The Steklov spectrum

asymptotics and invariants

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This talk is based on joint work with

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Iosif Polterovich (Université de Montréal)
David Sher (University of Michigan)

arXiv:1311.5533

1. The Steklov spectrum.
2. Can one hear the shape of a Steklov drum?
3. Main inverse spectral theorem.
4. Application: Spectral rigidity of the disk.
5. Sharp spectral asymptotics.
   The Dirichlet-to-Neumann operator as a ψDO
6. Geometric information from sharp asymptotics:
   The Dirichlet theorem on diophantine approximation
7. Open problems and conclusion

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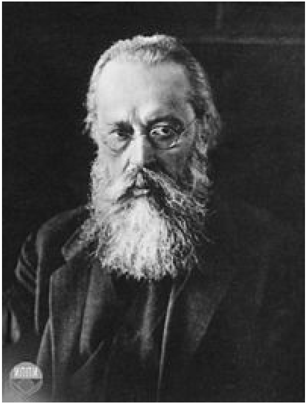
Boundary spectral parameters. . .

where introduced in 1902 by

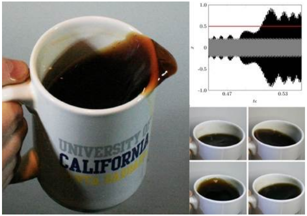


Vladimir Steklov

They are models for **linear water waves**, vibrations of membranes, heat diffusion, optimal design, etc.

Linear coffee waves

H. C. Mayer and R. Krechetnikov, ***Walking with coffee: Why does it spill?*** (2012)



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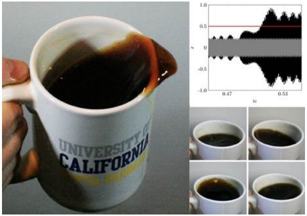

The velocity potential $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies

$$\begin{aligned} \Delta\Phi &= 0 && \text{inside the mug} \\ \partial_t^2\Phi + g\partial_n\Phi &= 0 && \text{on the free surface} \\ \partial_n\Phi &= 0 && \text{on the sides and bottom.} \end{aligned}$$

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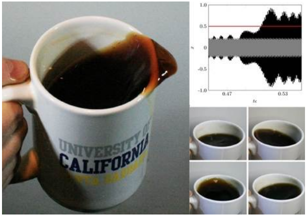

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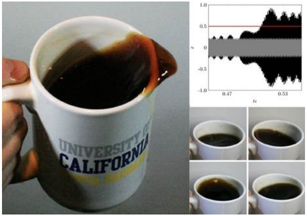

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Winner of the 2012 Ig Nobel prize in physics

http://www.improbable.com/ig/

Examples

1. The Steklov spectrum of the unit disk $\mathbb{D} \subset \mathbb{R}^2$.

Eigenvalues

$0, 1, 2, 3, \dots, k, \dots$

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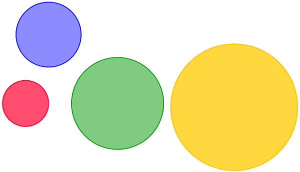
The **eigenfunctions** corresponding to $\sigma = k$ are $r^k u(\theta)$, where u a solution of

$$\Delta_{\mathbb{S}^{n-1}} u + \lambda_k u = 0, \quad \lambda_k = k(k+n-2).$$

Spherical harmonics

3. The Steklov spectrum of a disjoint union of disks.

Let Ω be the disjoint union of k disks of radii $r_1, \cdots, r_k$.



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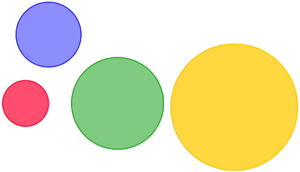

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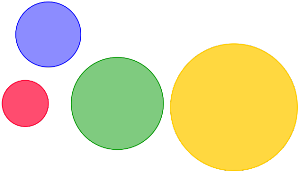

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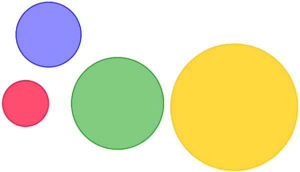

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The spectrum of Ω is a **union of arithmetic progressions**

$$S(G) := \overbrace{\{0, \ldots\ldots, 0\}}^{k} \cup \alpha_1\mathbb{N} \cup \alpha_1\mathbb{N} \cup \alpha_2\mathbb{N} \cup \alpha_2\mathbb{N} \cup \cdots \cup \alpha_k\mathbb{N} \cup \alpha_k\mathbb{N},$$

reordered as a monotone increasing sequence.

