# Redefining the maximum sustainable yield for the Schaefer population model including multiplicative environmental noise 

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#### Abstract

The focus of this article is to investigate the biological reference points, such as the Maximum Sustainable Yield (MSY), in a common Schaefer (logistic) surplus production model in the presence of a multiplicative environmental noise. This type of model is used in fisheries stock assessment as a first-hand tool for biomass modelling. Under the assumption that catches are proportional to the biomass, we derive new conditions on the environmental noise distribution such that stationarity exists and extinction is avoided. We then get new explicit results about the stationary behavior of the biomass distribution for a particular specification of the noise, namely the biomass distribution itself and a redefinition of the MSY and related quantities that now depend on the value of the variance of the noise. Consequently, we obtain a more precise vision of how less optimistic the stochastic version of the MSY can be than the traditionally used (deterministic) MSY. In addition, we give empirical conditions on the error variance to approximate our specific noise by a lognormal noise, the latter being more natural and leading to easier inference in this context. These conditions are mild enough to make the explicit results of this paper valid in a number of practical applications. The outcomes of two case-studies about northwest Atlantic haddock (Spencer and Collie 1997) and South Atlantic albacore tuna (Millar and Meyer 2000) are used to illustrate the impact of our results in bioeconomic terms.


Key words: Process error; lognormal distribution; Markov chain; biomass equilibrium distribution; proportional harvest.

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## 1 Introduction

In ecology, bioeconomics and more specifically in fisheries research, surplus production models (SPM) are a traditional first analytical tool for stock assessment when the only available data consist of a time series of catches (or harvest) and some index of abundance for an exploited population. Though they are less realistic than age-structured models, SPM are useful for illustrating relevant concepts in harvest strategies (Jensen 2002b). Moreover, sometimes they may even provide better estimates of biological landmarks than age-structured models (Laloe 1995). Besides, SPM are still used for species for which age classes are difficult to determine like tropical fish (Prager 2002), or simply because of a lack of research funds, especially in developing countries (Chavance et al. 2002). These models are also useful for an age-structured analysis for the recruitment of the first age population (Beard et al. 2003).

Let $\Delta$ denote the difference operator $\left(\Delta x_{t}=x_{t+1}-x_{t}\right)$. SPM are usually defined by the difference equation

$$
\begin{equation*}
\Delta B_{t}=f\left(B_{t}\right)-C_{t} \tag{1}
\end{equation*}
$$

where $B_{t}, C_{t} \geq 0$ denote the biomass and the catches at time $t$, respectively, and $f$ is a socalled production function. In this article, we are especially concerned with the simple Schaefer, or logistic, production function (Schaefer 1954)

$$
\begin{equation*}
f\left(B_{t}\right)=r B_{t}\left(1-\frac{B_{t}}{K}\right), t \geq 0 \tag{2}
\end{equation*}
$$

$r>0$ being the per capita growth rate and $K>0$ the carrying capacity (the point at which growth and decline rates are equal). Though some authors (Maunder 2002b, Prager 2002) do not consider that the Schaefer model is a tool powerful enough for effective stock assessment, it remains useful in preliminary efforts to evaluate harvesting strategies and assessment methods (Millar and Meyer 2000, Hammond and Trenkel 2005, de Valpine and Hilborn 2005).

As in numerous other studies in fisheries research, this study will be done within the framework of discrete time series analysis. Indeed, a discrete-time approach should be more relevant in our application of SPM to fisheries, since time steps between environmental measures are large in comparison to the lifetime cycle of the studied species.

One key purpose of SPM consists in the estimation of the maximum sustainable yield (MSY) and its associated biomass (or resource), which are biological reference parameters for fisheries management (Jensen 2002b). The MSY is defined as the maximal possible catch such that the population biomass can continue to regenerate. Its expression in function of the biological parameters is derived under an assumption of equilibrium for the biomass $\left(\Delta B_{t}=0\right)$. For instance, choosing $f$ as (2) in (1) yields MSY $=r K / 4$.

However, deterministic SPM have been criticized since they ignore the effect of environmental variability on the population dynamics, that is reflected in the observed data (Gore and Paranje, 2001, p. 207-208). In the past, MSY estimations from models that ignored biological and statistical uncertainties were certainly involved in overoptimistic decision making, leading to major fishery
collapses (Larkin 1977, Clark 1981, Lewis 1981, Amundsen and Bjørndal 1999), jointly with inabilities to enforce regulations or poor institutional arrangements (Patterson 1998, Hammond and Trenkel 2005). However, despite the warnings from scientists, the MSY is still viewed as a key parameter for the fisheries (Maunder 2002a, Jacobson et al. 2002) and a starting point for international environmental policies (Holt 2007). Thus, some authors advocate to set harvesting quotas as a fraction of the deterministic MSY (Doubleday 1976, Caddy and Mahon 1995, Roughgarden and Smith 1996, Jensen 2002a) to maximize short-term revenue and preserve ecological stability. They suggest that the MSY is in itself a relevant indicator, but that efforts should be focused on a) taking stochasticity into account in its calculation and b) avoiding the frequent disrespect of quotas by exploiters (Rosenberg et al. 1993).

To this end, it has become standard to incorporate an environmental noise (process error), say $\epsilon_{t}$, in analytical expressions of the biomass dynamics to reflect its natural variability, and thus to increase the relevance of the SPM (Penn and Caputi 1986, Gudmundsson 1994, Patterson et al. 2001 for a review).

Therefore, in this article, we will consider a stochastic form of the SPM: incorporating a multiplicative environmental noise $\epsilon_{t}$ into (1), namely considering the Markov chain

$$
\begin{equation*}
B_{t+1}=\left(B_{t}+r B_{t}\left\{1-\frac{B_{t}}{K}\right\}-C_{t}\right) \epsilon_{t+1}, t \geq 0 \tag{3}
\end{equation*}
$$

our aim will be to define a stochastic equilibrium in order to obtain explicit derivations of the MSY and some related quantities at this stochastic equilibrium. In recent years stochastic SPM with multiplicative noise have been largely used in fisheries research (see for instance de Valpine and Hastings 2002, Punt 2003), but their stochastic properties have not been studied extensively.

The biomass variable $B_{t}$ then becomes random with distribution $\mathcal{F}_{B_{t}}$ and the deterministic equilibrium assumption $\left(\Delta B_{t}=0\right)$ has to be replaced by a stochastic equilibrium assumption, such as stationarity $\left(\mathcal{F}_{B_{t+1}}=\mathcal{F}_{B_{t}}\right)$.

As we shall see, the "dispersion" of the noise $\epsilon_{t}$, typically quantified by its variance $\sigma^{2}$, should appear in the recalculation of the reference points and can have important effects. Such a concern is not new. Since the 1970's a great deal of attention has been devoted to the analysis of the impact of environmental fluctuations on harvesting strategies, in biological as well as in economical contexts. Most of the studies have been led in the frameworks of perturbation methods and diffusion theory, assuming continuous time and unpredictable additive perturbations; see Lande et al. (1997) for a survey of the main references, and Kirkwood (1980) and Lande et al. (1995) for detailed presentations. Beddington and May (1977), May et al. (1978), Ludwig (1979) and Ludwig and Varah (1979), Ludwig (1980), Pyndick (1984), Alvarez and Shepp (1998), Alvarez et al. (2001), Shah and Sharma (2003) have obtained exact or approximated recalculations of bioeconomic reference points when a stochastic term is added to the production function. When additive perturbations are not assumed to be stochastic, Jensen (2002a, 2005) derived similar reference points, including environmental trends. In the same way, understanding the impact of stochasticity on the time before extinction has become the subject of increasing research (Routledge and Irvine 1999, Dushoff 2000, Hakoyama and Iwasa 2000, Allen et al. 2005).

