Modelling Catastrophes and their Impact on Insurance Portfolios

Abstract

In this paper we propose a general individual catastrophe risk model that allows damage ratios to be random functions of the catastrophe intensity. We derive some distributional properties of the insured risks and of the aggregate catastrophic loss under this model. Through the model and ruin probability calculations, we give a formal illustration of the well known fact that the catastrophe risk cannot be diversified through premium collection alone, as is the case with the usual "day-to-day" risk, even for an arbitrary large portfolio. We also derive some risk orderings between different catastrophe portfolios, and show that the risk level of a realistic portfolio falls between that of a portfolio of comonotonic risks and that of a portfolio of independent risks. Finally, we illustrate our findings with a numerical example inspired from earthquake insurance.

1 Introduction

Earthquakes, droughts, floods, hurricanes, winter storms and tornado outbreaks are among the natural catastrophes that can produce large amounts of losses. For instance

> "The U.S. has sustained 49 weather-related disasters over the past 22 years in which overall damages/costs reached or exceeded \$1 billion. 42 of these disasters occurred during the 1988-2001 period with total damages/costs exceeding \$185 billion."[1]

An important proportion of these losses are insured losses, and this tendency should persist as there are more items at risk in the catastrophe-prone areas, a larger proportion of these risks are getting insured and the value of the items insured increases. In health insurance, the rise of health care related costs and of the density of populations in urban areas increase the potential for a costly epidemic.

Unfortunately, the evaluation of the probability distribution of the losses following from such disasters can be quite difficult, and simple actuarial models and methods are usually not appropriate for such calculations. One key element missing in the more traditional models is the intrinsic dependency between the risks exposed. For example, one hurricane will cause several correlated claims at the same time, and risks that are geographically close to each other are likely to produce highly correlated claim amounts.

In this paper we aim to fulfil several objectives. Our first goal is to propose individual catastrophe risk models that will be general and realistic, yet tractable. We do so by generalizing the classical individual risk model through a mathematical formalization of the catastrophe computer simulation models that can be found in the actuarial literature. We

[1]National Climatic Data Center, NOAA, "Billion Dollar U.S. Weather Disasters 1980-2001", June 27, 2001.

proceed in three steps: (i) we add catastrophic loss random variables to the individual risk model (ii) we add dependence between the catastrophic claim amounts by making them deterministic functions of the catastrophe intensity (iii) we make the model more realistic by letting the catastrophic loss amounts be random functions of the catastrophe intensity.

Our second objective is to examine the behavior of the aggregate catastrophic loss for a portfolio under these three individual catastrophe models, and to model the impact of the presence of catastrophes on an insurance portfolio. We tackle this problem by calculating ruin probabilities and by deriving stochastic orderings under the three catastrophe models. Interesting consequences of these calculations are that the usual methods of evaluation of the distribution of the aggregate claim amounts used in risk theory still apply in the context of catastrophe models, and we are able to illustrate in a formal manner the well known fact that the catastrophic risk cannot be diversified through premium collection alone, as is the case with a portfolio of independent risks.

The paper is organized as follows. Section 2 presents a review of the current developments in the research areas related to catastrophe insurance. The three sections that follow Section 2 contain the main contributions of the paper. We construct a general and realistic individual catastrophe risk model in three steps in Section 3. Ruin probability calculations and results on the non diversifiability of the catastrophe risk are obtained in Section 4. Section 5 compares the risk portfolios from Section 3 through risk measures and stochastic ordering. In Section 6, a numerical example based on earthquake insurance is given to illustrate the models, methods and results presented in sections 3 to 5. We conclude with a discussion in Section 7.

2 Review of the current developments in the area

Since the 1990's, various papers have considered dependency between risks. Among them, Dhaene and Goovaerts (1996) and Bäuerle and Müller (1998) study different orderings between two portfolios of dependent risks in the individual risk model. Based on the concept of comonotonicity (Dhaene et al, 2002ab), Wang and Dhaene (1998) find the riskiest stop-loss premiums. Denuit et al. (1999) construct stochastic bounds on sums of dependent risks. Wang (1998) (see also the discussion by Meyers, 1999) suggests a set of tools for modelling and combining correlated risk portfolios. Most methods proposed by Wang (1998) are also applicable to the collective risk model.

Another approach to the impact of catastrophes on insurance business is through extreme value theory. See Beirlant and Teugels (1992), Beirlant et al. (1996), Embrechts et al. (1997), McNeil (1997), Rootzén and Tajvidi (1997), Reiss and Thomas (1997), Resnick (1997). Their approach is mainly concerned with the effect of possible catastrophes on the probability distribution of the aggregate loss random variable.

Finally, a third approach considers modelling the loss portfolio at the individual risk level. This includes stochastic modelling, such as Brillinger (1993) who examines the development of appropriate premium rates in the case of a natural disaster, such as an earthquake, through temporal and spatial stochastic modelling of the frequency and intensity of earthquakes in a given region. Modelling is also done by directly simulating the effect of catastrophes on a portfolio with computer based models. As a matter of fact, earthquake and hurricane simulation models have been the tools of choice for actuaries who wanted to adapt their ratemaking methods to take the risk of catastrophes into account. Because of the importance of losses due to natural catastrophes in the last decade (more particularly hurricanes), there has already been a good amount of work done in the actuarial literature, and the Forums of the Casualty Actuarial Society on ratemaking have produced some very interesting proposals

(Clark, 1986, Burger et al., 1996, Walters and Morin, 1996, Chernick, 1998). Basically, these papers propose methods to assess the effect of catastrophes on an insurance portfolio through simulation methods based on recent meteorological or geological models. These methods represent a great improvement over the more traditional methods based on short-term loss data, as they are a better reflection of the mechanism that causes the claims and they make better use of the recent meteorological, demographic and engineering developments and data. Moreover, their implementation is relatively straightforward with the easy access to computational power available today.

