integral kernels are positive, which is very helpful when trying to accommodate rough $V$.
- the Tauberian theorems become more difficult.

Our strategy is similar to Avakumović's, but compared to Avakumović ($V = 0$) and Bojanić ($V$ bounded, Euclidean space), we need to work harder in both ingredients.

- There is an additional term in the parametrix estimates, which may give rise to the violation of the sharp pointwise Weyl law.
- This additional term affects the Tauberian theorem to leading order.

# Weyl law and Tauberian theorems

Reminder: Carleman's proof of the pointwise Weyl law. With $\rho(\lambda, x) = \mathbb{1}(-\Delta_g \leq \lambda)(x,x)$,

$$e^{t\Delta_g}(x,x) = \int_0^\infty e^{-t\lambda}\, d\rho(\lambda, x)\,, \qquad (-\Delta_g + \lambda)^{-2}(x,x) = \int_0^\infty \frac{d\rho(\mu, x)}{(\mu+\lambda)^2}\,.$$

- Parametrix estimates:

$$e^{t\Delta_g}(x,x) = (4\pi t)^{-3/2}(1+o(1)) \qquad \text{or} \qquad (-\Delta_g+\lambda)^{-2}(x,x) = (8\pi\sqrt{\lambda})^{-1}(1+o(1))$$

- Tauberian theorem of Hardy–Littlewood (with simple proof of Karamata)

Asymp of Laplace / Stieltjes trafo of pos measure $\Longrightarrow$ Asymp of pos measure

This implies $\mathbb{1}(-\Delta_g \leq \lambda)(x,x) = \rho(\lambda, x) = (6\pi^2)^{-1}\lambda^{3/2}(1+o(1))$.

Problem: Getting the next term in the parametrix estimates gives almost nothing for $\mathbb{1}(-\Delta_g \leq \lambda)(x,x)$. (We get $\mathcal{O}(1/\ln\lambda)$ instead of $o(1)$, Freud's theorem.)

**Avakumović's insight no. 1**: A remainder $\mathcal{O}(e^{-\epsilon/t})$ / $\mathcal{O}(e^{-\epsilon\sqrt{\lambda}})$ gives the sharp bound for $\mathbb{1}(-\Delta_g \leq \lambda)(x,x)$. (We get $\mathcal{O}(\lambda^{-1/2})$ instead of $o(1)$, complex Tauberian theorem.)

Avakumović (1952): For open sets $\Omega \subset \mathbb{R}^3$ in Euclidean space

$$\mathbb{1}(-\Delta_g \leq \lambda)(x,x) = (6\pi^2)^{-1}\lambda^{3/2} + \mathcal{O}(\lambda\, \mathrm{dist}(x, \partial\Omega)^{-1})\,.$$

(Follows also by finite speed of propagation for wave equation from Levitan's approach.)

# Weyl law and Tauberian theorems, cont'd

On manifolds, one cannot expect an exponentially small error, in fact,

$$e^{t\Delta_g}(x,x)=(4\pi t)^{-3/2}(1+ct+o(t)) \quad \text{or} \quad (-\Delta_g+\lambda)^{-2}(x,x)=(8\pi\sqrt{\lambda})^{-1}(1+c'\lambda^{-1}+o(\lambda^{-1}))$$

**Avakumović's insight no. 2**: The non-exponentially small terms have a special structure. They are the Stieltjes transform of an acceptable term,

$$(-\Delta_g+\lambda)^{-2}(x,x) = (8\pi\sqrt{\lambda})^{-1} + \int_0^\infty \frac{R(\mu,x)}{(\mu+\lambda)^3}\,d\mu + \mathcal{O}(e^{-\epsilon\sqrt{\lambda}}) \quad \text{with } |R(\mu,x)| \le C\mu\,.$$

Now take $B_1(\lambda) = \mathbb{1}(-\Delta_g \le \lambda)(x,x)$, $B_0 \equiv -(6\pi^2)^{-1}$, $B_2(\lambda) = R(\lambda,x)$ and apply

## Theorem (Tauberian theorem)

*Let $B_1$ be nondecreasing, $B_2(\lambda) = \mathcal{O}(\lambda)$ and $B_0$ bounded with*

$$B_0(v^2) - B_0(u^2) \ge -C/u, \quad 0 < u \le v \le u+1.$$

*Assume that for some $\epsilon_0 > 0$,*

$$\int_0^\infty \frac{B_0(\mu)\,\mu^{3/2} + B_1(\mu) + B_2(\mu)}{(\mu+\lambda)^3}\,d\mu = \mathcal{O}\left(e^{-\epsilon_0\sqrt{\lambda}}\right) \quad \text{as } \lambda \to +\infty.$$

*Then, $B_0(\lambda)\,\lambda^{3/2} + B_1(\lambda) + B_2(\lambda) = \mathcal{O}(\lambda)$ as $\lambda \to +\infty$.*