To our knowledge, the work about the transmission of environmental patterns into population behaviors is limited in discrete frameworks. Besides, it seems that cases where the noise is included multiplicatively (Eq. 3) have retained even less attention in stochastic studies. In the fishery literature such a noise is however largely used in articles which mainly focus on parameter estimation or biological case-studies (Millar and Meyer 2000, de Valpine 2002, Punt 2003, de Valpine and Hilborn 2005, among others). In his seminal work, Reed (1978) studied the stochastic behavior of the biomass under the SPM (3), considering cases where at low biomass levels the population size increases with probability one. He obtained general conditions on the noise leading to the stationarity of the biomass Markov chain (a bounded $\epsilon_{t}$ with bounded dispersion). Including this uncertainty in the computation of the reference points, he studied the optimality of some harvesting strategies (Reed 1978, 1979) and highlighted serious overestimations of the available resources resulting from using deterministic reference points, leading to unwelcomed economic and ecological decisions. This conclusion was shared with most of the previously cited authors. See Flåm (1981) for a more specific stochastic investigation, Clark and Kirkwood (1986) for a Bayesian approach to the problem and Sethi et al. (2005) for a recent continuance of this research, including other error sources, in a time-continuous framework using dynamic programming.

Thus, Reed's work draws the main steps of a general study, but it cannot be directly applied to stochastic logistic cases. In this article, we complement his approach to apply it to the Schaefer SPM. The novelty in our work can be summarized in three points which are treated in Section 2. First, we obtain necessary and sufficient conditions on the environmental noise to obtain the stochastic equilibrium. Second, for a given form of the noise (called "product-of-beta" throughout this article) and assuming an upper bound for its variance $\sigma^{2}$, we obtain explicitly the unique stationary biomass distribution. Third we study the impact of $\sigma$ on the MSY and related quantities by deriving explicit expressions for these reference points. A classic conclusion, shared with the majority of previously cited authors, is that harvesting according to the deterministic MSY rule is clearly an underoptimized strategy and can lead to strong decreases of the resource (see especially Larkin 1977, Reed 1978, Roughgarden and Smith 1996). As a corollary of our results from Section 2, we highlight the stronger conclusion that a deterministic long-term strategy is incompatible with the assumption of equilibrium: on average, one cannot hope to harvest more than the stochastic MSY. Furthermore, constant harvesting fatally leads to the extinction of the resource; thus optimizing the harvest must be done by optimizing the harvest rate, which implies that precise information about the levels of the stock through the timescale must be obtained, and not only a good estimation of the biological parameters. Another noticeable result is the substantial influence of $\sigma^{2}$ on reference point $B_{M S Y}$, the mean stock level in the optimized harvesting conditions. Since it is an indicator from which the notion of "overfishing" is usually derived in fisheries management, an explicit approximation of $B_{M S Y}$ as a function of $\sigma$ is emphasized. Finally, we indicate how to derive other reference points, for instance defined under bioeconomic constraints.

In Section 3, we give conditions under which the commonly used lognormal distribution can be approximated by a product-of-beta without having a serious impact on the biomass trajectories. This amounts to giving guidelines under which the new expressions for biological landmarks derived in Section 2 under a product-of-beta noise remain pertinent under a lognormal noise.

Finally, two numerical applications are presented in Section 4, using fishery data on the Georges Bank haddock in the northwest Atlantic and the albacore tuna in South Atlantic.These examples illustrate the possible shortfall, in bioeconomic terms, resulting from the use of a deterministic strategy to manage a stochastic population.

## 2 A stochastic Schaefer model

### 2.1 Description and stationarity

We assume that catches in Eq. (2) are always proportional to the biomass, i.e. $C_{t}=\phi B_{t}$, where the capture rate $\phi$ is supposed constant over time; this assumption can possibly be relaxed (see Section 5 for further discussion). Then (3) becomes

$$
\begin{equation*}
B_{t+1}=\left[(1-\phi+r) B_{t}-r B_{t}^{2} / K\right] \epsilon_{t+1}, \tag{4}
\end{equation*}
$$

where the $\epsilon_{t}$ are assumed to be positive independent and identically distributed (i.i.d.) random variables. Without loss of generality, we suppose that the expectancy $\mathrm{E}\left(\epsilon_{t}\right)=1$. Indeed, suppose that $\mathrm{E}\left(\epsilon_{t}\right)=\mu \neq 1$ and let $r_{\mu}=r \mu+\mu-1, \phi_{\mu}=\phi \mu$, and $K_{\mu}=K r_{\mu} /(r \mu)$. Then the model can be re-expressed as

$$
B_{t+1}=\left[\left(1-\phi_{\mu}+r_{\mu}\right) B_{t}-r_{\mu} B_{t}^{2} / K_{\mu}\right] \epsilon_{t} / \mu
$$

which has the same form as the original model, but with innovations $\epsilon_{t} / \mu$ that have mean 1 . To obtain a simpler writing of the dynamics, set

$$
Z_{t}=\frac{r B_{t}}{K(1-\phi+r)}, t=0,1 \ldots
$$

Then (4) is equivalently rewritten as

$$
\begin{equation*}
Z_{t+1}=(1-\phi+r) Z_{t}\left(1-Z_{t}\right) \epsilon_{t+1} \tag{5}
\end{equation*}
$$

We now derive some results about the stationary behavior of the Markov chain defined by (5). (All proofs are given in the Appendix.) Essential arguments of stationarity are irreducibility and aperiodicity (Robert and Casella 2004). Theorem 2.1 indicates that a necessary and sufficient condition of irreducibility is a bounded noise as a function of $(r, \phi)$, without presuming any form for its density. Aperiodicity is then obtained under a very mild condition, and this leads to a unique stationary distribution.

Theorem 2.1 Let $\left\{Z_{t}\right\}$ be the Markov chain defined by (5) and put $\gamma=r-\phi$. Assume that $\left\{\epsilon_{t}\right\}$ are continuous i.i.d random variables with probability density function $f_{\epsilon}$. Assume that the support of $f_{\epsilon}, \operatorname{Supp}\left(f_{\epsilon}\right)$, is a subset of $\mathbb{R}_{+}$.
(A) The chain is irreducible if and only if $\operatorname{Supp}\left(f_{\epsilon}\right) \subset\left[0,4(1+\gamma)^{-1}\right]$ and the initial state $\left.z_{0} \in\right] 0,1[$.
(B) Assume that $\left.z_{0} \in\right] 0,1\left[\right.$ and $\operatorname{Supp}\left(f_{\epsilon}\right)=\left[0,4(1+\gamma)^{-1}\right]$. Then the chain is irreducible and aperiodic. Consequently, for almost every $z_{0}$, the chain admits a unique stationary distribution.

Theorem 2.1 thus gives us a first hope to generate biomass series that do not threaten to collapse or explode. Notice, however, that unbounded lognormal noises cannot lead to stationarity.