In this paper, we propose a mathematical formalization of the computer-based models. We propose models that are suitable for several types of catastrophe insurance (natural catastrophes, epidemics, etc.) and that can be interpreted under both a macro- and a micro-perspective. (By macro-perspective we refer to approaches used in pricing catastrophe bonds, i.e., that we have information on the number of catastrophes and the total amount of losses by catastrophe, while the micro-perspective refers to modelling risks at the individual loss level.) The methods proposed in this paper can thus be viewed as a "compromise" between the stochastic approach of Brillinger (1993) and the computer-based approach from the Forums of the Casualty Actuarial Society. While the latter two approaches use catastrophe models especially for premium calculations, we further use catastrophe models to make formal statements on risk diversifiability and risk ordering.

3 Modelling catastrophe risks

In this section, we consider a group of n insurance contracts in a specific geographic area. We assume that these contracts are exposed to one specific catastrophe risk such as hurricanes, earthquakes or floods. Our approach consists in modelling the risks individually; this can be related to the so-called individual risk model which is presented in Panjer and Willmot (1992), Klugman et al. (1998) or Rolski et al. (1999), for example. We define the models over a short period of time (say, 1, 3, 6 or 12 months). We first give a general formulation of the portfolio of insured risks, then we consider three different models for the financial losses caused by catastrophes. While the general portfolio representation and the first two models for the catastrophic losses are not new, the third model for catastrophic losses is; we present all three models to better illustrate the financial impact of the risk of catastrophes later on.

3.1 Building a portfolio of risks

The total losses over a given fixed period (e.g. 1 year) for the ith contract are represented by the r.v. X_i^{TOT} with

$$X_i^{TOT} = X_i^{UR} + X_i^{CAT}, \qquad i = 1, 2, ..., n.$$

The r.v. X_i^{CAT} corresponds to the losses due to the catastrophe risk and the r.v. X_i^{UR} represents all other losses due to "usual risks". We define the r.v.'s X_i^{UR} and X_i^{CAT} as

$$X_i^{UR} = \begin{cases} \sum_{k=1}^{M_i} B_{i,k}, & M_i > 0 \\ 0, & M_i = 0 \end{cases}, \quad i = 1, \ldots, n$$

and

$$X_i^{CAT} = \begin{cases} \sum_{k=1}^{M_0} C_{i,k}, & M_0 > 0 \\ 0, & M_0 = 0 \end{cases}, \quad i = 1, \ldots, n,$$

where

- M_i is the number of "usual risk" losses for the contract i over one period;
- M_0 is the number of catastrophes in the specific area over one period;
- $B_{i,k}$ is the kth "usual risk" claim amount for contract i;

- $C_{i,k}$ is the kth catastrophe loss amount for contract i.

We make the following assumptions:

A1 For a given contract i, X_i^{UR} and X_i^{CAT} are independent;

A2 For a given contract i, $B_{i,1}, B_{i,2}, ...$ are independent and identically distributed (i.i.d.) and independent of M_i;

A3 For a given contract i, $C_{i,1}, C_{i,2}, ...$ are i.i.d. and independent of M_0;

A4 X_i^{UR} and $X_{i'}^{UR}$ are independent, $i \neq i'$.

For the "usual risk" part of the portfolio, the model proposed above amounts to the traditional individual risk model. For the catastrophic part of the portfolio, however, we get a more flexible model. Clearly, the r.v.'s $X_1^{CAT}, ..., X_n^{CAT}$ are not independent since they are all a function of M_0, the number of occurrences of the catastrophe over one period. Moreover, for a given catastrophe, $C_{1k}, \ldots, C_{nk}$ are *not* necessarily independent. It follows that $X_1^{TOT}, ..., X_n^{TOT}$ are not independent either.

We define the total (aggregate) financial losses for the whole portfolio of n contracts as

$$S_n^{TOT} = \sum_{i=1}^{n} X_i^{TOT} = \sum_{i=1}^{n} X_i^{UR} + \sum_{i=1}^{n} X_i^{CAT} = S_n^{UR} + S_n^{CAT}.$$

In this paper, our focus will be on the aggregate amount of the catastrophic claims, S_n^{CAT}, and modelling of S_n^{UR} will not be addressed. We thus consider models for

$$S_n^{CAT} = X_1^{CAT} + X_2^{CAT} + ... + X_n^{CAT}. \tag{1}$$

We can rewrite (1) as follows

$$S_n^{CAT} = \begin{cases} C_1^{CAT} + ... + C_n^{CAT}, & M_0 = 1 \\ 0, & M_0 = 0. \end{cases} \tag{2}$$

An equivalent representation of S_n^{CAT} is given by

$$S_n^{CAT} = \begin{cases} D_n^{CAT}, & M_0 = 1 \\ 0, & M_0 = 0, \end{cases} \tag{3}$$

where $D_n^{CAT} = C_1^{CAT} + ... + C_n^{CAT}$ represents the total amount of losses due to a catastrophe. The representation in (2) and (3) clearly shows that S_n^{CAT} can be seen as a single risk, i.e., if a catastrophe occurs, then one large financial loss for the company occurs.

We therefore have two different perspectives in which to approach the modelling of S_n^{CAT}. The first, given by equation (1) is a "micro-perspective". In the pricing of individual insurance contracts, we need to use this "micro-perspective" approach, as we have to model the distribution of the individual catastrophic loss r.v.'s $C_1^{CAT}, ..., C_n^{CAT}$. This is the approach used when one simulates the effect of catastrophes on insurance portfolios using computer catastrophe models (e.g., Walters and Morin, 1996 or Chernick, 1998).

The second scale, given by equation (3), is a "macro-perspective" scale. When this macro-perspective approach is used, the distribution of the r.v. D_n^{CAT} is directly modelled. Such an approach is taken when the data on total losses by catastrophe (such as PCS index, etc.) are available. In the pricing of catastrophe bonds or other financial catastrophe insurance derivatives, we are interested in directly modelling the distribution of M_0 and D_n^{CAT} (see e.g. Schmock (1999), Harrington and Niehaus (1999), Christensen and Schmidli (2000), Cox and Pedersen (2000)).

