(Séminaire EDP$s^2$)

# Shape optimization for Neumann and Steklov eigenvalues


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# References

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# Neumann and Steklov eigenvalue problems

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected Lipschitz domain.

**Neumann problem:**

$$-\Delta u = \mu u \text{ in } \Omega \text{ and } \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0.$$

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**Steklov problem:**

$$\Delta u = 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} = \sigma u \text{ on } \partial\Omega.$$

The Steklov problem describes the vibration of a free membrane with its whole mass uniformly distributed on the **boundary**.

# Neumann and Steklov spectra

The spectra of Neumann and Steklov problems are **discrete**:

$$0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots \nearrow \infty,$$

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Both Neumann and Steklov spectra start with the simple eigenvalue

$$\mu_0 = \sigma_0 = 0,$$

and the corresponding eigenfunctions are **constant**.

# Variational characterization

$$\mu_k(\Omega) = \inf_{U_k} \sup_{0 \neq u \in U_k} \frac{\int_\Omega |\nabla u|^2 \, dz}{\int_\Omega u^2 \, dz}, \quad k = 1, 2, \ldots$$

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$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 \, dz}{\int_{\partial\Omega} u^2 \, ds}, \quad k = 1, 2, \ldots$$

Here $U_k$ and $E_k$ denote $k$–dimensional subspaces of the Sobolev space $H^1(\Omega)$, such that $U_k$ are orthogonal to constants on $\Omega$ and $E_k$ are orthogonal to constants on $\partial\Omega$.

# Questions

**Question 1.** How large can the $k$–th **Neumann** eigenvalue be on a simply-connected planar domain of a given **area**?

**Question 2.** How large can the $k$–th **Steklov** eigenvalue be on a simply-connected planar domain of a given **perimeter**?

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**Question 2.** How large can the $k$–th **Steklov** eigenvalue be on a simply-connected planar domain of a given **perimeter**?

The mass of the membrane $\Omega$ equals $\mathrm{area}(\Omega)$ in the Neumann case and the perimeter $\mathbf{L}(\partial\Omega)$ in the Steklov case.

Therefore, Questions 1 and 2 can be reformulated as follows:

How large can $\mu_k$ and $\sigma_k$ be on a membrane of a given mass?

# Neumann eigenvalues

In **1954**, this problem was solved for $\mu_1$ by **Szegő**:

$$\mu_1(\Omega)\,\mathrm{area}(\Omega) \leq \pi\mu_1(\mathbf{D}) \approx 3.39\pi,$$

where $\mathbf{D}$ is the unit disk.

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Two years later, **Weinberger** generalized this result to arbitrary (not necessarily simply–connected) domains in any dimension.

# Pólya conjecture for Neumann eigenvalues (1954)

$$\mu_k(\Omega)\,\text{area}(\Omega) \le 4\pi k, \quad k = 1, 2, \ldots$$

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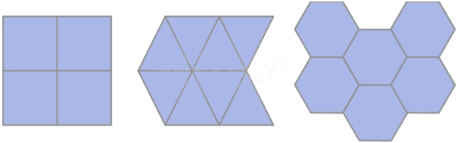

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# Second Neumann eigenvalue

**Theorem 1** (**G.–Nadirashvili–Polterovich** '09)

– If $\Omega$ is simply-connected then

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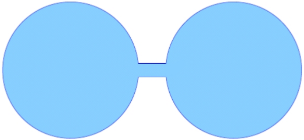

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Note that equality is not attained.

**Corollary** Pólya's conjecture holds for $k = 2$ on simply–connected planar domains with Neumann boundary conditions.

# Maximizing Neumann eigenvalues for $k \geq 3$

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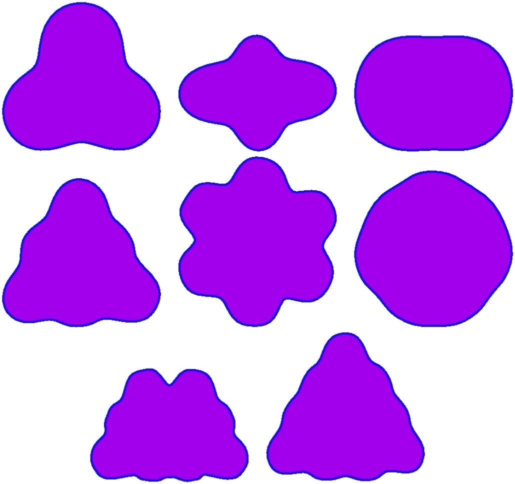
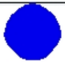
**Antunes–Freitas** 2011. Numerical star shaped maximizers for $k = 3, \cdots, 10$.