If Theorem 2.1 stays very general about the form of the noise $f_{\epsilon}$, Theorem 2.2 is specific to a particular choice of $f_{\epsilon}$ which we will term "product-of-beta" and define as follows:

Definition 2.1 Denote $\Gamma$ the gamma function and $\mathcal{B}_{e}(\alpha, \beta)$ the beta distribution with density

$$
f(x \mid \alpha, \beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{\{0 \leq x \leq 1\}}
$$

defined for strictly positive $\alpha$ and $\beta$. Let

$$
\begin{aligned}
U_{t} & \sim \mathcal{B}_{e}\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta-1}{2}\right), \\
V_{t} & \sim \mathcal{B}_{e}\left(\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}\right)
\end{aligned}
$$

be two independent random variables (with $\alpha+\beta>1$ and $\alpha<\beta$ ). The random variable $\epsilon_{t}$ is said to follow a product-of-beta distribution with parameter $(\alpha, \beta)$ if

$$
\begin{equation*}
\epsilon_{t}=\frac{4 \beta}{1+\alpha+\beta} U_{t} V_{t} \tag{6}
\end{equation*}
$$

Theorem 2.2 Let $\gamma=r-\phi>0$ be the difference between the per capita growth rate $r$ and the fishing rate $\phi$ and $\sigma^{2} \in\left[0, \sigma_{M}^{2}[\right.$ be the innovation variance where

$$
\begin{equation*}
\sigma_{M}^{2}=1-\frac{1+\gamma+2 \gamma^{3}}{(1+\gamma)^{3}} \tag{7}
\end{equation*}
$$

Define $\alpha=\gamma \beta-1$ where

$$
\begin{equation*}
\beta=\frac{1}{2(1+\gamma) \sigma^{2}}\left(\left\{2-\gamma-(1+\gamma) \sigma^{2}\right\}+\sqrt{8 \sigma^{2}+\left\{2-\gamma-(1+\gamma) \sigma^{2}\right\}^{2}}\right) . \tag{8}
\end{equation*}
$$

Suppose $\epsilon_{t}$ is defined by (6). Then
a) $E\left[\epsilon_{t}\right]=1$ and $\operatorname{Var}\left[\epsilon_{t}\right]=\sigma^{2}$;
b) for any initial state $\left.z_{0} \in\right] 0,1\left[\right.$, the process $\left\{Z_{t}\right\}$ admits the $\mathcal{B}_{e}(\alpha, \beta)$ as its unique stationary distribution.

The upper bound (7) on the innovation variance is a re-expression, in terms of $\left(\gamma, \sigma^{2}\right)$, of the condition that $\alpha>0$ in Definition 2.1. Theorem 2.2 indicates that by choosing the noise according to (6) (i) the conditions of Theorem 2.1 are respected and (ii) the stationary distribution of the biomass is explicitly known and is, up to a scaling constant, a nondegenerate beta distribution for
any initial state $\left.z_{0} \in\right] 0,1\left[\right.$. In an illustrative purpose, the upper bound $\sigma_{M}$ is plotted as a function of $\gamma$ in Figure B.1. Notice that when $\phi=r(\gamma=0)$, there is no stationnary distribution and extinction is unavoidable.

## PLACE FIGURE B. 1 ABOUT HERE

Extinction condition. The condition $\sigma^{2}<\sigma_{M}^{2}$ is a necessary and sufficient condition for nonextinction. If it is not met the process $B_{t}$ converge to 0 with probability 1 . It is stronger than the local condition given by Reed (1978, Eq. (2.6)) on the noise dispersion. In our framework, Reed's necessary condition for non-extinction of the population can be written as

$$
\begin{equation*}
V=\exp \left(-\mathrm{E}\left[\log \epsilon_{t}\right]\right)-1 \leq g^{\prime}(0)-1, \tag{9}
\end{equation*}
$$

where $g(x)=(1+\gamma) x(1-x)$ (in Reed's original work $g$ does not incorporate the capture rate since he first considers an unexploited population). Thus $g^{\prime}(0)=1+\gamma$. We obtain the following proposition, whose proof is given in Appendix.

Proposition 2.1 Assume a product-of-beta environmental noise $\epsilon_{t}$. Then the non-extinction condition $\sigma^{2}<\sigma_{M}^{2}$ implies (9).

Finally note that $\sigma^{2}<\sigma_{M}^{2}$ implies (9) for all $\gamma>0$ in the case of a lognormal noise (cf. Section $3)$, where $V=\sqrt{1+\sigma^{2}}-1$.

### 2.2 Derivation of biological reference points

Assume for the biomass the stochastic equilibrium given in Theorem 2.2. The expectation of the stationary distribution for $B_{t}$ is then

$$
\mathrm{E}\left(B_{t}\right)=\frac{K(1+r-\phi)}{r} \frac{\alpha}{\alpha+\beta}=\frac{K(r-\phi)}{r}\left\{1-\frac{(r-\phi)^{-1} \sigma^{2}}{(2-r+\phi)}+o\left(\sigma^{3}\right)\right\},
$$

where $\alpha$ and $\beta$ are defined in Theorem 2.2 and $g(\sigma)=o\left(\sigma^{n}\right)$ means that $g(\sigma) / \sigma^{n} \rightarrow 0$ as $\sigma \rightarrow 0$. The expected catch $\phi \mathrm{E}\left(B_{t}\right)$ is thus maximal when

$$
\begin{equation*}
\phi_{o p t}=\frac{r}{2}-\frac{2(2-r)}{(4-r)^{2}} \sigma^{2}+o\left(\sigma^{3}\right) \tag{10}
\end{equation*}
$$

The difference is obvious with the well known deterministic case where $\phi_{o p t}=r / 2$ (e.g., Jensen 2002b). Substituting (10) into (7) yields a more general upper bound for $\sigma$ in terms of $r$ which is displayed in Figure B.2. The MSY and the mean biomass under maximal exploitation are respectively $M S Y=\phi_{o p t} \mathrm{E}\left[B_{t} \mid \phi=\phi_{o p t}\right]$ and $B_{M S Y}=\mathrm{E}\left[B_{t} \mid \phi=\phi_{o p t}\right]$. Both expressions are
unexplicit too but can be accurately approximated using Taylor expansion in $\sigma$. The following computations were performed by using Maple ${ }^{\mathrm{TM}}$ (Monagan et al. 2005). We obtain

$$
\begin{align*}
& M S Y(\sigma)=\frac{r K}{4}\left(1-\frac{\sigma^{2}}{r(1-r / 4)}+\frac{4 \sigma^{4}}{r^{2}(4-r)^{4}}\left[r^{4}-4 r^{3}-12 r^{2}+48 r-16\right]+o\left(\sigma^{5}\right)\right),  \tag{11}\\
& B_{M S Y}(\sigma)=\frac{K}{2}\left(1-\frac{8 \sigma^{2}}{r(4-r)^{2}}-\frac{8 \sigma^{4}}{r(4-r)^{5}}\left[3 r^{3}-18 r^{2}+12 r+32\right]+o\left(\sigma^{5}\right)\right) \tag{12}
\end{align*}
$$

For some chosen values of $r$, the functions $M S Y(\sigma) / M S Y(0)$ and $B_{M S Y}(\sigma) / B_{M S Y}(0)$ are displayed on Figures B. 3 and B.4. Notice that for a small $r$ and a value of $\sigma$ large enough, taking the stochastic environment into account can reduce the the deterministic $M S Y(0)=r K / 4$ more than $50 \%$ ! As a matter of comparison, authors like Pindyck (1984) expressed MSY as a decreasing function of $\sigma$, proportional to $1-\sigma^{2} / r$, in a time-continuous framework with additive environmental noises of variance $\sigma^{2}$. A less conventional result is the fact that $B_{M S Y}(\sigma)$ can be significantly lower than $K / 2$, even if the decrease of $B_{M S Y}(\sigma)$ as a function of $\sigma$ appears to be slower than the decrease of $\operatorname{MSY}(\sigma)$. We discuss further the bioeconomical meaning of these results in the discussion section.

