Nodal Sets of Eigenfunctions: Progress via Optimal Transport

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Spectral Geometry in the Clouds

Outline

- Sturm-Liouville Theory
- What we expect in higher dimensions
- Two Slides of Optimal Transport
- Some Results and Many Problems

Sturm-Liouville Theory: Some History

Sturm-Liouville theory dates from 1836.

> *In 1833 both* **Sturm** *and* **Liouville** *and their common friend* **Duhamel** *applied for the seat vacated by the death of* **Legendre**. *A fourth applicant was* **Libri-Carucci** *[...] (Lützen, 1984)*

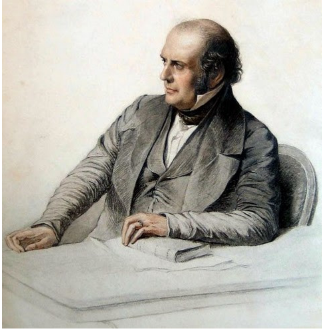
French Academy, 1833

The outcome of the elections is

1. Sturm: 0
2. Liouville: 1
3. Duhamel: 16
4. Libri-Carucci: 37

Who is this mysterious Libri-Carucci?

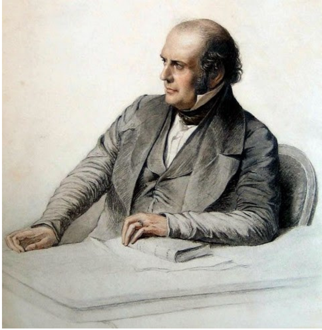
Guglielmo Libri Carucci dalla Sommaja (1803–1869)



Libri-Carucci

In 1841, Libri obtained an appointment as Chief Inspector of French Libraries through his friendship with influential French Chief of Police Francois Guizot. This job, involving in part the cataloguing of valuable books and precious manuscripts allowed Count Libri to indulge his collecting passion **by stealing them.**

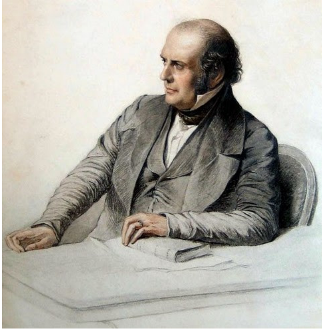
Guglielmo Libri Carucci dalla Sommaja (1803–1869)



Libri-Carucci

In 1848, as France was involved in a liberal revolution and the government fell, a warrant was issued for Libri's arrest. [...] However he received a tip-off and fled to London, shipping 18 large trunks of books and manuscripts, about 30,000 items, before doing so.

Guglielmo Libri Carucci dalla Sommaja (1803–1869)



Libri-Carucci

In June 2010, one of the stolen items, a letter from Descartes to Father Marin Mersenne, dated May 27, 1641 concerning the publication of Meditations on First Philosophy, was discovered in the library of Haverford College. The college returned the letter to the Institut de France on June 8, 2010.

Thee year laters: French Academy, 1836

After the death of Ampere, there are again elections. Sturm, Liouville and Duhamel compete once more for the seat (together with several others).

> *Three weeks before the election of* **Ampere***'s successor,* **Liouville** *presented a paper to the Academy [1837a] in which he praised* **Sturm***'s two memoires on the Sturm-Liouville theory as ranking with the best works of* **Lagrange***. Supporting a rival in this way was rather unusual in the competitive Parisian academic circles [...] (Lützen, 1984)*

Thee year laters: French Academy, 1836

> *[...] and it must have been shocking when on the day of the election, December 5-th,* **Liouville** *and* **Duhamel** *withdrew their candidacies to secure the seat for their friend.* **Sturm** *was elected with an overwhelming majority. (Lützen, 1984)*

Now Sturm is one of the 72 names in the Eiffel tower.