Basic properties of the distribution of S_n^{CAT} can be derived. From (1), we deduce

$$\begin{aligned} E\left[S_n^{CAT}\right] &= E\left[X_1^{CAT} + X_2^{CAT} + ... + X_n^{CAT}\right] = \sum_{i=1}^{n} E\left[X_i^{CAT}\right] \\ &= \sum_{i=1}^{n} E\left[M_0\right] E\left[C_i^{CAT}\right]. \end{aligned}$$

According to (2) and (3), we also have

$$E\left[S_n^{CAT}\right] = E_{M_0}\left[E\left[S_n^{CAT}\middle| M_0\right]\right] = E_{M_0}\left[M_0 \times E\left[D_n^{CAT}\right]\right]$$

$$
\begin{aligned}
&= E\left[M_0\right] E\left[D_n^{CAT}\right] = E\left[M_0\right] E\left[C_1^{CAT} + ... + C_n^{CAT}\right] \\
&= E\left[M_0\right] \sum_{i=1}^{n} E\left[C_i^{CAT}\right].
\end{aligned}
$$

From (1), the variance of S_n^{CAT} is given by

$$
\begin{aligned}
Var\left(S_n^{CAT}\right) &= \sum_{i=1}^{n} Var\left(X_i^{CAT}\right) \\
&\quad + \sum_{i=1}^{n} \sum_{\substack{i'=1 \\ i' \neq i}}^{n} Cov\left(X_i^{CAT}, X_{i'}^{CAT}\right).
\end{aligned}
$$

We can also deduce the variance of S_n^{CAT} from (2) and (3)

$$
\begin{aligned}
Var\left(S_n^{CAT}\right) &= E\left[M_0\right] Var\left(D_n^{CAT}\right) + Var\left(M_0\right) E\left[D_n^{CAT}\right]^2 \\
&= E\left[M_0\right] Var\left(C_1^{CAT} + ... + C_n^{CAT}\right) \\
&\quad + Var\left(M_0\right) E\left[C_1^{CAT} + ... + C_n^{CAT}\right]^2.
\end{aligned}
$$

The cumulative distribution function of S_n^{CAT} is obtained from (3):

$$
\begin{aligned}
F_{S_n^{CAT}}(x) &= \Pr\left(M_0 = 0\right) + \Pr\left(M_0 = 1\right) F_{D_n^{CAT}}(x) \\
&= (1-q) + q F_{C_1^{CAT} + ... + C_n^{CAT}}(x), \quad x \geq 0.
\end{aligned}
$$

The stop-loss premium, which is defined as

$$
\pi_{S_n^{CAT}}(d) = E\left[\left(S_n^{CAT} - d\right)_+\right] = qE\left[\left(D_n^{CAT} - d\right)_+\right],
$$

is the pure premium for a stop-loss reinsurance contract with a given retention level $d \geq 0$. Finally, from (3), the moment generating function (m.g.f.) of S_n^{CAT} is given by

$$
\phi_{S_n^{CAT}}(t) = \phi_{M_0}\left\{\ln\left(\phi_{D_n^{CAT}}(t)\right)\right\} = 1 - q + q\left\{\left(\phi_{D_n^{CAT}}(t)\right)\right\}. \tag{4}
$$

3.2 Models for the individual catastrophic claims

We will consider three different models for the financial losses due a catastrophe. As we shall see, the third approach is the more realistic one; it is a mathematical formalization of some of the computer simulation models used for catastrophe insurance pricing (e.g., Walters and Morin, 1996 or Chernick, 1998). The first two approaches can be considered as opposite extremes with respect to the level of dependence between the catastrophic claim amounts: the first approach assumes independence between these amounts, while the second approach assumes complete dependence. In order to simplify the presentation, we assume that only one catastrophe can occur in a specific area over a year. The models can be easily adapted to the case where more than one catastrophe can occur but, in most practical applications, the probability of more than one catastrophe in a region in a year is negligible.

In each model j ($j = 1, 2, 3$), the r.v. $X_i^{CAT(j)}$ represents the costs related to the catastrophe protection, which is defined by

$$X_i^{CAT(j)} = \begin{cases} C_i^{CAT(j)}, & M_0 = 1 \\ 0, & M_0 = 0, \end{cases}$$

where M_0 is a Bernoulli r.v. with mean q and $C_i^{CAT(j)}$ represents the financial losses for the contract i if a catastrophe occurs. We assume that the financial loss $C_i^{CAT(j)}$ is expressed as a proportion of the property value, i.e.

$$C_i^{CAT(j)} = U_i^{CAT(j)} \times b_i$$

where b_i is the value of the property i and $U_i^{CAT(j)} \in [0, 1]$ is called the loss proportion or the damage ratio. (In subsections 3.2.1 to 3.2.3, we give a different formulation of $U_i^{CAT(j)}$ for each model j ($j = 1, 2, 3$).) We can thus write $X_i^{CAT(j)}$ as $X_i^{CAT(j)} = b_i \times Y_i^{CAT(j)}$, with

$$Y_i^{CAT(j)} = \begin{cases} U_i^{CAT(j)}, & M_0 = 1 \\ 0, & M_0 = 0 \end{cases}.$$