> PLACE FIGURE B.2 ABOUT HERE

$$
\text { PLACE FIGURE B. } 3 \text { ABOUT HERE }
$$



It is essential to notice that $\operatorname{MSY}(\sigma)$ represents the maximal harvest which can be expected on average in the long run when the harvesting strategy is optimized through the choice of the rate $\phi$ (the effect of a non-optimized rate can be perceived on Figure B.5). Thus, linking $\phi_{\text {opt }}$ to an optimal harvest strategy requires the knowledge of the immediate stock level $B_{t}$, as $C_{t}^{\text {opt }}=\phi_{\text {opt }} B_{t}$. Therefore information-gathering for the purpose of biomass estimation (e.g., abundance indexes) is an indispensable prerequisite for strategy-making.


The implications of Theorems 2.1 and 2.2 actually go beyond the determination of an optimal harvest strategy. A first corollary is that any choice of $C_{t}$ that is not based on both an estimation of $B_{t}$ and $\phi$ will fatally lead to extinction. Another corollary is that no effort rate $\phi^{*}$ can be
found such that both a) the biomass distribution remains stationary and b) on average, the expected harvest reaches the deterministic level $r K / 4$. Indeed, such a $\phi^{*}$ would be the solution of $\phi^{*} \mathrm{E}\left[B_{t} \mid \phi=\phi^{*}\right]=r K / 4$, which would imply, by definition, that

$$
M S Y(0)=r K / 4 \leq \phi_{o p t} \mathrm{E}\left[B_{t} \mid \phi=\phi_{o p t}\right]=M S Y(\sigma)
$$

which is true only when $\sigma=0$.

Other reference points. The MSY is the most common biological reference point derived from SPM. An alternative management policy could maximize catches subject to the constraint that $B_{t}$ does not drop below a fraction $0<\rho<1$ of $K$ with a high probability, say $1-\alpha$. Traditionally, one chooses $\rho=0.2$ (Katsukawa 2004). A new reference point $M Y C$ (maximum yield under constraint) can then be defined as

$$
M Y C(\sigma)=\max _{0 \leq \phi \leq r} \phi E\left(B_{t} \mid \phi\right)
$$

under the constraint $P\left(B_{t}<\rho K \mid \phi\right)=\alpha$. Though an approximate expression for $M Y C(\sigma)$ cannot be calculated easily, the new $\phi_{\text {opt }}$ and $M Y C(\sigma)$ as functions of $\sigma$ can easily be numerically derived. Figures (B.6) and (B.7) illustrate these values with $\rho=0.2$ and $\alpha=0.05$. The difference between $M S Y(\sigma)$ and $M Y C(\sigma)$ (or between the optimized harvest rates) is not excessive when $\sigma<0.2$, but it can severely increase otherwise. In some cases where a unique standard stochastic MSY exists, a nonzero MYC might not exist.

$$
\text { PLACE FIGURE B. } 6 \text { ABOUT HERE }
$$

## PLACE FIGURE B. 7 ABOUT HERE

In economic frameworks, constraints can be introduced under the form of a concave profitability function (or the opposite of a cost function) $I(q)$, where $q$ denotes the fished quantity. One can write $I(q)=p(q)-c(q)$, where $p$ is the selling price of the resource and $c$ is the (economic/environmental) fishing cost. Fishing effort $\phi$ is optimized if $\mathrm{E}\left[I\left(\phi B_{t}\right)\right]$ is maximal, i.e., when $\phi=\phi^{*}$ with

$$
\left.\frac{\partial \mathrm{E}\left[I\left(\phi B_{t}\right)\right]}{\partial \phi}\right|_{\phi=\phi^{*}}=0
$$

## 3 Lognormal approximation

The lognormal is a common choice of distribution for the multiplicative process error terms in fishery research, and is probably more practical than the product-of-beta distribution for statistical
inference. The biological rationale for this is as follows: the stock-recruitment process producing the biomass at the end of the time period (one-year cycle) can be seen as the outcome of a series of successive survivorships from the birth to the recruit. After $n$ life stages, the complete survivorship probability is the product of $n$ survivorships. By a log transformation, the central limit theorem suggests than the log-survivorship at each end of time period (assuming a large $n$ ) is normally distributed. See Ricker (1975), Reed (1978) or Haddon (2001) for more precisions. Thus, our aim in this section is to assess whether the explicit results established in Section 2 can be used when a lognormal distribution is assumed for the process error instead of the product-of-beta. When $\sigma$ is small, the product-of-beta (bounded) distribution $\mathcal{F}_{\epsilon}$ with mean 1 and variance $\sigma^{2}$ can be approximated by a lognormal (unbounded) distribution $\tilde{\mathcal{F}}_{\epsilon}$ with same mean and variance (though we cannot theoretically obtain stationarity of the Markov chain under the lognormal, cf. Theorem 2.1).

A reasonable approximation should be reflected by the fact that the difference between $\mathcal{F}_{\epsilon}$ and $\tilde{\mathcal{F}}_{\epsilon}$ results in a difference between the related biomass distributions $\mathcal{B}_{\mathcal{F}_{\epsilon}}$ and $\mathcal{B}_{\tilde{\mathcal{F}}_{\epsilon}}$ that is not significant at any time $t$. Thus deciding whether the approximation is reasonable or wrong can be formalized by means of the following test of hypothesis:

$$
\mathcal{H}_{0}: \mathcal{B}_{\mathcal{F}_{\epsilon}}=\mathcal{B}_{\tilde{\mathcal{F}}_{\epsilon}} \quad \text { vs } \quad \mathcal{H}_{1}: \mathcal{B}_{\mathcal{F}_{\epsilon}} \neq \mathcal{B}_{\tilde{\mathcal{F}}_{\epsilon}} .
$$

We propose a simple nonparametric Mann-Whitney implementation of this test (Wasserman 2005) to compare the distributions of biomasses simulated independently, with similar initial conditions, from both noises. We generate $M=8,000$ biomass trajectories of 2,000 years for both distributions, and at each year, we compute the p-value of the Mann-Whitney test that the two samples of 8,000 biomasses have the same distribution. When this p-value becomes small, we reject the hypothesis of similarity between the biomass distributions. For the simulations we chose values of $r$ and $\sigma$ respecting condition (7) and set $\phi=\phi_{M S Y}$ according to (10). Initial states $\left(z_{0}\right)_{M}$ were sampled from the stationary $\mathcal{B}(\alpha, \beta)$ distribution. Then the mean and standard deviation of the p-value, computed on 2,000 years, are displayed on Figures B. 8 and B.9. We noticed no extinction in the lognormal case. When $\sigma$ is close to 0 , the distribution of the p-value appears uniform with mean 0.5 and standard deviation $1 /(2 \sqrt{3}) \simeq 0.29$. A sensible break appears after $\sigma=0.25$, on average on $r$, where the difference between biomass distributions becomes more and more statistically significant with time and increasing $\sigma$. A very small standard deviation means that the discrepancy between the distributions comes very early in the timescale.

$$
\text { PLACE FIGURE B. } 8 \text { ABOUT HERE }
$$



Thus we suggest that, in terms of statistical inference, without knowing $r$, the lognormal approximation could be considered when $\sigma \leq 0.25$. For values of $r$ higher than 0.4 , the approximation
remains satisfactory until $\sigma=0.35$. Refinements of this result could certainly be the subject of longer studies. These estimated upper limits can be compared with the typical values encountered in the dedicated literature (converted to the scale used in this paper), and appear rather reasonable. For instance, in a Bayesian framework, Meyer and Millar (1999) and Millar and Meyer (2000) proposed prior estimates of $\sigma$ (based on biological considerations) in the $95 \%$ domain ( $0.0498,0.0717$ ) and obtained posterior estimates close to 0.06 . Hammond and Trenkel (2005) obtained estimates beetwen 0.033 and 0.092. de Valpine and Hilborn (2005) got estimates between 0.022 and 0.14 . However, no real consensus exists about the definition of a "low" or "high" value of $\sigma$. Thus, de Valpine and Hastings (2002, for a Ricker model) or Punt (2003) proposed higher values of $\sigma$ (up to 0.3 ) for simulation studies, considering that they can be seen as being low or high in a variety of biological situations.