Sturm-Liouville Theory

Sturm-Liouville Theory covers general second order linear operators. For simplicity, we will always discuss the Laplacian

$$-\Delta u_k = \lambda_k u_k \qquad \text{on } [a, b]$$

with Dirichlet boundary conditions.

Theorem (Sturm Oscillation Theorem)

u_k has $k - 1$ roots.

Theorem (Sturm Oscillation Theorem, 1836)

u_k has $k-1$ roots.

In our special case of the Laplacian, this is not all that surprising: just count the roots of the sign.

In fact, stronger results are possible.

Theorem (Sturm-Hurwitz Theorem, 1903)

The function

$$f(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$$

has at least $2n$ distinct roots in $[0, 2\pi)$.

Sturm-Hurwitz

Theorem (Sturm-Hurwitz Theorem, 1903)

The function

$$f(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$$

has at least $2n$ distinct roots in $[0, 2\pi)$.

Proof.

$f(x)$ is the imaginary part of the holomorphic function

$$g(z) = \sum_{k=n}^{\infty} a_k z^k$$

when evaluated on the boundary of the unit circle $g(e^{it})$. It has at least n roots in the origin and thus at least n roots inside the unit disk. By the argument principle, $g(e^{it})$ winds around the origin at least n times creating at least $2n$ roots. □

Sturm-Hurwitz

Theorem (Sturm-Hurwitz Theorem, 1903)

The function

$$f(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$$

has at least $2n$ distinct roots in $\mathbb{T}$.

So, unsurprisingly, we can strenghten the Sturm Oscillation Theorem in the case of the Laplacian: surely that is because the trigonometric system is special.....

The REAL Sturm Oscillation Theorem

Theorem (Sturm Oscillation Theorem, 1836)

u_k has $k-1$ roots.

Theorem (**Sturm Oscillation Theorem**, 1836)

Unless all coefficients vanish, the function

$$\sum_{k=m}^{n} a_k u_k$$

has at least $m-1$ roots and at most $n-1$ roots.

This is a remarkable statement, even for sines and cosines alone. A sum of oscillating functions is still oscillating.

Berard & Helffer, 2017

Although well known in the nineteenth century, this theorem seems to have been ignored or forgotten by some of the specialists in spectral theory since the second half of the twentieth-century.

Timeline from Berard & Helffer, 2017

- **1833.** *Sturms Memoir presented to the Paris Academy of sciences in September.*
- **1836.** *Sturms papers published.*
- **1877.** *Lord Rayleigh writes a beautiful theorem has been discovered by Sturm*
- **1891.** *F. Pockels [30, pp. 68-73] gives a summary of Sturms results [...] provided by Sturm, Liouville and Rayleigh.*
- **1903.** *Hurwitz gives a lower bound for the number of zeros of the sum of a trigonometric series with a spectral gap and refers, somewhat inaccurately, to Sturms Theorems. This result, known as the Sturm-Hurwitz theorem, already appears in a more general framework in Liouvilles paper.*

Timeline from Berard & Helffer, 2017

- **1931.** *Courant & Hilbert extensively mention the Sturm-Liouville problem. They do not refer to the original papers of Sturm, but to Bochers book [8] which does not include Theorem 1.4 [the full result].*
- **1956.** *Pleijel mentions Sturms Theorem 1.4, somewhat inaccurately [...]*
- **...**

Sturm-Liouville Theory in Higher Dimensions

If it exists, what does it look like?

Sturm-Liouville Theory in Higher Dimensions

Let us assume (M, g) is a nice, smooth compact Riemannian manifold and that

$$-\Delta u_k = \lambda_k u_k.$$

Theorem (Sturm Oscillation Theorem, 1836)

u_k has $k-1$ roots.

How would one generalize this result to higher dimensions? It depends on how you interpret the notion of a root.

- **Topological.** $M \setminus \{x : u_k(x) = 0\}$ has at most ? connected components.
- **Metric.** $\mathcal{H}^{d-1}(\{x : u_k(x) = 0\})$ has at most size ?.