The expectation and the variance of $X_i^{CAT(j)}$ are

$$E\left[X_i^{CAT(j)}\right] = E\left[M_0\right] E\left[C_i^{CAT(j)}\right] = b_i E\left[M_0\right] E\left[U_i^{CAT(j)}\right],$$

and

$$\begin{aligned} Var\left(X_i^{CAT(j)}\right) &= E\left[M_0\right] Var\left(C_i^{CAT(j)}\right) + Var\left(M_0\right) E\left[C_i^{CAT(j)}\right]^2 \\ &= b_i^2 \times \left\{E\left[M_0\right] Var\left(U_i^{CAT(j)}\right) + Var\left(M_0\right) E\left[U_i^{CAT(j)}\right]^2\right\}, \end{aligned}$$

for $i = 1, 2, ..., n$. The m.g.f of $X_i^{CAT(j)}$ is given by

$$\phi_{X_i^{CAT(j)}}\left(t\right) = \phi_{M_0}\left(\ln\left\{\phi_{C_i^{CAT(j)}}\left(t\right)\right\}\right),$$

where ϕ_{M_0} denotes the m.g.f. of M_0. We also have that

$$Cov\left(X_i^{CAT(j)}, X_{i'}^{CAT(j)}\right) = E\left[X_i^{CAT(j)} X_{i'}^{CAT(j)}\right] - E\left[X_i^{CAT(j)}\right] E\left[X_{i'}^{CAT(j)}\right],$$

for $i \neq i' \in \{1, 2, ..., n\}$. First,

$$E\left[X_i^{CAT(j)} X_{i'}^{CAT(j)}\right] = E_{M_0}\left[E\left[\left. X_i^{CAT(j)} X_{i'}^{CAT(j)}\right| M_0\right]\right]$$

with

$$E\left[\left. X_i^{CAT(j)} X_{i'}^{CAT(j)}\right| M_0 = 1\right] = E\left[C_i^{CAT(j)} C_{i'}^{CAT(j)}\right] = b_i b_{i'} E\left[U_i^{CAT(j)} U_{i'}^{CAT(j)}\right]$$

and

$$E\left[\left. X_i^{CAT(j)} X_{i'}^{CAT(j)}\right| M_0 = 0\right] = 0.$$

Then,

$$\begin{aligned} E\left[X_i^{CAT(j)} X_{i'}^{CAT(j)}\right] &= E_{M_0}\left[M_0 \times E\left[C_i^{CAT(j)} C_{i'}^{CAT(j)}\right]\right] \\ &= b_i b_{i'} E\left[M_0\right] E\left[U_i^{CAT(j)} U_{i'}^{CAT(j)}\right]. \end{aligned}$$

It follows that

$$Cov\left(X_i^{CAT(j)}, X_{i'}^{CAT(j)}\right) = b_i b_{i'} \left\{E\left[M_0\right] E\left[U_i^{CAT(j)} U_{i'}^{CAT(j)}\right] - E\left[M_0\right]^2 E\left[U_i^{CAT(j)}\right] E\left[U_{i'}^{CAT(j)}\right]\right\},$$

for $i \neq i' \in \{1, 2, ..., n\}$.

Note that under this specification of $X_i^{CAT(j)}$, the stop-loss premium for $S_n^{CAT(j)}$ becomes

$$\pi_{S_n^{CAT(j)}}(d) = q b_{TOT} E\left[\left(V_n^{CAT(j)} - \frac{d}{b_{TOT}}\right)_+\right],$$

where $b_{TOT} = b_1 + ... + b_n$ corresponds to the total exposure in insured property value of an insurance company in a given area and

$$V_n^{CAT(j)} = U_1^{CAT(j)} \frac{b_1}{b_{TOT}} + ... + U_n^{CAT(j)} \frac{b_n}{b_{TOT}}$$

can be seen as an aggregate measure of the damage ratio for the whole portfolio.

3.2.1 Model with independent damage ratios

In the first model ($j = 1$), the r.v.'s $U_i^{CAT(1)}$, $i = 1, \dots, n$ that represent the damage ratios are assumed independent. This implies that $Cov\left(U_i^{CAT(j)}, U_{i'}^{CAT(j)}\right) = 0$ and

$$Cov\left(C_i^{CAT(j)}, C_{i'}^{CAT(j)}\right) = b_i b_{i'} Cov\left(U_i^{CAT(j)}, U_{i'}^{CAT(j)}\right) = 0$$

for $i \neq i' \in \{1, 2, ..., n\}$. It follows that

$$E\left[X_i^{CAT(j)} X_{i'}^{CAT(j)}\right] = b_i b_{i'} E\left[M_0\right] E\left[U_i^{CAT(j)}\right] E\left[U_{i'}^{CAT(j)}\right]$$

and

$$\begin{aligned} Cov\left(X_i^{CAT(j)}, X_{i'}^{CAT(j)}\right) &= b_i b_{i'} \left\{E\left[M_0\right] E\left[U_i^{CAT(j)} U_{i'}^{CAT(j)}\right]\right. \\ &\quad \left. - E\left[M_0\right]^2 E\left[U_i^{CAT(j)}\right] E\left[U_{i'}^{CAT(j)}\right]\right\} \\ &= b_i b_{i'} \left\{E\left[M_0\right] E\left[U_i^{CAT(j)}\right] E\left[U_{i'}^{CAT(j)}\right]\right. \\ &\quad \left. - E\left[M_0\right]^2 E\left[U_i^{CAT(j)}\right] E\left[U_{i'}^{CAT(j)}\right]\right\} \\ &= b_i b_{i'} E\left[U_i^{CAT(j)}\right] E\left[U_{i'}^{CAT(j)}\right] \left\{E\left[M_0\right] - E\left[M_0\right]^2\right\}, \end{aligned}$$

for $i \neq i' \in \{1, 2, ..., n\}$. Also, $D_n^{CAT(1)} = C_1^{CAT(1)} + ... + C_n^{CAT(1)}$ corresponds to a sum of independent r.v's. Therefore, the m.g.f. of $S_n^{CAT(1)}$ in (4) becomes

$$\begin{aligned}
\phi_{S_n^{CAT(1)}}(t) &= \phi_{M_0}\left\{\ln\left(\phi_{D_n^{CAT(1)}}(t)\right)\right\} = \phi_{M_0}\left\{\ln\left(\prod_{i=1}^{n}\phi_{C_i^{CAT(1)}}(t)\right)\right\} \\
&= \phi_{M_0}\left\{\sum_{i=1}^{n}\ln\left(\phi_{C_i^{CAT(1)}}(t)\right)\right\}
\end{aligned}$$

Obviously this model is not completely realistic since by assuming that the damage ratios are independent, they are not influenced by the intensity of a catastrophe. In the following subsection, a first attempt is made to take the intensity of the catastrophe into account.