## 4 Illustrations

To illustrate the impact of our results, we look back at two case-studies that have been formerly treated in the literature. The first one is a bioeconomic study presented by Spencer and Collie (1997), who considered the stocks of Georges Bank haddock (Melanogrammus aeglefinus) in the northwest Atlantic during the years 1931 to 1993. Statistical estimations of parameters can today be considered as crude, but it is a useful example to obtain orders of magnitude of market prices and to observe the repercussions of an overoptimistic fishing policy. Besides, this is one of the rare real case-studies where the stochastic SPM obeys to our specifications (especially $E\left[\epsilon_{t}\right]=$ 1). The second case-study has been treated by Millar and Meyer (2000), about the Bayesian assessment of a stochastic Schaefer SPM to fit the stocks of South Atlantic albacore tuna (Thunnus alalunga) during the years 1967 to 1978, formely studied by Polacheck et al. (1993) . The estimation procedure appears to be more reasonable than in the first case-study. Reproducing the Bayesian posterior parameter simulation of Millar and Meyer (2000), we can correct their posterior mean of the MSY.

### 4.1 Example 1: Georges Bank haddock

Spencer and Collie (1997) used the Schaefer SPM in parallel with a Steele-Henderson model that incorporates a predation term. A multiplicative lognormal environmental noise $\epsilon_{t}=\exp \left(\nu_{t}\right)$ is assumed, such that

$$
\nu_{t} \sim \mathcal{N}\left(-\frac{\sigma_{\nu}^{2}}{2}, \sigma_{\nu}^{2}\right)
$$

(thus $\mathrm{E}\left[\epsilon_{t}\right]=1$ ). Real catch and estimated biomass data are used to fit the models with the "total least squares" method of Ludwig et al. (1988). A residual measurement lognormal error $\mu_{t}$ with variance $\sigma_{\mu}^{2}$ (similar to a nuisance parameter at each time $t$ ) between the estimated and (unknown) real biomass is introduced into the fitting criterion. However, the framework remains very close to a "pure process error" case-study since $\lambda=\sigma_{\mu}^{2} /\left(\sigma_{\mu}^{2}+\sigma_{\nu}^{2}\right)=0.01$. Using biological arguments,

Spencer and Collie concluded that the Schaefer model is relevant for the years 1976-1993 and produced estimates $\hat{r}=0.4, \hat{K}=129 \mathrm{kt}$ (kilotonnes) and $\sigma_{\mu}=0.31$, which implies $\sigma=0.3176$.

Following the conclusions of Section 3, we may use the results of Section 2.2. The MSY and its associated biomass $B_{M S Y}$ are presented in Table 1, in the deterministic and stochastic frameworks. Standard deviations have been reestimated using the Delta method (Patterson et al. 2001). Some simplified economic consequences are derived in the two cases for the years after 1993: the observed price (U.S. dollars per kilogram) per year is given by (Spencer and Collie 1997)

$$
P=a+b Q+c T
$$

where $Q$ is the quantity landed (kilotonnes) in the northeastern United States and $T$ is a time index ( $T=30$ from year 1993). The authors provided estimations $a=1.82, b=-0.022, c=0.033$ to obtain a maximum revenue policy, using dynamic programming. A simplified projection over the years following 1993 is assuming that all the landed stocks will be sold. From the results of Table 1, one can crudely estimate that on average, the use of a deterministic strategy will predict an estimated gain of 8.3 million dollars above the maximal possible income.

## PLACE TABLE B. 1 ABOUT HERE

Even though the estimations of $\left(r, K, \sigma_{\nu}\right)$ should probably be improved using recently developed techniques (de Valpine and Hilborn 2005), these results do highlight the need of studying carefully the influence of $\sigma$ on the management landmarks. It is a convergence point between ecological and economic stakes.

### 4.2 Example 2: South Atlantic albacore tuna

Millar and Meyer (2000) used a multiplicative lognormal environmental noise $\epsilon_{t}=\exp \left(\nu_{t}\right)$ such that

$$
\nu_{t} \sim \mathcal{N}\left(0, \sigma_{\nu}^{2}\right)
$$

thus $\mathrm{E}\left[\epsilon_{t}\right]=\exp \left(\sigma_{\nu}^{2} / 2\right) \neq 1$. The authors provided prior distributions for biological and variance parameters (observation and process errors) and obtained samples from posterior distributions using Gibbs sampling. Focusing on the posterior sensitivity to prior modifications, Millar (2004) concluded to the relative stability of these distributions, the posterior information being carried by the data for the most part.

Removing the model bias due to the choice of $\epsilon_{t}$, as explained in the beginning of Section 2.1, and using the same priors and techniques, we obtained posterior means for the parameters that are close to those of Millar and Meyer's. Our results are summarized in Table 2.

$$
\text { PLACE TABLE B. } 2 \text { ABOUT HERE }
$$

With $\mathrm{E}[r]=0.330$ and $E\left[\sigma_{\nu}\right]=0.0632$, following the conclusions of Section 3 the results of Section 2.2 can be applied with relevance. Thus the posterior mean of the stochastic MSY is $\mathrm{E}[M S Y]=19.18(\mathrm{kt})$ which, given the stochastic model, is a more appropriate quota value than the deterministic value of 19.42 kt . These results differ slightly from those reported in Millar and Meyer (2000) since their parametrization do not satisfy the constraint $E\left[\epsilon_{t}\right]=1$. The gap between the deterministic and stochastic MSY appears more pronounced than the gap between the two $B_{M S Y}$, as predicted by the results in § 2.2. Notice that, though the point estimates are well-differentiated, the posterior credibility intervals of $M S Y$ and $B_{M S Y}$ are quite similar for this example.

## 5 Discussion

In this article we have investigated the stochastic properties of a Schaefer SPM with a constant catch rate, which implies constant catchability and fishing effort. Our main results are necessary and sufficient conditions on the features of an environmental multiplicative noise to obtain stochastic stationarity, which is the primary condition to derive the maximum sustainable yield (MSY) and related biological reference points. Slight modifications to the statistical justifications should be enough to obtain similar results for other SPM, especially the Pella-Tomlinson model (Pella and Tomlinson 1969). For a given form of the noise, which can often be approximated by a common lognormal distribution, the stationary distribution is known and the landmarks can be accurately derived. Important conditions on the noise dispersion are given to avoid extinction. We show that the optimization of the harvesting rate and the precise knowledge of the biomass size are the key components of an optimized time-step to time-step harvesting strategy; besides, the deterministic MSY appears to be an unreachable objective, that is incompatible with the conditions of stationarity. Thus, our study has reinforced the conviction shared by numerous researchers that biological reference points calculated in a deterministic framework can be far from optimal in stochastic settings.

The approximated expressions of the reference points as functions of $\sigma$ are noticeable results, especially $B_{M S Y}$ which is a prominent indicator of the stock level for fisheries management. In the United States, the Magnusson-Stevens Act demands $B_{M S Y}$ values for any exploited stock. Therefore an important point is the fact that the stochastic $B_{M S Y}$ can be significantly lower than the deterministic value $K / 2$. A consequential implication for fisheries management is that the traditional way of considering the stocks as "overfished", namely when they are depleted to a prespecified fraction of $K$, could be modified to account for the estimation of $\sigma$. For instance setting the limit of "overfishing" to a pre-specified fraction of the stochastic $B_{M S Y}$ should be considered. Allowing a small probability $\alpha$ of overfishing, an optimized long-term fishing strategy should refer to the constrained MYC proposed at the end of Section 2.2.