# Numerical maximizers

**Antunes–Freitas** 2011.

| i | $\Omega$ | multiplicity | $\mu_i^\star$ | $\mu_i^\star(C)$ | $\mu_i^\star(UD)$ |
|---|---|---|---|---|---|
| 3 |  | 3 | **32.79** | | 31.95 |
| 4 |  | 5 | **43.43** | 43.20 | 42.60 |
| 5 |  | 7 | **54.08** | 52.97 | 53.25 |
| 6 |  | 4 | **67.04** | | 63.90 |
| 7 |  | 6 | **77.68** | 77.24 | 74.55 |
| 8 |  | 4 | **89.22** | | 88.85 |
| 9 |  | 4 | **101.73** | | 99.50 |
| 10 |  | 5 | **113.86** | | 110.15 |

# Steklov eigenvalues

Example On the unit disk **D**, the eigenvalues of the Steklov problem are equal to

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and the corresponding eigenfunctions are given by

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Weinstock inequality '54:

$$\sigma_1(\Omega)\, L(\partial\Omega) \leq 2\pi\sigma_1(\mathbf{D}) = 2\pi.$$

Equality is attained **iff** $\Omega$ is a **disk**.

# Hersch–Payne–Schiffer inequalities

In 1974, Hersch, Payne and Schiffer proved that

$$\sigma_k(\Omega)\,\sigma_n(\Omega)\,L(\partial\Omega)^2 \leq \begin{cases} (k+n-1)^2\,\pi^2 & \text{if } k+n \text{ is odd,} \\ (k+n)^2\,\pi^2 & \text{if } k+n \text{ is even.} \end{cases}$$

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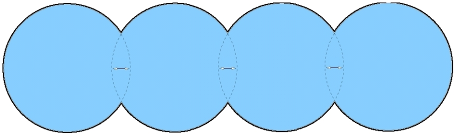
**In fact, a much stronger statement holds!**

**Theorem 2** (**G.–Polterovich** '10) There exists a family of simply-connected domains, degenerating to the disjoint union of $n$ identical disks as $\varepsilon \to 0+$, such that



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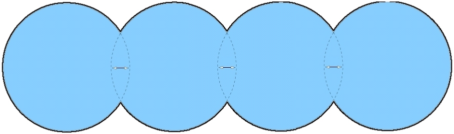

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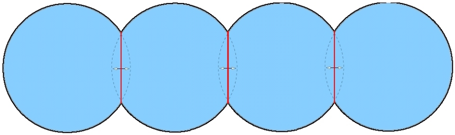

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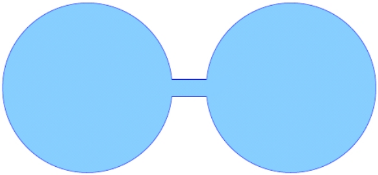
# Why pull the disks apart?

One could ask why in Theorem 2 we do not apply the same construction used in Theorem 1:



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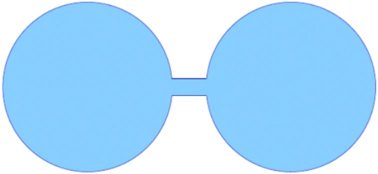

The reason is that convergence of Steklov eigenvalues in this case is very different from that of Neumann eigenvalues.