3.2.2 Damage ratios as deterministic functions of catastrophe intensity

In this second model ($j = 2$), the damage ratio $U_i^{CAT(2)}$ is a deterministic function of a r.v. I that represents the intensity of the catastrophe felt in the specific geographic area of the n insured risks of the portfolio, i.e. $U_i^{CAT(2)} = \psi_i(I)$, where $\psi_i : \ \Omega \to [0, 1]$, with Ω being the range of I. We suppose that the r.v. I has c.d.f. F_I and is independent of M_0. In reality, the intensity may be a function of several random factors relating to a catastrophe. The definition of $\psi_i(\cdot)$ depends on the characteristics of the covered property i (e.g. type of the building structure), but usually ψ_i is a positive, increasing function of the catastrophe intensity. This implies that if two properties i and i' have the same characteristics, then $\psi_i = \psi_{i'}$ and thus $P[U_i^{CAT(2)}(x) = U_{i'}^{CAT(2)}(x)|I = x] = 1$.

We have that $E\left[U_i^{CAT(2)}U_{i'}^{CAT(2)}\right] = E_I[\psi_i(I)\psi_{i'}(I)]$ or $i \neq i' \in \{1, 2, ..., n\}$. It follows that

$$Cov\left(C_i^{CAT(2)}, C_{i'}^{CAT(2)}\right) = b_i b_{i'} E[M_0]\left\{E_I[\psi_i(I)\psi_{i'}(I)] - E_I[\psi_i(I)]E_I[\psi_{i'}(I)]\right\}.$$

The c.d.f. of $S_n^{CAT(2)}$ is

$$F_{S_n^{CAT(2)}}(x) = \Pr(M_0 = 0) + \Pr(M_0 = 1) F_{D_n^{CAT(2)}}(x)$$

$$= \quad (1-q) + qF_I(\psi(x)),$$

where $\psi(x) = \{y: \sum_{i=1}^n b_i\psi_i(y) = x\}$. It follows that the m.g.f. of $S_n^{CAT(2)}$ is

$$\begin{aligned} \phi_{S_n^{CAT(2)}}(t) &= \phi_{M_0}\left\{\ln\left(\phi_{D_n^{CAT(2)}}(t)\right)\right\} = \phi_{M_0}\left\{\ln E_I\left[e^{t\sum_{i=1}^n b_i\psi_i(I)}\right]\right\} \\ &= (1-q) + q\left\{\int_\Omega \left(e^{t\sum_{i=1}^n b_i\psi_i(\theta)}\right) dF_I(\theta)\right\}. \end{aligned}$$

This model is still not quite realistic, as two properties with the same characteristics are unlikely to incur the exact same damage ratio upon occurrence of a catastrophe.

3.2.3 Damage ratios as random functions of catastrophe intensity

For the third approach ($j = 3$), the proportion $U_i^{CAT(3)}$ is not a deterministic function of the catastrophe intensity anymore, but rather a r.v. whose distribution is conditional on the intensity of the catastrophe. Of course, this conditional distribution will also depend on the characteristics of the risk insured. For example, the ATC-13 report (Applied Technology Council, 1985) gives probability mass functions for the damage ratios as a function of building type and earthquake intensity on the modified Mercali scale. Mathematically, this can be written as

$$P[U_i^{CAT(3)} = u|I = x] = p_{iux}, \ u \in \{u_1, \ldots, u_k\}, \ x \in \Omega, \ \sum_u p_{iux} = 1.$$

We let p_{iux} depend on i to make explicit the fact that these probabilities will depend on the characteristics (e.g., building type, age) of the ith risk. Note that the conditional distribution of $U_i^{CAT(3)}$ need not be discrete; Cossette et al. (2002) give a similar model with a continuous conditional beta distribution for the damage ratios caused by wind. Obviously, two properties with the same characteristics will not necessarily incur the same damage ratio, i.e., $P[U_i^{CAT(3)}(x) = U_{i'}^{CAT(3)}(x)|I = x]$ is not necessarily equal to 1 for two properties i and i' with identical characteristics, as was the case with the second model.

We make the assumption that conditional on I, $U_1^{CAT(3)}, \ldots, U_n^{CAT(3)}$ are independent. We then have

$$\begin{aligned} E\left[U_i^{CAT(3)} U_{i'}^{CAT(3)}\right] &= E_I\left[E\left[U_i^{CAT(3)} U_{i'}^{CAT(3)} | I\right]\right] \\ &= E_I\left[E\left[U_i^{CAT(3)} | I\right] E\left[U_{i'}^{CAT(3)} | I\right]\right] \end{aligned}$$

for $i \neq i' \in \{1, 2, ..., n\}$. It follows that

$$\begin{aligned} Cov\left(C_i^{CAT(3)}, C_{i'}^{CAT(3)}\right) &= b_i b_{i'} E\left[M_0\right] E_I\left[E\left[U_i^{CAT(3)} | I\right] E\left[U_{i'}^{CAT(3)} | I\right]\right] \\ &\quad - b_i b_{i'} E\left[M_0\right]^2 E\left[U_i^{CAT(3)}\right] E\left[U_{i'}^{CAT(3)}\right], \end{aligned}$$

for $i \neq i' \in \{1, 2, ..., n\}$. The c.d.f. of $S_n^{CAT(3)}$ is

$$\begin{aligned} F_{S_n^{CAT(3)}}(x) &= \Pr\left(M_0 = 0\right) + \Pr\left(M_0 = 1\right) F_{D_n^{CAT(3)}}(x) \\ &= (1-q) + q F_{C_1^{CAT(3)} + \ldots + C_n^{CAT(3)}}(x) \\ &= (1-q) + q \int_\Omega F_{C_1^{CAT(3)} + \ldots + C_n^{CAT(3)} | I = \theta}(x)\, dF_I(\theta). \end{aligned}$$