These explicit results can be directly used by fisheries and other resource exploiters when Schaefer model is a reasonable choice. Clearly, this model made our calculations tractable and further study for larger classes of SPM (Pella and Tomlinson 1969) or more complicated models with age-dependence (Lassen and Medley 2000) might be more challenging. Since there is a consensus
among researchers that the practical relevance of the Schaefer model is questionable, even in its stochastic form, this study seems to be highly relevant.

One of the major conditions to obtain stationarity is to assume a bounded process error. This was considered by Reed $(1978,1979)$ as a sufficient condition, but it becomes a necessary condition in our work. Biological circumstances can probably help to justify this statistical requirement. For example, a geographically limited habitat should yield an upper limit for the biomass. In cases where the process error does not satisfy our stationarity conditions, then no equilibrium assumption can be derived and landmarks such as the MSY should not be derived. However reasoning with quasi-stationary rather than stationary distributions in those cases might also lead to certain management criteria (Lande et al. 1995).

Although it has not been treated in this article, a complete study of the stochastic process, viewed as a prediction tool of the future state of stocks, should incorporate a survey of its forecasting properties. Studying autocorrelation patterns can give an idea of the potential for forecasting the future values of the process $B_{t}$ (Box et al. 1994). Our numerical experimentations have shown that, on average on $r \in[0,1]$, when $\phi=\phi_{\text {opt }}$, the process becomes unpredictable after 5 years. More studies have to be done, but these first results call for a regular renewal of parameter and biomass stock estimations in short periods of time. Actually, this estimation updating has become central to decision-making and our results support this policy. Finally, a more rigorous treatment of the predictability of the biomass series may possibly be given using tools from dynamics processing such as the Lyapunov exponent (Chaudhuri 2006).

An issue which deserves to be discussed is the validity of the constant catch rate assumption. Again, this choice made our calculations simpler, but it is probably unrealistic in the long term, since numerous fishery jurisdictions have adopted threshold management strategies which reduce the harvest rate (possibly to zero values) for stocks depleted to below a threshold level. See for instance Zheng et al. (1993), Myers et al. (1994), Jonzén et al. (2003), Katsukawa (2004) and Iskin da Silveira Costa (2007). Thus, in practice the harvest rate likely evolves through time according to a piecewise constant function. In this context, however, our results remain valuable. First, the harvest rate linked to the stochastic MSY provides an upper bound for this series of rates. Second, notice that a modification of $\phi$ at time $t$, when the biomass follows the stationary distribution before $t$, makes the biomass enter into a new stable Markov chain (if $\phi_{\text {new }} \leq \phi_{M S Y}$ ), for which the probability of coming back to high levels of the stocks (or conversely to decrease to low stock levels) can be easily reestimated, thanks to the knowledge of the explicit biomass distribution. (We have started to focus on the relaxation time between these two equilibrium distributions and we hope to present detailed results of different management procedures in a future article.) Thus our results can help to optimize the interplay between harvesting and growth of exploited populations.

Note that inferentially speaking, the constant catch rate assumption could otherwise be softened by assuming that $C_{t}^{*}=\phi B_{t} \nu_{t}$ where $C_{t}^{*}$ means the observed catches and $\nu_{t}$ is an observation error. This choice could help to obtain a simple definition of the likelihood (and even to consider that observed catches are censored values, following the recent idea of Hammond and Trenkel 2005). Model estimation and fitting tests should be presented in future work.

Finally, although SPM can be seen as basic representations of the reality (and thus as first-hand tools), they are still used as decision tools by international commissions. Recently (in 2006), the European Commission proposed to amend the Common Fisheries Policy according to the MSY principle, making a fundamental objective of the restoration and maintenance of fish stocks to levels and conditions in which they are capable of providing maximum sustainable yields. Faced with the overoptimism of this principle, Holt (2007) claimed the necessity of redefining the MSY. Studying carefully the stochastic behavior of an even simple SPM can help to improve this policy.

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## A Appendix

## A. 1 Proofs of Theorems 2.1 and 2.2 and of Proposition 2.1

We use some of the notation from Meyn and Tweedie (1993, chapters 4 to 7), henceforth referred to as MT. Our stochastic Schaefer model (SSM) is a typical example of a scalar nonlinear state-space model whose associated deterministic control model $\mathrm{CM}(F)$ can be written as

$$
F_{k}\left(z_{0}, \epsilon_{1}, \ldots, \epsilon_{k}\right)=F\left(F_{k-1}\left(z_{0}, \epsilon_{1}, \ldots, \epsilon_{k-1}\right), \epsilon_{k}\right)
$$

with $F(z, \epsilon)=F_{\gamma}(z, \epsilon)=(1+\gamma) z(1-z) \epsilon$. We denote the state space $\mathbf{Z}=\{z\}$ and let $\AA$ represent the interior of a set $A$. We put $O_{\epsilon}=\operatorname{Supp}\left(f_{\epsilon}\right)$ and denote the set of all states reachable from $z$ at time $k$ by $\mathrm{CM}(F)$ as $A_{+}^{0}(z)=z$ and for $k \geq 1$,

$$
A_{+}^{k}(z)=\left\{F_{k}\left(z, \epsilon_{1}, \ldots, \epsilon_{k}\right): \epsilon_{i} \in O_{\epsilon}, 1 \leq i \geq k\right\}
$$

Proof of Theorem 2.1. The Markov chain cannot be irreducible if $\{0,1\} \in \mathbf{Z}$ : no state except 0 can be reached after 0 or 1 . Then we will suppose for simplicity that $\mathbf{Z} \subset \mathbb{R}-\{0,1\}$. Now we use some results about irreducibility. For all $z \in \mathbf{Z}$ and $A \in \mathcal{B}(\mathbf{Z})$ denote the Markovian kernel of the chain $P(z, A)=P\left(Z_{t+1} \in A \mid Z_{t}=z\right)$. Then

$$
P(z, A)=\frac{1}{\lambda(z)} \int_{A} f_{\epsilon}\left(\frac{\epsilon}{\lambda(z)}\right) d \epsilon
$$

where $\lambda(z)=(1+\gamma) z(1-z)$. The chain is $\psi$-irreducible if there exists a measure $\psi$ on $\mathcal{B}(\mathbf{Z})$ such that for all $z \in \mathbf{Z}$, whenever $\psi(A)>0$, there exists some $n \in \mathbb{N}_{+}$(possibly depending on both $A$ and $z$ ) such that $P^{n}(z, A)>0$ where, for any $0 \leq m \leq n$,

$$
P^{n}(z, A)=\int_{\mathbf{Z}} P^{m}(z, d y) P^{n-m}(y, A)
$$

Since

$$
\begin{equation*}
P^{n}(z, A)=\lambda^{-1}(z) \underbrace{\int_{\mathbf{z}} \ldots \int_{\mathbf{z}}}_{n-1} Q_{\lambda, \mathbf{y}} R_{\lambda, \mathbf{y}} d y_{1} \ldots d y_{n-1} \tag{A.1}
\end{equation*}
$$

with $R_{\lambda, \mathbf{y}}=\int_{A} f_{\epsilon}\left(\frac{y_{n}}{\lambda\left(y_{n-1}\right)}\right)$ and

$$
Q_{\lambda, \mathbf{y}}=\frac{f_{\epsilon}\left(\frac{y_{1}}{\lambda(z)}\right)}{\lambda\left(y_{n-1}\right)} \prod_{i=1}^{n-2} \lambda^{-1}\left(y_{i}\right) f_{\epsilon}\left(\frac{y_{i+1}}{\lambda\left(y_{i}\right)}\right),
$$