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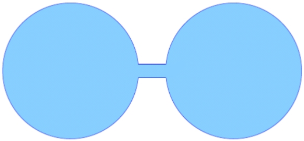

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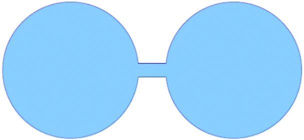

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For every fixed $m$, the numerator in the Rayleigh quotient tends to zero as $\varepsilon \to 0+$, while the denominator does not. This implies

$$\lim_{\varepsilon \to 0+} \sigma_k(\varepsilon) = 0$$

for **all** $k = 1, 2, \ldots$!

# Isoperimetric control in higher dimension

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**Corollary** Each $\sigma_k$ is bounded above among domains with boundary of fixed measure.

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Since the Dirichlet energy is conformally invariant,

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The infimum is taken over all two–dimensional subspaces $E \subset H^1(\mathbf{D})$ such that

$$\int_{\mathbf{D}} f \, d\rho = 0 \quad \text{for all } f \in E.$$

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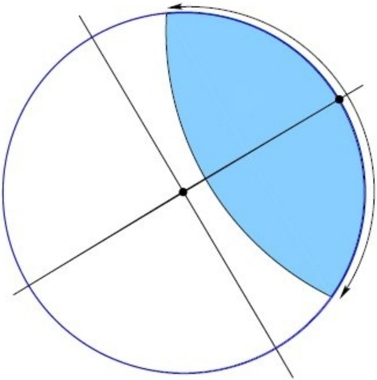
Set

$$K = \int_{\mathbf{D}} X_t^2(z)\,dz = \frac{1}{2}\int_{\mathbf{D}} f^2(|z|)\,dz.$$

# Hyperbolic caps



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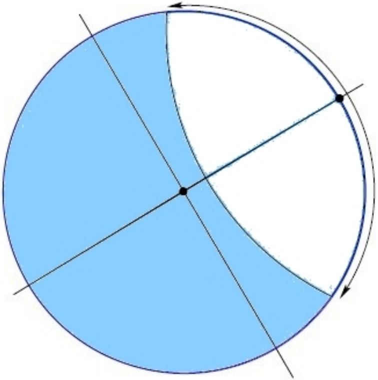

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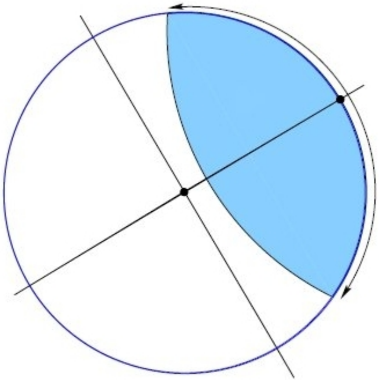
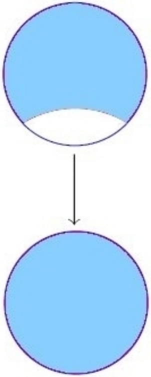


- Let $\gamma$ be an hyperbolic geodesic.
- A **hyperbolic cap** is a connected component of $\mathbf{D} \setminus \gamma$.
- $\tau : \mathbf{D} \to \mathbf{D}$ is the reflection across the geodesic $\gamma$

The **lift** of $u : a \to \mathbf{R}$ is $\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in a, \\ u(\tau z) & \text{if } z \in a^* = \tau(a). \end{cases}$

# Test functions for $\mu_2(\Omega)$



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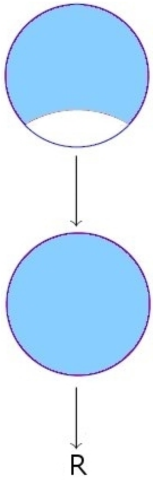


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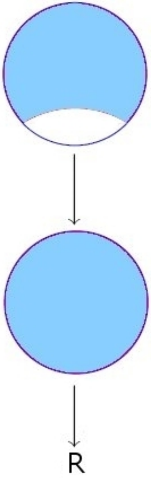


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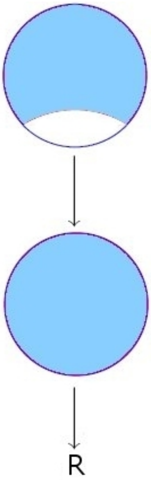


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The function $\tilde{u}_a^t$ is linear in $t$.