It follows that the m.g.f. of $S_n^{CAT(3)}$ is

$$\begin{aligned} \phi_{S_n^{CAT(3)}}(t) &= \phi_{M_0}\left\{\ln\left(\phi_{D_n^{CAT(3)}}(t)\right)\right\} = \phi_{M_0}\left\{\ln E_I\left[\prod_{i=1}^n \phi_{C_i^{CAT(3)}|I}(t)\right]\right\} \\ &= (1-q) + q\left\{\ln \int_\Omega \left(\prod_{i=1}^n \phi_{C_i^{CAT(3)}|I}(t)\right) dF_I(\theta)\right\}. \end{aligned}$$

The third model is the most realistic of the three models introduced in this section. Under this model, the damage ratio is influenced by the intensity of the catastrophe (which was not the case with the first model), but is still allowed to fluctuate from property to property even if these properties share the same characteristics (which was not the case with the second model). Now that we can model the catastrophe risk, we can evaluate its impact on the financial risk posed by an insurance portfolio.

4 Portfolio risk management

We consider the global risk of a portfolio of an insurance company. An approach based on risk measures will be presented in Section 5. In this section, we first give an argument that looking at individual premiums will not detect the risk induced by catastrophes in a portfolio. We then examine the behavior of the aggregate financial losses as the number of contracts within the portfolio increases, and illustrate how the catastrophic risk cannot be diversified by increasing the size of the portfolio.

4.1 Individual premiums and ruin probability

Let $\pi_i^{CAT(j)} = \pi\left(X_i^{CAT(j)}\right)$ denote the loaded premium associated to catastrophe coverages of contract i under model j ($j = 1, 2, 3$). We exclude expenses and profit components from the premium calculations. We assume that the $\pi_i^{CAT(j)}$'s are computed under separate premium principles. (For a survey of the premium calculation principles, see e.g. Gerber (1979), Daykin et al. (1994) or Rolski et al. (1999).) Generally, premium calculations are presented in the context of coverages excluding catastrophe risks (see CAS (1996) for a survey on the computation of such premiums). In this section, we apply these principles to the computation of $\pi_i^{CAT(j)}$.

The simplest principle is the pure premium principle where $\pi_i^{CAT(j)} = E\left[X_i^{CAT(j)}\right]$ for $i = 1, 2, ..., n$. More generally, the loaded premium is greater than the pure premium and the difference is called the safety margin (or safety loading): $\gamma_i^{CAT(j)} = \pi_i^{CAT(j)} - E\left[X_i^{CAT(j)}\right]$ where $\gamma_i^{CAT(j)}$ are assumed positive. The relative safety margin $\eta_i^{CAT(j)}$ is defined by

$$\eta_i^{CAT(j)} = \frac{\gamma_i^{CAT(j)}}{E\left[X_i^{CAT(j)}\right]}.$$

Among the premium calculation principles, we find

- the expected value principle: $\pi_i^{CAT(j)} = \left(1 + \alpha_i^{CAT(j)}\right) E\left[X_i^{CAT(j)}\right]$, with $\alpha_i^{CAT(j)} > 0$;
- the variance principle: $\pi_i^{CAT(j)} = E\left[X_i^{CAT(j)}\right] + \alpha_i^{CAT(j)} Var\left[X_i^{CAT(j)}\right]$, with $\alpha_i^{CAT(j)} > 0$;
- the standard deviation principle: $\pi_i^{CAT(j)} = E\left[X_i^{CAT(j)}\right] + \alpha_i^{CAT(j)} \sqrt{Var\left[X_i^{CAT(j)}\right]}$, with $\alpha_i^c > 0$.

It is worth pointing out that any of these premium principles will yield identical premiums for each of the three risk models considered in Section 3. This is due to the fact that the difference between the risk models is not in the marginal distributions of each insured risk but in their joint distribution. Hence, the above premium principles fail to detect the difference between the three models. We must therefore consider the stochastic behavior of the aggregate financial losses. We do so through ruin probability calculations, which formally illustrates the well known fact that independent risks are diversifiable, whereas catastrophic risks are not.

In this paper, by *ruin probability* we mean the probability that the insurance company does not meet its financial commitments over a fixed period of time (e.g., the next year). Note that we do not take into account any risk reserve or special allocation from the surplus (or capital) in our definition of the ruin probability.

One question of interest is the behavior of the ruin probability as the number of insured contracts within the portfolio of the insurance company increases. We assume that the first two moments of any claim amount random variables X_i are finite and strictly positive. These assumptions are reasonable in practice since amounts insured on individual contracts are finite. To simplify the presentation, we also assume that the relative safety margins in the premiums are equal for all contracts of the portfolio.

4.2 Diversification under independent risks

We show in Proposition 2 that if the relative safety loading η is positive, the probability of not having enough money to cover the total claims tends to 0 as the number of risks tends to infinity, whereas Proposition 1 shows that this same probability converges to 1 when the safety loading is negative. In both propositions, we consider a sequence of positive independent individual contract loss amount random variables $Y_1, \ldots, Y_n$. We assume that the first two moments are finite and strictly positive, i.e., there exist real numbers a_1, a_2, b_1, b_2 such that $0 < a_1 \leq E[Y_i] \leq a_2$ and $0 < b_1 \leq Var(Y_i) \leq b_2$. Let T_n be the aggregate financial loss random variable, defined as $T_n = Y_1 + ... + Y_n$, with

$$E[T_n] = \sum_{i=1}^{n} E[Y_i] \qquad (5)$$

$$Var(T_n) = \sum_{i=1}^{n} Var(Y_i). \qquad (6)$$

Define the ruin probability ζ_n for the portfolio of n contracts as

$$\zeta_n = \Pr\left(T_n > \sum_{i=1}^{n} \pi_i\right),$$

where $\pi = E[(1+\eta)Y_i]$ is the premium for the ith contract.