a necessary condition of Lebesgue irreducibility is having $\lambda(z)>0 \forall z \in \mathbf{Z}$, which means $\mathbf{Z} \subset] 0,1[$. Then, for all $z \in \mathbf{Z}$, we have $P(z] 0,,1[)=1$. It leads to

$$
\begin{equation*}
\int_{0}^{\frac{4}{1+\gamma}} f_{\epsilon}(\epsilon) d \epsilon+\int_{\frac{4}{1+\gamma}}^{\frac{(1+\gamma)^{-1}}{z(1-z)}} f_{\epsilon}(\epsilon) d \epsilon=1 \tag{A.2}
\end{equation*}
$$

since $z(1-z) \leq 1 / 4$ in $] 0,1[$. Because this equation is true for all $z \in \mathbf{Z} \subset] 0,1[$, it is especially true for $z=1 / 2$. Thus the second term of the left-hand side of (A.2) becomes zero. Since $f_{\epsilon}$ is a density, we have necessarily $\operatorname{Supp}(f) \subset\left[0,4(1+\gamma)^{-1}\right]$. Reciprocally, the conditions of Theorem 2.1 insure a) that if $\left.z_{0} \in\right] 0,1\left[\right.$ then $\left.A_{+}\left(z_{0}\right) \subset\right] 0,1[; \mathrm{b})$ that integral terms in (A.1) are all finite and strictly positive, and $0<\lambda(z)<(1+\gamma)^{2} / 4$. Thus the conditions are sufficient for irreducibility.

To show aperiodicity we use arguments proposed in MT (sections 7.2.1 and 7.3). If (B) of Theorem 2.1 is satisfied, since we assume a continuous $f_{\epsilon}$, then there exists a unique (topologically) closed minimal set $M$ for the deterministic control model $\mathrm{CM}(F)$ such that $M \subset A_{+}(M)$ (Theorem 7.2.6). Moreover, $O_{\epsilon}$ and $M$ are connected. Since the chain is irreducible and necessarily forward accessible (MT p.151), it is aperiodic from Theorem 7.3.2.

Proof of Theorem 2.2. First assume $\epsilon_{t}$ follows a product-of-beta distribution with parameters $(\alpha, \beta)$. Elementary calculations show that the constraint $\mathrm{E}\left(\epsilon_{t}\right)=1$ is respected and that the process error variance is given by

$$
\begin{align*}
\operatorname{Var}\left(\epsilon_{t}\right) & =\frac{2 \beta^{2}+\beta(\alpha+3)-(\alpha+1)^{2}}{(\beta+1)(\alpha+\beta+1)^{2}} \\
& =\frac{2+\beta(2-\gamma)(1+\gamma)}{\beta(\beta+1)(1+\gamma)^{2}} \tag{A.3}
\end{align*}
$$

if $(\alpha+1) / \beta=\gamma$. Now suppose that $\phi, r$ (and thus $\gamma$ ) and $\sigma^{2}=\operatorname{Var}\left(\epsilon_{t}\right)$ are fixed. We can then easily get an expression for $\beta$ by solving (A.3). Thus, a) can be proved.

Second, we prove the Harris recurrence of the chain, which means that the chain converges to a unique stationary distribution for every starting point $\left.z_{0} \in\right] 0,1\left[\right.$. Third we show that the $\mathcal{B}_{e}(\alpha, \beta)$ distribution is invariant for the chain. Necessarily, it is the stationary distribution.
a) Harris recurrence. We note easily that $O_{\epsilon}$ is the interior of $\left(0,4(1+\gamma)^{-1}\right)$ and $f_{\epsilon}$ is non-zero on $O_{\epsilon}$. Thus we satisfy (B) and the chain is irreducible and aperiodic. Now we prove a so-called drift condition to obtain the Harris recurrence. For a given $b<\infty$ denote

$$
f\left(z_{t}\right)=\mathrm{E}\left[Z_{t+1} \mid z_{t}\right]-z_{t}+1-b I_{\left\{z_{t} \in\right] 0,1[ \}} .
$$

Since $\mathrm{E}\left[\epsilon_{t}\right]=1$, we have

$$
f\left(z_{t}\right)=\gamma z_{t}-(1+\gamma) z_{t}^{2}+1-b I_{\left\{z_{t} \in\right] 0,1[ \}} .
$$

This function takes its maximum value in $] 0,1\left[\right.$ in $z_{t}^{*}=\gamma(1+\gamma)^{-1} / 2$ and

$$
f\left(z_{t}^{*}\right)=1+\frac{\gamma^{2}}{4(1+\gamma)}-b
$$

We will now use the fact that if a Markov chain has a $\psi$-irreducible kernel taking its values in a separable space, any set with positive measure $\psi$ admits a petite set (Neveu 1972). If we choose $b \geq 1+\gamma^{2} /(4(1+\gamma))$, then, for any petite subset $C$ of $\mathcal{B}(\mathbf{Z})$ we satisfy the drift condition

$$
\mathrm{E}\left[V\left(Z_{t+1}\right) \mid z_{t}\right] \leq V\left(z_{t}\right)-1+b \mathbb{1}_{\left\{z_{t} \in C\right\}},
$$

where $V: \mathbf{Z} \rightarrow[0, \infty)$ is the identity function which is everywhere finite and bounded in $C$. From Theorem 11.3.4 in MT, it is enough to prove the Harris recurrence of process $Z_{t}$.
b) Invariance of the $\mathcal{B}_{e}(\alpha, \beta)$ distribution. Let $Z_{t} \sim \mathcal{B}_{e}(\alpha, \beta)$ and suppose that innovations satisfy (6), then we prove that $Z_{t+1}$ given by (5) follows the same $\mathcal{B}_{e}(\alpha, \beta)$ distribution. We prove this by showing that all the moments of the two sides of (5) coincide. Indeed, a bounded distribution is uniquely determined by its moments (Shobat and Tamarkin 1943, p.11). The $n$th moment of a $\mathcal{B}_{e}(\alpha, \beta)$ is

$$
\mathrm{E}\left(Z_{t}^{n}\right)=\frac{\Gamma(\alpha+\beta) \Gamma(\alpha+n)}{\Gamma(\alpha+\beta+n) \Gamma(\alpha)} .
$$

In addition

$$
\mathrm{E}\left\{Z_{t}^{n}\left(1-Z_{t}\right)^{n}\right\}=\frac{\Gamma(\alpha+\beta) \Gamma(\beta+n) \Gamma(\alpha+n)}{\Gamma(\alpha+\beta+2 n) \Gamma(\alpha) \Gamma(\beta)} .
$$

From (6),

$$
\mathrm{E}\left\{(1-\phi+r)^{n} \epsilon_{t}^{n}\right\}=4^{n} \frac{\Gamma(\alpha+\beta) \Gamma\left(\frac{\beta+\alpha+1}{2}+n\right)}{\Gamma(\alpha+\beta+n) \Gamma\left(\frac{\beta+\alpha+1}{2}\right)} \frac{\Gamma(\beta) \Gamma\left(\frac{\beta+\alpha}{2}+n\right)}{\Gamma(\beta+n) \Gamma\left(\frac{\beta+\alpha}{2}\right)} .
$$

The duplication formula for the gamma function gives $\Gamma(2 z)=2^{2 z-1} \Gamma(z) \Gamma(z+1 / 2) / \Gamma(1 / 2)$. Thus

$$
\mathrm{E}\left\{(1-\phi+r)^{n} \epsilon_{t}^{n}\right\}=\frac{\Gamma(\beta+\alpha+2 n)}{\Gamma(\alpha+\beta+n)} \frac{\Gamma(\beta)}{\Gamma(\beta+n)},
$$

and $\mathrm{E}\left\{(1-\phi+r)^{n} \epsilon_{t}^{n}\right\} \times \mathrm{E}\left\{Z_{t}^{n}\left(1-Z_{t}\right)^{n}\right\}$ gives the moments of a beta random variable. Finally, there are conditions on $(\gamma, \sigma)$ such that parameters of the considered Beta distributions are all strictly positive. Moreover, we must have $\beta>\alpha>0$ and $\alpha+\beta>1$. It leads to the sufficient conditions

$$
0<\gamma \leq 1 \quad \text { and } \quad \beta>\gamma^{-1}
$$

From (8), we obtain $\sigma^{2}<1-\frac{1+\gamma+2 \gamma^{3}}{(1+\gamma)^{3}}$.