This will be useful later

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For instance, on $[-L, L] \times S^1$ these are

$$0,\ 1/L,\ \underbrace{k\tanh(kL),\ k\coth(kL)}_{\text{multiplicity}=2}$$

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1. Isospectral cylinders.

If Σ_1 and Σ_2 are two **isospectral closed Riemannian manifolds**, then $\Omega_1 = [-L, L] \times \Sigma_1$ and $\Omega_2 = [-L, L] \times \Sigma_2$ are Steklov isospectral. This follows from Example 4.

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Numerous examples of isospectral non-isometric closed manifolds are known.

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The infimum is over all k-dimensional subspaces E_k of the Sobolev space $H^1(\Omega)$ which are orthogonal to constants on $M = \partial\Omega$.

Open problem

In both constructions of Steklov isospectral manifolds:

1. cylinders over Laplace isospectral closed manifolds

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For surfaces, we will give a complete answer to this question.

3. Main results

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$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty.$$

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Theorem (A. Girouard, L. Parnovski, I. Polterovich, D. Sher)

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Corollary

Steklov isospectral surfaces have Laplace isospectral boundaries. (Indeed, ***they are isometric****!)*

Algorithmic extraction of the lengths

Our result is actually stronger. We have obtained an explicit procedure for extracting the geometric information from the spectrum. For instance...

Theorem

Let $L_1 \geq L_2 \geq \cdots \geq L_k$ be the lenghts of the boundary components of Ω. Then

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The procedure for extracting the other lengths is recursive.

The analogous result is false in higher dimension

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P.G. Doyle and J. P. Rossetti, 2011

The surfaces

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The collection of areas of boundary components

is not a Steklov invariant.

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Proof. Let $\Omega \subset \mathbb{R}^n$ be such that $\lambda_j(\Omega) = \lambda_j(\mathbb{D})$ for each $j \in \mathbb{N}$.

1. The **Weyl Law**

$$\mathcal{N}(\lambda) := \#\{j \in \mathbb{N} \,:\, \lambda_j < \lambda\}$$

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3. The conclusion follows from the **Weinstock Inequality**

$$\lambda_1(\Omega)L(\partial\Omega) \leq \lambda_1(\mathbb{D})2\pi$$

   with equality iff Ω is a disk. **QED**

Strategy of the proof

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This is proved in three steps:

Strategy of the proof

Theorem

Let $L_1 \geq L_2 \geq \cdots \geq L_k$ be the lenghts of the boundary components of Ω. Then

$$L_1 = \frac{2\pi}{\limsup_{j\to\infty}(\lambda_{j+1} - \lambda_j)}.$$

This is proved in three steps:

1. Reformulate the Steklov problem in the setting of a pseudodifferential operator.
2. Use symbol computation to obtain obtain a very precise description of the spectral asymptotics of λ_j as $j \nearrow \infty$.
3. Extract information using the Dirichlet diophantine approximation theorem.

5. The Dirichlet–to–Neumann map

Let Ω is a **compact manifold of dimension** n with smooth boundary $M = \partial\Omega$ of dimension $n-1$.

The **Dirichlet-to-Neumann map** $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ is defined by

$$\mathcal{D}f = \partial_n(Hf),$$

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The DtN map is **not local**. It is not a differential operator...

Smooth perturbations of pseudodifferential operators

A bounded (pseudodifferential) operator S acting on M is **smoothing** if

$$S : H^s(M) \to H^t(M) \quad \forall s, t \in \mathbb{R}$$

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Lemma (Folklore)

Let P and Q be elliptic, bounded below, self-adjoint ψDOs. ***If the difference*** $P - Q$ ***is smoothing****, then*

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This point of view is useful, because the DtN map on $M = \partial\Omega$ is a smooth perturbation of an operator that we understand very well.

Let Ω be a **compact surface with boundary**.

Theorem (A. Girouard–L. Parnovski–I. Polterovich–D. Sher)

The square $\mathcal{D}^2$ of the DtN map is a smooth perturbation of the Laplace operator on $M = \partial\Omega$.

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$$\lambda_j(\mathcal{D}) = \sqrt{\lambda_j(-\Delta_M)} + \mathcal{O}(j^{-\infty}).$$

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Corollary

Let $L_1, \cdots, L_k$ be the lengths of the components of $M = \partial\Omega$.

Let $G = \{\alpha_1, \cdots, \alpha_k\}$, where $\alpha_i = \frac{2\pi}{L_i} \quad i = 1, \cdots, k$.

$$\lambda_j = S(G)_j + \mathcal{O}(j^{-\infty}).$$

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This is a tremendous improvement from the abstract Weyl Law for ΨDOs.