Proposition 1 *If the safety margin $\eta > 0$, then the ruin probability ζ_n goes to 0 when $n \to \infty$.*

Proposition 2 *If the safety margin $\eta < 0$, then the ruin probability ζ_n goes to 1 when $n \to \infty$.*

The previous two propositions confirm the common knowledge that independent risks are diversifiable, as long as the number of risks is large and the premium charged for each risk is superior to its expected value. The proofs of all the propositions of Section 4 are relegated to Appendix A.

4.3 Diversification under catastrophic risks

In the next three propositions, we consider a sequence of positive catastrophic individual contract loss amount random variables $X_1^{CAT(j)}, \ldots, X_n^{CAT(j)}$, $j = 1, 2, 3$. We assume that the first two moments are finite and strictly positive, i.e., there exist real numbers a_1, a_2, b_1, b_2 such that $0 < a_1 \leq E\left[X_i^{CAT(j)}\right] \leq a_2$ and $0 < b_1 \leq Var\left(X_i^{CAT(j)}\right) \leq b_2$. Let $S_n^{CAT(j)}$ be the aggregate financial loss random variable, defined as $S_n^{CAT(j)} = X_1^{CAT(j)} + ... + S_n^{CAT(j)}$ with

$$E\left[S_n^{CAT(j)}\right] = \sum_{i=1}^{n} E\left[X_i^{CAT(j)}\right].$$

Define the ruin probability $\zeta_n^{CAT(j)}$ for the portfolio of n contracts as

$$\zeta_n^{CAT(j)} = \Pr\left(S_n^{CAT(j)} > \sum_{i=1}^{n} \pi_i^{CAT(j)}\right),$$

where $\pi_i^{CAT(j)} = (1+\eta)E\left[X_i^{CAT(j)}\right]$ is the premium for the catastrophe coverage for the ith contract. In the remainder of Section 4.3 we examine the behavior of $\zeta_n^{CAT(j)}$ for the catastrophe models with independent damage ratios ($j = 1$), with damage ratios as deterministic functions of catastrophe intensity ($j = 2$) and with damage ratios as random functions of catastrophe intensity ($j = 3$). This will provide a formal argument to support the fact that catastrophe risk cannot be diversified in the same fashion as the risk of usual "day-to-day" business. Indeed, we will prove that an insurance or reinsurance company cannot diversify the financial risk paused by catastrophes through premium income alone, even with an arbitrary large portfolio of insured risks.

4.3.1 Independent damage ratios

In the following proposition, we demonstrate that under the model with independent damage ratios, the ruin probability tends to q, the probability that a catastrophe occurs:

Proposition 3 *If* $0 < \eta < \left(\frac{E\left[C_i^{CAT(1)}\right]}{E\left[X_i^{CAT(1)}\right]} - 1 \right) = \frac{1}{q} - 1$*, then the ruin probability* $\zeta_n^{CAT(1)}$ *tends, when* $n \to \infty$*, to the probability that at least one catastrophe occurs, i.e.*

$$\lim_{n\to\infty} \zeta_n^{CAT(1)} = 1 - \Pr\left(M_0 = 0\right) = q.$$

Notice that the upper bound $\left(\frac{E\left[C_i^{CAT(1)}\right]}{E\left[X_i^{CAT(1)}\right]} - 1 \right)$ on the relative safety margin η corresponds to the case where the premium $\pi_i^{CAT(1)}$ is equal to $E\left[C_i^{CAT(1)}\right]$, the expected individual catastrophic loss given a catastrophe happens. This is a reasonable bound as it is hard to imagine an insurance company charging a premium greater than $E\left[C_i^{CAT(1)}\right]$. Within the context of this simple model, we have shown that the catastrophe risks cannot be fully diversified like the non-catastrophe risks can.

4.3.2 Damage ratios as deterministic functions of catastrophe intensity

Recall that the damage ratio r.v.'s $U_1^{CAT(2)}$, ...,$U_n^{CAT(2)}$ are positive increasing functions ψ_i of a r.v. I, the intensity of the catastrophe. We now show in this case that the ruin probability is greater than a strictly positive fraction of the probability that a catastrophe occurs.

Proposition 4 *If* $0 < \eta < \left(\frac{E\left[C_i^{CAT(2)}\right]}{E\left[X_i^{CAT(2)}\right]} - 1 \right) = \frac{1}{q} - 1$*, there exists a strictly positive real number* c *with* $0 < c \leq 1$*, such that*

$$\lim_{n\to\infty} \zeta_n^{CAT(2)} \geq q \times c > 0.$$

To illustrate the result of the previous proposition, consider a portfolio of property insurance where $b_i = b$ and the damage ratio r.v.'s $U_i^{CAT(2)}$ are identically distributed

$$U_i^{CAT(2)} \sim U^{CAT(2)},$$

which means that $\psi_i(I) = \psi(I)$ for $i = 1, 2, ..., n$. Then, for any n,

$$\begin{aligned}\zeta_n^{CAT(2)} = \Pr\left(S_n^{CAT(2)} > \sum_{i=1}^{n} \pi_i^{CAT(2)}\right) &= q \Pr\left(nb\psi(I) > (1+\eta) \times q \times nbE[\psi(I)]\right) \\ &= q \Pr\left(\psi(I) > (1+\eta) \times q \times E[\psi(I)]\right).\end{aligned}$$

For the positive and increasing function $\psi(u)$, define $\psi^{-1}()$ as its inverse. It leads to

$$\begin{aligned}\zeta_n^{CAT(2)} &= q \Pr\left(\psi(I) > (1+\eta) \times q \times E[\psi(I)]\right), \quad \forall n \\ &= q \Pr\left(I > \psi^{-1}\left((1+\eta) \times q \times E[\psi(I)]\right)\right), \quad \forall n\end{aligned}$$

where $(1+\eta) \times q \times E[\psi(I)] < E[\psi(I)]$ by assumption and $\Pr\left(I > \psi^{-1}\left((1+\eta) \times q \times E[\psi(I)]\right)\right)$ is strictly positive. Then, we obtain the ruin probability

$$\begin{aligned}\zeta_n^{CAT(2)} &= q \Pr\left(I > \psi^{-1}\left((1+\eta) \times q \times E[\psi(I)]\right)\right), \quad \forall n \\ &= qc, \quad \forall n,\end{aligned}$$

where $c = \Pr\left(I > \psi^{-1}\left((1+\eta) \times q \times E[\psi(I)]\right)\right)$ can be computed given the function ψ.