Proof of Proposition 2.1. Let $\Psi$ be the digamma function $(\Psi(x)=d \log \Gamma(x) / d x)$. Assume $Z \sim \mathcal{B}_{e}(\lambda, \nu)$. We have

$$
\begin{equation*}
E[\log Z]=\Psi(\lambda)-\Psi(\lambda+\nu) \tag{A.4}
\end{equation*}
$$

To prove this statement, denote $\mathcal{G}(\lambda, \theta)$ the gamma distribution with mean $\lambda \theta$ and variance $\lambda \theta^{2}$. Then one can write $\log Z=\log X-\log (X+Y)$ where $(X, Y)$ are independent and $X \sim \mathcal{G}(\lambda, \theta)$ and $Y \sim \mathcal{G}(\nu, \theta)$ with $0<\theta<\infty$. Then $X+Y \sim \mathcal{G}(\lambda+\nu, \theta)$. Since $\mathrm{E}[\log X]=\Psi(\lambda)-\log \theta$ (see for instance Penny 2001), we obtain (A.4). Then, from (6),

$$
\mathrm{E}\left[\log \epsilon_{t}\right]=\log \frac{4}{1+\gamma}+\Psi\left(\frac{\alpha+\beta+1}{2}\right)-\Psi(\alpha+\beta)+\Psi\left(\frac{\alpha+\beta}{2}\right)-\Psi(\beta) .
$$

The duplication formula for the gamma function yields the following formula for the digamma function,

$$
\Psi(2 x)=\frac{1}{2}(\Psi(x)+\Psi(x+1 / 2))+\log 2 .
$$

Then, replacing $x$ by $(\alpha+\beta) / 2$, we obtain

$$
\mathrm{E}\left[\log \epsilon_{t}\right]=-\log (1+\gamma)+\Psi(\alpha+\beta)-\Psi(\beta)
$$

then

$$
V=(1+\gamma) \exp \{\Psi(\beta)-\Psi(\alpha+\beta)\}-1
$$

Because $\Psi(x)$ is a strictly increasing function on $\mathbb{R}_{+}, \Psi(\beta)-\Psi(\alpha+\beta)<0$ for strictly positive $(\alpha, \beta)$. Then

$$
V<\gamma
$$

and (9) is automatically verified when $(\alpha, \beta)$ are well defined, assuming (7).

## B Tables and figures

|  | Deterministic case | Stochastic case |
| :--- | :--- | :--- |
| $M S Y(\mathrm{kt})$ | $12.9(3.2)$ | $9.326(3.6)$ |
| $B_{M S Y}(\mathrm{kt})$ | $64.5(19.9)$ | $53.85(22.5)$ |
| Market price $(\$ / \mathrm{kg})$ | $2.5262(0.07)$ | $2.6048(0.08)$ |
| Expected revenue $\left(10^{6} \$\right)$ | $32.58(0.22)$ | $24.29(0.28)$ |

Table B. 1
Bioeconomic landmarks and expected revenue per year after 1993. Standard deviations are given in parentheses.

| $r$ | 0.330 |  |
| :--- | :--- | :--- |
| $K$ | $240.2(\mathrm{kt})$ |  |
| $\sigma_{\nu}$ | 0.0632 |  |
| Deterministic MSY | $19.50(\mathrm{kt})$ | $(14.0,24.2)$ |
| Stochastic MSY | $19.18(\mathrm{kt})$ | $(13.8,23.9)$ |
| Deterministic $B_{M S Y}$ | $120.1(\mathrm{kt})$ | $(65.8,186.1)$ |
| Stochastic $B_{M S Y}$ | $118.6(\mathrm{kt})$ | $(61.2,178.4)$ |

Table B. 2
Posterior means and $95 \%$ credibility intervals (2.5th and 97.5 th percentiles, between parentheses) of the stochastic Schaefer parameters from Millar and Meyer (2000). Estimations are corrected to get an innovation variance of 1 .


Figure B.1. Plots of the upper limit (7) for $\sigma$ as a function of $\gamma=r-\phi$.


Figure B.3. Plots of the ratio $M S Y(\sigma) / M S Y(0)$ as a function of $\sigma$, for some representative values of $r$ and $\sigma \in\left(0, \sigma_{M}\right)$,see (7). $M S Y(0)$ is the deterministic yield $r K / 4$. Quantities are calculated using a higher order approximation than (11).


Figure B.2. Plots of the upper limit (7) for $\sigma$ as a function of $r$, assuming $\phi=\phi_{\text {opt }}$ (Eq. 10).


Figure B.4. Plots of the ratio $B_{M S Y}(\sigma) / B_{M S Y}(0)$ as a function of $\sigma$, for some representative values of $r$ and $\sigma \in\left(0, \sigma_{M}\right)$ ,see (7). $B_{M S Y}(0)$ corresponds to the deterministic maximally exploited biomass $K / 2$. Plots are drawn using a higher order approximation than (12).


Figure B.5. Plots of the ratio of mean catches over the deterministic maximum sustainable yield $\phi \mathrm{E}\left[B_{t}\right] / M S Y(0)$ as a function of the relative fishing effort $\phi / r$, for $r=0.1$ and some values of $\sigma$. There are no catches for large ratio $\phi / r$ and values of $\sigma$ larger than (7). A non-optimized $\phi$ can induce a severe shortfall especially when $\sigma$ is large.


Figure B.6. Plots of the ratio $\phi_{o p t}(\sigma) / \phi_{o p t}(0)$ as a function of $\sigma$, for some representative values of $r$. $\phi_{\text {opt }}(0)$ is the deterministic fraction $r / 2$. Two definitions of the stochastic $\phi_{\text {opt }}$ are used: the standard $\phi_{\text {opt }}$ (10) which corresponds to the fishing rate of the MSY, and the $\phi_{\text {opt }}$ calculated under the constraint $P\left(B_{t}<0.2 K \mid \phi=\phi_{\text {opt }}\right)=0.05$.


Figure B.7. Plots of the ratios $M S Y(\sigma) / M S Y(0) \quad$ and $\quad M Y C(\sigma) / M S Y(0)$ as functions of $\sigma$, for some representative values of $r$. $\operatorname{MSY}(0)$ is the deterministic yield $r K / 4$. The stochastic $M S Y(\sigma)$ refers to (11). $M Y C(\sigma)$ is defined as the maximal yield under the constraint $P\left(B_{t}<0.2 K \mid \phi=\phi_{\text {opt }}\right)=0.05$.


Figure B.8. Plots of the mean of the Mann-Whitney p-values averaged on 2,000 time steps as a function of $\sigma$ and $r$.


Figure B.9. Plots of the standard deviation of the Mann-Whitney p-values averaged on 2,000 time steps as a function of $\sigma$ and $r$.


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