Shape optimization problems and spectral theory
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Local and global spectral stability of the Dirichlet problem

with Bruno Colbois and Mette Iversen

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Geometric stability

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Burenkov–Lamberti, 2008

If Ω and Ω' are suitable $C^{1,1}$ domains, then

$$|\lambda_k(\Omega) - \lambda_k(\Omega')| \leq c_k |\Omega \Delta \Omega'|$$

where $\Omega \Delta \Omega'$ is the symmetric difference.

Inner perturbations

Suppose the domain Ω satisfies a Hardy inequality

$$\int_\Omega \frac{u^2}{\delta^2} \le a^2 \int_\Omega \left(|\nabla u|^2 + b u^2 \right) \quad \forall u \in C_0^\infty(\Omega),$$

where δ is the distance function to the boundary.

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Davies, 1993

If inradius$(\Omega) \geq r_0$, then there exist constants ε_k and C_k depending only on a, b, r_0 such that, for any $0 < \varepsilon < \varepsilon_k$,

$$0 \leq \lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \leq C_k \varepsilon^{2/a},$$

where $\Omega_\varepsilon = \{x \in \Omega : d(x, \Omega^c) > \varepsilon\}$.

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The Hardy inequality holds for convex domains, domains of class C^2, and simply connected planar domains.

Our project

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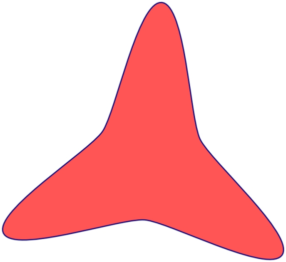

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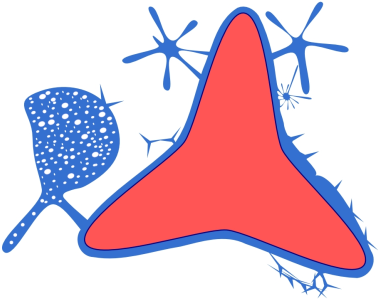

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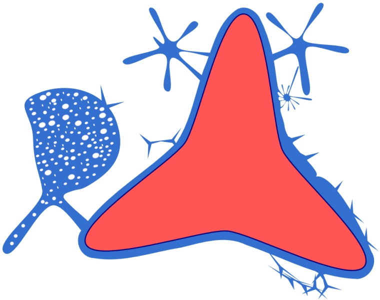

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Philosophy

If $\lambda_1(\Omega' \setminus \overline{\Omega})$ is large, then $|\lambda_k(\Omega) - \lambda_k(\Omega')|$ is small.

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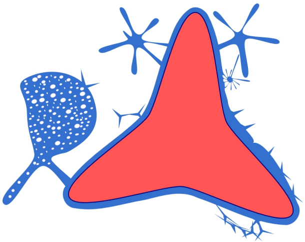

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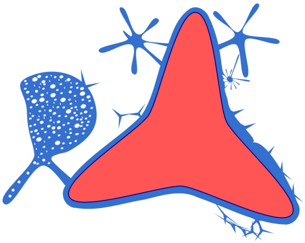

Let $N, \varepsilon_0, r_0 > 0$. Suppose

1. $\text{inradius}(\Omega) \geq r_0$
2. $(\Omega^\varepsilon)_{N\varepsilon} \subset \Omega$ for each $\varepsilon \leq \varepsilon_0$
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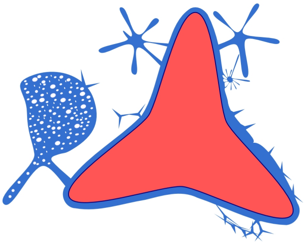

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Colbois–Girouard–Iversen, 2012

There exist $a_k, C = C(\Omega)$, $\varepsilon_k = \varepsilon_k(n, r_0, \varepsilon_0)$ such that:

If $\mu \geq \lambda\, a_k$ and

$$\left(\frac{\lambda}{\mu}\right)^{1/6} \frac{1}{\sqrt{\lambda}} \leq \varepsilon_k,$$

then

$$0 \leq \lambda_k(\Omega) - \lambda_k(\Omega') \leq C\mu^{-1/6}$$

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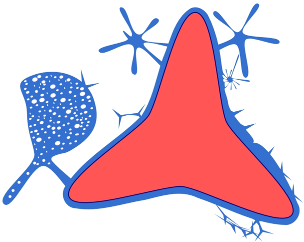

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$C = (b_k \lambda^{7/6} + C_k N \lambda^{-1/3})$ where C_k depends on n and r_0 only.

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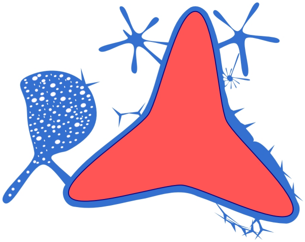

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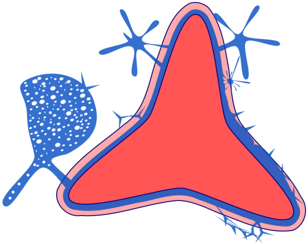

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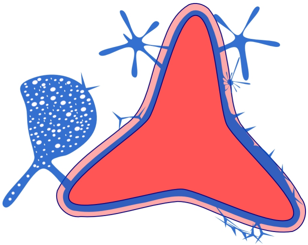

Because $\Omega \subset \Omega^\varepsilon \cap \Omega' \subset \Omega'$, monotonicity implies

$$0 \leq \lambda_k(\Omega) - \lambda_k(\Omega') = \overbrace{\lambda_k(\Omega) - \lambda_k(\Omega^\varepsilon \cap \Omega')} + \overbrace{\lambda_k(\Omega^\varepsilon \cap \Omega') - \lambda_k(\Omega')}$$

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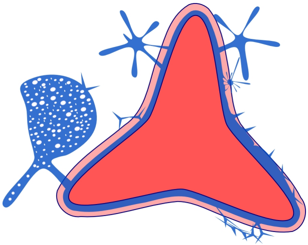

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We estimate the local and global term separately.

Local estimate for $\Omega \subset \Omega^\varepsilon \subset \mathbb{R}^n$

Let $N, \varepsilon_0, r_0 > 0$. Suppose that

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Idea of proof

The regularity asumption (3) $\Longrightarrow$ Ω^ε satisfies Hardy inequality

$$\int_\Omega \frac{u^2}{\delta^2} \leq a^2 \left(\int_\Omega |\nabla u|^2 + bu^2 \right) \quad \forall u \in C_0^\infty(\Omega),$$

with $a = 2$. Conclude using Davies results.

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In order to have uniform upper bounds, it is necessary to control the inner radius.

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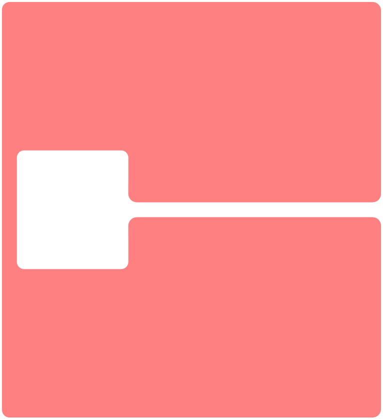

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Thank you !