SL Theory in Higher Dimensions: the topological version

The topological version of the statement essentially says that

$M \setminus \{x : u_k(x) = 0\}$ doesn't have many connected components.

Theorem (Courant, 1923)

Let $\Omega \subset \mathbb{R}^2$ be a bounded planar domain. Then

$$\# \text{ connected components of } M \setminus \{x : u_k(x) = 0\} \leq k.$$

SL Theory in Higher Dimensions: the topological version

Just as in the one-dimensional case, one could wonder whether this actually holds true in general.

This was originally claimed in Courant & Hilbert

> *[the Courant Nodal Line Theorem] may be generalized as follows: Any linear combination of the first n eigenfunctions divides the domain, by means of its nodes, into no more than n subdomains. See the Gottingen dissertation of H. Herrmann, Beitrage zur Theorie der Eigenwerten und Eigenfunktionen, 1932.*

SL Theory in Higher Dimensions: the topological version

This result attracted the attention of Gelfand.

> *(Gelfand:) I thought that, except for me, nobody paid attention to Courants remarkable assertion. But I was so surprised that I delved into it and found a proof. [...] However, I could prove this theorem of Courant only for oscillations of one-dimensional media, where $m = 1$. (Arnold)*

Arnold (2011) recalls

> *Having read all this, I wrote a letter to Courant, Where can I find this proof now, 40 years after Courant announced the theorem? Courant answered that one can never trust ones students: to any question they answer either that the problem is too easy to waste time on, or that it is beyond their weak powers.*

SL Theory in Higher Dimensions: the metric version

Conjecture (Yau)

$$\mathcal{H}^{d-1}(\{x : u_k(x) = 0\}) \sim \sqrt{\lambda_k}.$$

Theorem (Logunov, 2016)

$$\mathcal{H}^{d-1}(\{x : u_k(x) = 0\}) \gtrsim \sqrt{\lambda_k}.$$

This can be regarded as a really advanced (metric) Sturm Oscillation Theorem. But in the one-dimensional case, the Sturm Oscillation Theorem also holds for linear combinations ...

SL Theory in Higher Dimensions: the metric version

Theorem (Logunov, 2016)

$$\mathcal{H}^{d-1}\left(\{x : u_k(x) = 0\}\right) \gtrsim \sqrt{\lambda_k}.$$

Main Question. What about

$$\mathcal{H}^{d-1}\left(\left\{x : \sum_{k=n}^{\infty} a_n u_n(x) = 0\right\}\right) \gtrsim ?$$

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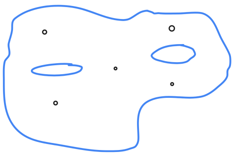
An obstruction: find a measure $\mu = \sum \delta_{x_i}$ such that

$$\langle u_j, \mu \rangle = 0 \qquad \text{for all } 1 \leq j \leq n.$$



Then let f be the function that is (a) mean 0, (b) constant and large in small balls around the points x_i, (c) constant and negative outside, (d) orthogonal to the first n eigenfunctions.

Thus any estimate of the flavor

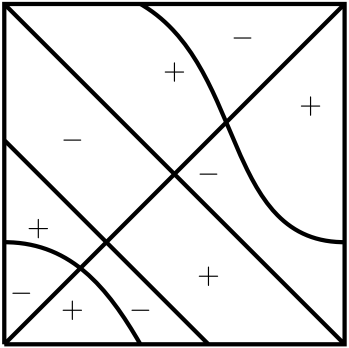
$$\mathcal{H}^{d-1}\left(\left\{x : \sum_{k=n}^{\infty} a_n u_n(x) = 0\right\}\right) \gtrsim ?$$

has to depend on the function

$$\sum_{k=n}^{\infty} a_n u_n(x).$$

But surely some sort of estimate must be possible!

