Spectral minimal partitions of metric graphs: what and why?

James Kennedy

University of Lisbon

February 19, 2024

SMPs offer a way of dividing a given object (domain, manifold or graph) into a given number of pieces in an "analytically optimal" way: typically, one attempts to minimise an energy functional defined on all k-partitions built out of some norm of (say) the first Dirichlet Laplacian eigenvalue on each piece. They have deep connections both to fine spectral properties of the Laplacian defined on the whole object, and also to purely geometric functionals/partitions such as so-called Cheeger cuts.

We will start by giving a brief overview of SMPs in the particular context of metric graphs. These are a useful sandbox since the existence theory is far easier than on domains or manifolds, but SMPs on metric graphs still enjoy the same (or even more) connections to spectral theory as their higher-dimensional counterparts. We will attempt to illustrate this principle with two new results for metric graphs that should also hold, mutatis mutandis, on domains (with more difficult proofs).

First, we will discuss the problem of partitioning an unbounded graph, possibly equipped with an underlying potential. The existence or non-existence of a minimising k-partition, for given k, is closely related to the infimum of the essential spectrum of the operator on the whole graph, and in particular whether there exists a "test" k-partition of energy below this infimum. This directly parallels min-max-type principles for discrete eigenvalues of the operator at energies below its essential spectrum.

Second, we will introduce partitions of compact graphs based on Robin Laplacian-type first eigenvalues, where a Robin parameter $\alpha > 0$ is imposed at all boundary points between partition pieces; $\alpha = \infty$ corresponds, formally, to the Dirichlet case. But as $\alpha \to 0$, the minimal partition energies, suitably normalised, converge to the purely geometric k-Cheeger constant of the graph; and, up to a subsequence, the minimising partitions also converge in a natural Hausdorff sense to a k-Cheeger cut of the graph.

This talk will be based on the results of several projects with multiple different co-authors: Pavel Kurasov, Corentin Léna and Delio Mugnolo; Matthias Hofmann and Andrea Serio; and João Ribeiro.