Hersch's lemma allows to choose $\phi_a$ so that $\int_{\mathbf{D}} \tilde{u}_a^t \, d\rho = 0$.

Using $E = E_a = \left\{ \tilde{u}_a^t \mid t \in \mathbf{R}^2 \right\}$ in

$$\mu_2(\Omega) = \inf_E \sup_{0 \neq f \in E} \frac{\int_{\mathbf{D}} |\nabla f|^2 \, dz}{\int_{\mathbf{D}} f^2 \, d\rho}$$

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**Lemma**

$$\int_{\mathbf{D}} |\nabla \tilde{u}_a^t|^2 \, dz = 2\mu_1(\mathbf{D})K.$$

The proof uses conformal invariance of the Dirichlet energy. The factor 2 appears because of the **lift**.

Previous lemma implies:

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**Lemma** For each cap $a$ there exists $t \in \mathbf{S}^1$ such that

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Indeed, since $\Omega$ is **planar**, the Gaussian curvature is zero, and hence $\Delta \log \delta = 0$.

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Since $f$ is increasing, for any $s, t \in \mathbf{S}^1$ with $s \perp t$ we have:

$$\int_{\mathbf{D}} \left(\left(\tilde{u}_a^s\right)^2 + \left(\tilde{u}_a^t\right)^2\right) \, d\rho = \int_{\mathbf{D}} (\overbrace{X_s^2 + X_t^2}^{f^2(|z|)})\, \delta(z)dz \geq \int_{\mathbf{D}} f^2(|z|)\, dz.$$

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Since for each cap $a \subset \mathbf{D}$, the integral

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is quadratic in $t$, there exists a unique maximizing direction $\pm t \in \mathbf{S}^1$ such that for each $s \neq \pm t$

$$\int_{\mathbf{D}} \left(\tilde{u}_a^t\right)^2 \, d\rho > \int_{\mathbf{D}} \left(\tilde{u}_a^s\right)^2 \, d\rho.$$

We obtain a continuous map

$$m : \mathcal{HC} \to \mathbf{R}P^1$$

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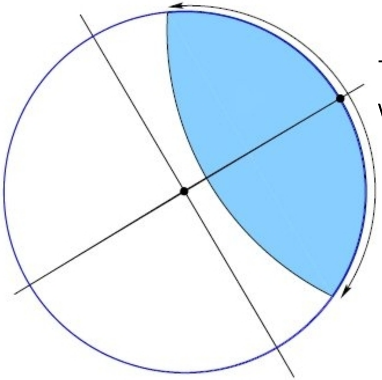

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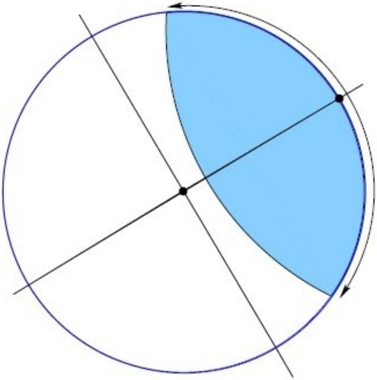

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first coordinate: midpoint of $\partial a \cap \partial \mathbf{D}$,
second coordinate: length of $\partial a \cap \partial \mathbf{D}$.

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**2.** Results for Steklov eigenvalues hold also if the mass on the boundary is distributed with a non–constant **density**. Theorem 1 holds for *log–subharmonic* densities, i.e. for domains in non-positively curved surfaces (**G. –Polterovich** '10).

**3.** Do maximizers exist? How regular are they? Understanding the geometry of maximizers for higher Neumann eigenvalues is challenging.

# In conclusion

**Conjecture** Among all domains of fixed area, the product of the first two non zero Neumann eigenvalues is maximal for the disk.
In other words...

$$\mu_1(\Omega)\mu_2(\Omega)\,\mathrm{area}(\Omega)^2 \leq \mu_1(\mathbf{D})^2\,\pi^2.$$

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For simply–connected domains, it follows from Szegő inequality and Theorem 1 that

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