An extreme example: suppose we are on $\mathbb{T}^2$ and $f : \mathbb{T}^2 \to \{-1, 1\}$ looks as follows



Surely such a function is **NOT** orthogonal to

$$e^{i\langle k,x\rangle} \qquad \text{for all } k \in \mathbb{Z}^2 \text{ with } \|k\| \leq 10^{10^{100}}.$$

Theorem (S., 2018)

Abbreviating $f = \sum_{k=n}^{\infty} a_n u_n(x)$,

$$\mathcal{H}^{d-1}\left(\{x : f(x) = 0\}\right) \gtrsim \frac{\sqrt{\lambda}}{\log^A \lambda} \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right)^{2-\frac{1}{d}}$$

- Idea: linear combinations of high-frequency eigenfunctions decay rapidly under the heat equation.
- $\mathbb{T}^2$ shows that the growth in λ is optimal
- The power $2 - 1/d$ is probably not optimal.
- 2020: Carroll, Massaneda & Ortega-Cerda: removed log

Conjecture

Abbreviating $f = \sum_{k=n}^{\infty} a_n u_n(x)$,

$$\mathcal{H}^{d-1}\left(\{x : f(x) = 0\}\right) \gtrsim \sqrt{\lambda} \frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}.$$

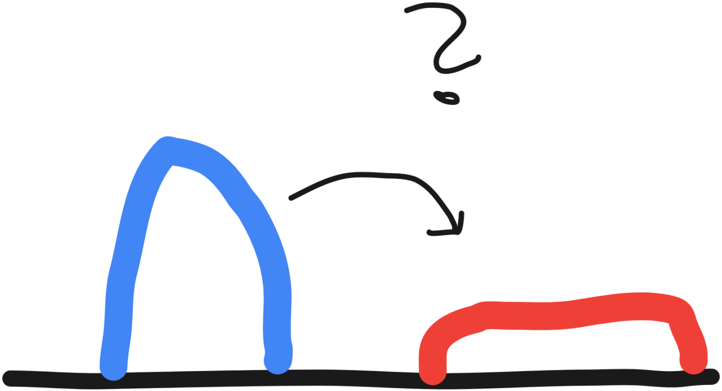
This is only known for $d = 2$ by an argument coming from *Optimal Transport*.

Optimal Transport

A very, very basic introduction.

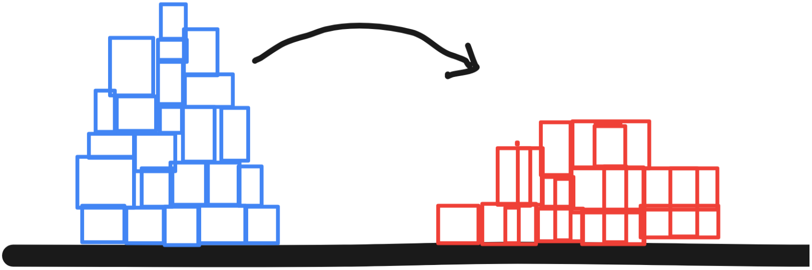
Optimal Transport

Suppose we are given two measure μ and ν having same total mass and want to transport one to the other.