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–Bad– There is very little geometric information contained contained in spectral asymptotics at ***polynomial scale***. In terms of ***inverse problems*** this looks like a bad feature.

For instance, it is impossible to determine the genus of a surface from the polynomial asymptotic scale.

6. Extracting informations from asymptotics

Let Ω be a compact surface with k boundary components of lengths

$$L_1 \geq L_2 \geq \cdots \geq L_k$$

Theorem

The length of the longest boundary component is

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In other words

$$\limsup_{j\to\infty}(\lambda_{j+1} - \lambda_j) = \frac{2\pi}{L_1}.$$

Proof...

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$$S = \overbrace{\{0, \ldots \ldots, 0\}}^{k} \cup \alpha_1 \mathbb{N} \cup \alpha_1 \mathbb{N} \cup \alpha_2 \mathbb{N} \cup \alpha_2 \mathbb{N} \cup \cdots \cup \alpha_k \mathbb{N} \cup \alpha_k \mathbb{N}$$

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$$\limsup_{j \to \infty} (\lambda_{j+1} - \lambda_j) = \limsup_{j \to \infty} (S_{j+1} - S_j) \leq \alpha_1.$$

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We need to prove

$$\limsup_{j\to\infty}(S_{j+1} - S_j) \geq \alpha_1.$$

1. Suppose **all generators** $\alpha_1 \leq \alpha_2 \leq \cdots \alpha_k$ **are rational numbers**.

Say $\alpha_i = p_i / q_i$.

Let $X = p_1 p_2 \cdots p_k$ be the product of their numerators.

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Therefore, the sequence

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Since this gap occurs infinitely often, this shows

$$\limsup_{j \to \infty} (S_{j+1} - S_j) \geq \alpha_1.$$

and we are done.

1. Suppose **some generators are irrational numbers**.

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Multiples of X still occur in the sequence S_j, but now, they are surrounded by a cloud of irrational numbers.

Dirichlet diophantine approximation

Let $\beta_1, \cdots, \beta_m$ be **irrational numbers**.

There exists an infinite set $K \subset \mathbb{N}$ such that for each $q \in K$, there exist $p_{1,q}, \cdots, p_{m,q} \in \mathbb{N}$ such that

$$\left| \beta_i - \frac{p_{i,q}}{q} \right| \leq \frac{1}{q^{1+1/m}}$$

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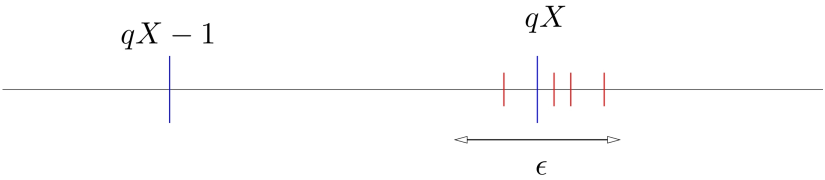
Using the irrational numbers $\beta_m = X/\alpha_{i_m}$ leads to

$$|qX - p_{i,q}\alpha_{i_m}| \leq \frac{\alpha_{i_m}}{q^{1/m}} \cdots$$



...hence, to the opening of gaps in the spectrum! **QED**

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For surfaces, we just showed this is true.

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This question is completely open.

Question 3. Find geometric invariants determined by the Steklov spectrum.
We have seen a variety of answers to this question.
Is the genus of a surface a Steklov spectral invariant?

Conclusion and references

1. N. Kuznetsov, T. Kulczycki, M. Kwasnicki, A. Nazarov, S. Poborchi, I. Polterovich, B. Siudeja.
   The Legacy of Vladimir Andreevich Steklov
   Notices AMS, January 2014.

2. A. Girouard, L. Parnovski, I. Polterovich, D. Sher
   The Steklov spectrum of surfaces: asymptotics and invariants
   preprint: arXiv:1311.5533

Thank you for your attention!

Part II

The Dirichlet-to-Neumann operator

from a ψDO perspective

In this follow up lecture, the following topics will be discussed:

1. The DtN map on the disk $\mathbb{D}$ and on the upper half-plane $\mathbb{H}$.
2. Asymptotic expansions and smooth perturbations.
3. Inverse problems and symbol computation.
   (The work of Lee and Uhlmann)
4. The full symbol of the DtN map on simply connected planar domains.
5. The links with trace formulae.
   (Chazarain, Duistermaat–Guillemin)