# Side remarks

- Open question: Recall that

$$(-\Delta_g+\lambda)^{-2}(x,x) = (8\pi\sqrt{\lambda})^{-1}+\int_0^\infty \frac{R(\mu,x)}{(\mu+\lambda)^3}\,d\mu+\mathcal{O}(e^{-\epsilon\sqrt{\lambda}}) \quad \text{with } |R(\mu,x)| \leq C\mu\,.$$

  Is it true that

$$e^{t\Delta_g}(x,x) = (4\pi t)^{-3/2} + \int_0^\infty te^{-t\mu}\tilde{R}(\mu,x)\,d\mu + \mathcal{O}(e^{-\epsilon/t}) \quad \text{with } |\tilde{R}(\mu,x)| \leq C\mu\,?$$

  A footnote in Avakumović's paper seems to suggest that yes. This might make it possible to extend the method to higher dimensions.

- Instead of $(-\Delta_g+\lambda)^{-2}(x,x)$, Avakumović works with

$$\lim_{y\to x}\left((-\Delta+\lambda)^{-1}(x,y) - \lim_{\lambda'\to 0}\left((-\Delta+\lambda')^{-1}(x,y) - \frac{1}{\lambda' \operatorname{Vol}_g M}\right)\right).$$

- For Avakumović's proof the simpler version of the Tauberian theorem with $B_0 \equiv \text{const.}$ suffices, but it does not for us due to the 'additional terms' mentioned before. We haven't seen this theorem in the literature, but it might be known. The strategy of the proof is to reduce it to the Ingham–Karamata Tauberian theorem.

## Parametrix estimates

### Proposition

Let $V : M \to \mathbb{R}$ be in the Kato class. There are $C < \infty$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, all $x \in M$ and $\lambda \geq C\epsilon^{-2}$, with some explicitly given $r_{0,\epsilon}^V$ and $R_\epsilon^V$,

$$\left| (-\Delta_g + V + \lambda)^{-2}(x,x) + \frac{1}{8\pi\sqrt{\lambda}} - \int_0^\infty \frac{t^{3/2}\, r_{0,\epsilon}^V(t,x) + R_\epsilon^V(t,x)}{(t+\lambda)^3}\, dt \right| \leq C\, e^{-\epsilon\sqrt{\lambda}/4},$$

$$\left| R_\epsilon^V(t,x) \right| \leq C\, t\,, \qquad \left| r_{0,\epsilon}^V(t,x) \right| \leq C\, \|V\|_{\mathcal{K}(\epsilon)} \qquad \text{for all } t \geq 0\,,$$

$$\left| r_{0,\epsilon}^V(t,x) - r_{0,\epsilon}^V(t',x) \right| \leq C\, \|V\|_{\mathcal{K}(\epsilon)} \frac{\sqrt{t'} - \sqrt{t}}{\sqrt{t}} \qquad \text{for all } 0 \leq t \leq t'\,.$$

Feeding this into our Tauberian theorem with $B_0(\lambda) = (-(6\pi^2)^{-1} + 2^{-1} r_{0,\epsilon}^V(\lambda, x))\lambda^{3/2}$ gives

### Corollary

Let $V : M \to \mathbb{R}$ be in the Kato class. Then for all $x \in M$ and $\epsilon \in (0, \epsilon_0]$,

$$\left| \mathbb{1}(-\Delta_g + V \leq \lambda)(x,x) - (6\pi^2)^{-1}\lambda^{3/2} - 2^{-1} r_{0,\epsilon}^V(\lambda,x)\, \lambda^{3/2} \right| \leq C_\epsilon(1+\lambda)\,.$$

This corollary and the explicit form of $r_{0,\epsilon}^V$ gives more or less directly our main results.

# Proof of parametrix estimates

## Goal

$$\left|(-\Delta_g + V + \lambda)^{-2}(x,x) + \frac{1}{8\pi\sqrt{\lambda}} - \int_0^\infty \frac{t^{3/2}\, r^V_{0,\epsilon}(t,x) + R^V_\epsilon(t,x)}{(t+\lambda)^3}\, dt\right| \le C\, e^{-\epsilon\sqrt{\lambda}/4}$$

We work with a $V$-independent parametrix for $(-\Delta_g + V + \lambda)^{-1}$,

$$T_{\lambda,\epsilon}(x,y) := \frac{e^{-\sqrt{\lambda} d_g(x,y)}}{4\pi d_g(x,y)}\; U_0(x,y)\; \chi(\epsilon^{-1} d_g(x,y))\,.$$

Set $\gamma^V_{\lambda,\epsilon}(x,y) := T_{\lambda,\epsilon}(x,y) - (-\Delta_g + V + \lambda)^{-1}(x,y)$ and

$$R^V_{\lambda,\epsilon}(x,y) := (-\Delta_g + V + \lambda)\gamma^V_{\lambda,\epsilon} = R_{\lambda,\epsilon} + V T_{\lambda,\epsilon}\,.$$