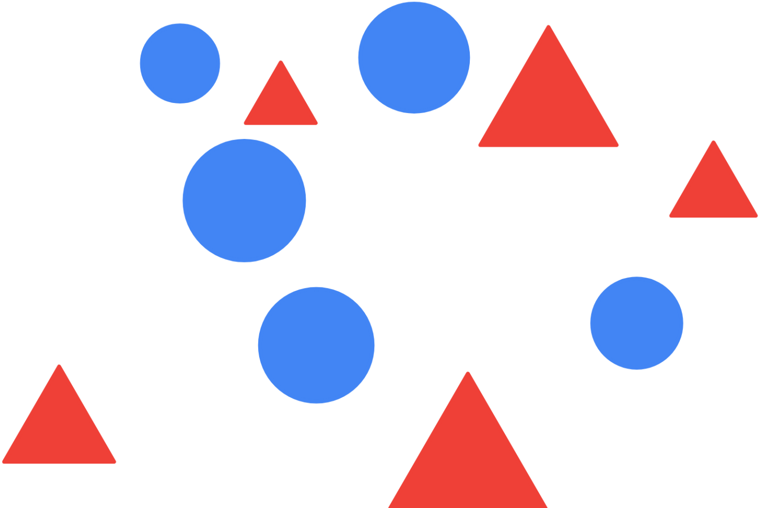
Optimal Transport

Think of both measures as being a collection of little boxes. Suppose it costs $\delta \cdot \varepsilon$ to move a box of weight ε distance δ. What is the cheapest way to move the boxes to the desired goal?



Optimal Transport

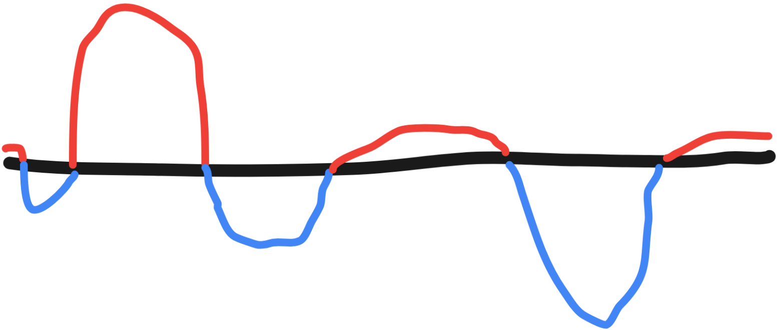
As it turns out, this problem is not really an issue in one dimension but it is already quite tricky in $d = 2$. We call the answer the 1-Wasserstein distance of μ, ν, denoted by $W_1(\mu, \nu)$.



Optimal Transport

This quantity is even funky for $d = 1$. Let $f : \mathbb{T} \to \mathbb{R}$ be a function of mean value 0 and set

$$\mu = f^+ dx \qquad \text{and} \qquad \nu = f^- dx.$$



A couple of years ago, I was playing with whether you can understand optimal transport in terms of Fourier coefficients of a function f. The following fun inequality popped up.

Theorem (S. 2018)

Let $f : [0, 2\pi] \to \mathbb{R}$ be a function with mean value 0. Then

$$\# \{x : f(x) = 0\} \cdot \left(\sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^2}{|k|^2} \right)^{\frac{1}{2}} \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

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- A function that has many large coefficients has the second term (which is merely the Sobolev norm $\|\cdot\|_{\dot{H}^{-1}}$) small: thus the first term has to be large.
- One of very few *lower* bounds on Fourier coefficients.
- I would be interested in any related estimates of this flavor.
- The proof is completely Optimal Transport.

Main Result

A bit later, I realized what I was really looking for.

Wasserstein Uncertainty Principle. *If there are very few post offices, some of your letters will have to travel a very long time. If letters arrive quickly, there must be many post offices.*

Main Result

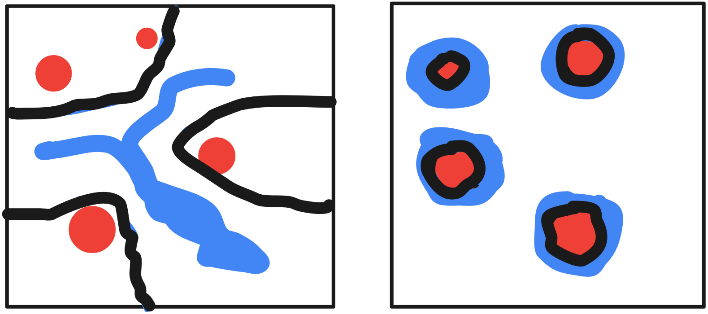
Wasserstein Uncertainty Principle. *If there are very few post offices, some of your letters will have to travel a very long time. If letters arrive quickly, there must be many post offices.*

Theorem (S. 2019)