Then $\gamma^V_{\lambda,\epsilon}$ satisfies

$$\gamma^V_{\lambda,\epsilon}(x,y) = \int_M T_{\lambda,\epsilon}(x,z) R^V_{\lambda,\epsilon}(z,y)\, dv_g(z) - \int_M \gamma^V_{\lambda,\epsilon}(x,z) R^V_{\lambda,\epsilon}(z,y)\, dv_g(z)\,.$$

If $\lambda\epsilon^2$ is large enough and $\epsilon$ small enough, this can be solved by iteration,

$$\gamma^V_{\lambda,\epsilon} = \sum_{n=1}^\infty (-1)^{n-1}\, T_{\lambda,\epsilon} \left(R^V_{\lambda,\epsilon}\right)^n\,.$$

We want to see that $\gamma^V_{\lambda,\epsilon}$ leads to the terms $t^{3/2}\, r^V_{0,\epsilon}(t,x) + R^V_\epsilon(t,x)$ in the claimed bound.

# Proof of parametrix estimates, cont'd

$$\gamma^V_{\lambda,\epsilon} = \sum_{n=1}^{\infty} (-1)^{n-1} \gamma^{(n,V)}_{\lambda,\epsilon} \quad \text{where } \gamma^{(n,V)}_{\lambda,\epsilon} = T_{\lambda,\epsilon} \left( R^V_{\lambda,\epsilon} \right)^n \text{ and } R^V_{\lambda,\epsilon}(x,y) = R_{\lambda,\epsilon} + V T_{\lambda,\epsilon}$$

Recall Avakumović's two insights. Accordingly, we decompose

$$\gamma^{(n,V)}_{\lambda,\epsilon} = \text{additional term} + \text{special structure} + \text{exponentially small}$$

- The 'additional' term is simply $T_{\lambda,\epsilon} \left( V T_{\lambda,\epsilon} \right)^n$.
- The 'special structure' term comes from the region

  $$\{(z_1, \ldots, z_n) : \ d_g(x, z_1) \le \epsilon\,,\ d_g(z_1, z_2) \le \epsilon\,, \ldots, d_g(z_{n-1}, z_n) \le \epsilon\,,\ d_g(z_n, y) \le \epsilon\}$$

  in the $n$-fold integral $T_{\lambda,\epsilon} \left( R^V_{\lambda,\epsilon} \right)^n - T_{\lambda,\epsilon} \left( V T_{\lambda,\epsilon} \right)^n$.
- The rest is exponentially small.

The $\lambda$-derivatives of the 'additional' and 'special structure' terms need to be shown to be Stieltjes transforms of 'nice' functions. The only $\lambda$-dependence in these terms is through

$$\frac{e^{-\sqrt{\lambda}\delta}}{\sqrt{\lambda}} = \int_0^\infty \frac{t^{3/2} \kappa(\delta \sqrt{t})}{(t+\lambda)^3}\, dt\,, \quad \delta = d_g(x, z_1) + d_g(z_1, z_2) + \ldots + d_g(z_{n-1}, z_n) + d_g(z_n, y)$$

with $\kappa(r) = (8/\pi)(\sin r - r \cos r)/r^3$. This is where we use $d = 3$! We find

$$r^{(n,V)}_{0,\epsilon}(x,y) = \tfrac{1}{2} \int \cdots \int \delta \kappa(\delta \sqrt{t})\, T_{0,\epsilon}(x, z_1)\, V(z_1)\, T_{0,\epsilon}(z_1, z_2) \cdots V(z_n)\, T_{0,\epsilon}(z_n, y)\, dv_g(z_1) \cdots dv_g(z_n)$$

# Summary

- We have discussed various forms of Weyl's law for $-\Delta_g + V$ on 3D manifolds $(M, g)$.
- We have seen that for Kato class potentials $V$,

$$\mathbb{1}(-\Delta_g + V \leq \lambda)(x,x) = \frac{\lambda^{3/2}}{6\pi^2} + o(\lambda^{3/2})$$

  and that the remainder cannot be replaced by $\mathcal{O}(\lambda^{3/2-\epsilon})$ for any $\epsilon > 0$.
- We have shown that

$$\sup_{x \in M} \int_{d_g(y,x)<\epsilon} d_g(y,x)^{-2} |V(y)| \, dv_g(y) < \infty$$

  is sufficient for a $\mathcal{O}(\lambda)$ remainder.
- We have argued that Avakumović's method, which is based on $(-\Delta_g + V + \lambda)^{-2}$ for large positive $\lambda$, is well suited for the inclusion of rough potentials $V$.
- It remains an open problem to extend this method to higher dimensions.
- It would be interesting to recover and to extend to $V \neq 0$ Seeley's estimate in the presence of a boundary.

THANK YOU FOR YOUR ATTENTION!