Let $f : [0,1]^2 \to \mathbb{R}$ be a function with mean value 0. Then

$$W^1(f^+ dx, f^- dx) \cdot \mathcal{H}^1\left(x : f(x) = 0\right) \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

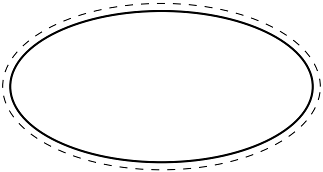
Main Result



Theorem (S. 2019)

$$W^1(f^+ dx, f^- dx) \cdot \mathcal{H}^1\left(x : f(x) = 0\right) \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

Sketch of the Argument



$$|\varepsilon - \text{neighborhood}(\Omega)| \leq \varepsilon|\partial\Omega|$$

when $\varepsilon \ll \text{diam}(\Omega)$

Main Result

Theorem (S. 2019)

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^1\left(x : f(x) = 0\right) \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

How does this connect to Sturm-Liouville Theory?

1. The linear combination of high-frequency eigenfunction has mean 0 and decays quickly under the heat equation.
2. The heat equation can be understood as inducing a transport: positive mass is spread evenly and negative mass is spread evenly.
3. Thus we get an upper bound on the transport cost.
4. The uncertainty principle implies a lower bound on the $\mathcal{H}^1-$size.

Theorem for $[0,1]^d$ (Amir Sagiv and S. 2019)

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right)^{4-\frac{1}{d}} \|f\|_{L^1}.$$

Theorem for $[0,1]^d$ (Carroll, Massaneda, Ortega-Cerda, 2020)

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right)^{2-\frac{1}{d}} \|f\|_{L^1}.$$

Conjecture

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right) \|f\|_{L^1}.$$

Theorem for $[0,1]^d$ (Amir Sagiv and S. 2019)

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right)^{4-\frac{1}{d}} \|f\|_{L^1}.$$

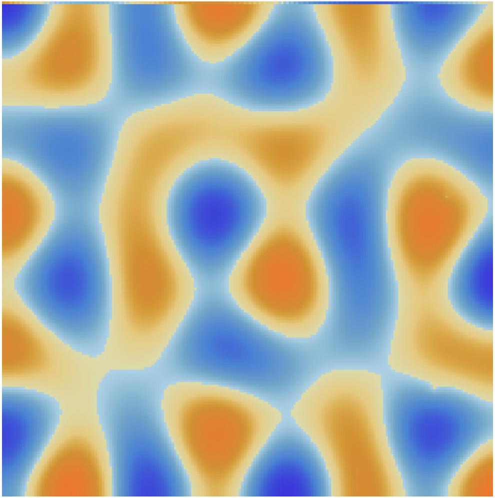
1. Divide into many small boxes. $\|f\|_{L^1}/\|f\|_{L^\infty}$ tells you how many boxes have to have nontrivial supply/demand.
2. If such a box is mainly supply or mainly demand, then there has to be transport.
3. If such a box has evenly matched supply demand, then

Relative Isoperimetric Inequality

Let $\Omega \subset [0,1]^d$ with $|\Omega| \le 1/2$. Then

$$\partial\Omega \cap (0,1)^d \gtrsim_d |\Omega|^{\frac{d-1}{d}}.$$

Supply and Demand



Wasserstein Spectral Geometry

Let $-\Delta u_k = \lambda_k u_k$. u_k has mean value 0. How much does it cost to move the positive part to the negative part?

Theorem (S. 2018)

$$W^1(u_k^+ dx, u_k^- dx) \lesssim \frac{\log^A \lambda_k}{\sqrt{\lambda_k}} \|u_k\|_{L^1}.$$

Carroll, Massaneda, Ortega-Cerda (2020) removed the log *therefore obtaining the sharp result*.

$$W^1(u_k^+ dx, u_k^- dx) \lesssim \frac{1}{\sqrt{\lambda_k}} \|u_k\|_{L^1}.$$

Is there a corresponding lower bound? I don't even know a bad bound.

Thank you!