4.3.3 Damage ratios as random functions of catastrophe intensity

Recall that the damage ratio r.v.'s $U_1^{CAT(3)}$, ...,$U_n^{CAT(3)}$ are positive random functions ψ_i of a r.v. I, the intensity of the catastrophe. We now show in this case that the ruin probability tends to a strictly positive fraction of the probability that a catastrophe occurs.

Proposition 5 *If* $0 < \eta < \frac{E\left[C_i^{CAT(3)}\right]}{E\left[X_i^{CAT(3)}\right]} - 1 = \frac{1}{q} - 1$*, there exists a strictly positive real number* c *with* $0 < c \leq 1$*, such that*

$$\lim_{n\to\infty} \zeta_n^{CAT(3)} \to q \times c > 0.$$

4.3.4 Illustration

The results of Propositions 4 and 5 may feel counterintuitive at first, since the ruin probability under Model 2 is lower than the probability of ruin under Model 3. The following example helps to better understand the meaning of these propositions.

For simplicity, suppose that the catastrophe intensity is a discrete random variable that may take on one of 7 distinct values. The damage ratios are assumed discrete on $\{1/5, 2/5, \ldots, 5/5\}$ (respective probabilities 0.29, ...). Using the results from Appendix A, we can compute the c.d.f.'s of $S_n^{CAT(j)}$, $j = 1, 2, 3$. These c.d.f.'s are plotted for a portfolio of 150 risks, each of insured value 10, with a probability of catastrophe in the year equal to 0.1 in Figure 1. The model is such that $E[S_n^{CAT(j)}] = 70.6$ and $E[D_n^{CAT(j)}] = 706$, $j = 1, 2, 3$.

[Insert Figure 1 here ...]

The most obvious characteristic of the distribution of $S_n^{CAT(j)}$ is that it has a probability mass of 0.9 at 0, since the probability of no catastrophe is 0.9. In the independent damage ratio case, we see that the distribution of $S_n^{CAT(1)}$ given a catastrophe occurs is concentrated around $E[S_n^{CAT(1)}]$, as the law of large numbers would suggest. In the case of deterministic damage ratios, when a catastrophe occurs all damage ratios take on the same value and we get a c.d.f. for $S_n^{CAT(2)}$ with a stair case appearance. For instance if the damage ratios all take on the value 1/5, the expected value of $S_n^{CAT(2)}$ is 300, the point of the first jump in the stair case c.d.f. Note that given a catastrophe occurs, the distribution of $S_n^{CAT(2)}$ is a lot more spread out around $E[D_n^{CAT}] = 706$ than is the case with independent damage ratios. For random damage ratios, we see that the c.d.f. of $S_n^{CAT(3)}$ given a catastrophe occurs is concentrated around 7 points, each point corresponding to a possible value of the catastrophe intensity. This makes sense as given a catastrophe of a specific intensity, $S_n^{CAT(3)}$ amounts to a sum of independent random variables. For example, for the smallest of the 7 values of the catastrophe intensity, the expected value of $S_n^{CAT(3)}$ is 480, the center point of the first climb of the c.d.f. of $S_n^{CAT(3)}$. Notice that the spread of the distribution of $S_n^{CAT(3)}$ about $E[D_n^{CAT}]$ given a catastrophe is intermediate between that of the distributions of $S_n^{CAT(1)}$ and $S_n^{CAT(2)}$.

From the c.d.f.'s, we can calculate ruin probabilities for various values of the risk loading η. When $\eta = 200\%$, all three ruin probabilities $\zeta_n^{CAT(j)} = 0.10$, $j = 1, 2, 3$. When $\eta = 400\%$, $\zeta_n^{CAT(2)} = 0.074$ while $\zeta_n^{CAT(j)} = 0.10$, $j = 1, 3$. When $\eta = 600\%$, $\zeta_n^{CAT(3)} = 0.076$, $\zeta_n^{CAT(2)} = 0.074$ and $\zeta_n^{CAT(1)} = 0.10$. Hence, despite very large safety margins, the ruin probabilities remain positive.

The results derived in Section 4.3 strongly suggest that companies selling protection against the risk of catastrophes seek protection themselves, either through reinsurance or insurance derivatives (see e.g. Schmock (1999), Harrington and Niehaus (1999), Christensen and Schmidli (2000), Cox and Pedersen (2000), and Cox, Fairchild and Pedersen (2000)). However ruin probabilities alone cannot be used to assess the dangerousness of a portfolio. Indeed, the portfolio with deterministic damage ratios seems to be more risky given the spread of its c.d.f. in Figure 1, but its ruin probability is the lowest. In Section 5 we quantify the effect of the spread in the c.d.f.'s of $S_n^{CAT(j)}$ in terms of measures of the risk level of the portfolios.

5 Comparison between the catastrophe models

We now want to assess the riskiness of a portfolio when catastrophes are possible. We will do so by comparing the risk level of the realistic catastrophe model (third model) proposed in Section 3 to the risk level of portfolios based on the other two more extreme models (first model and second model) of Section 3. We will rank the risk levels of these portfolios using risk measures, to be reviewed in Section 5.1 for completeness. To make sure that the portfolios are comparable, we will construct the portfolios so that the n risks insured against catastrophes have the same marginal distributions in all three portfolios in Section 5.2. We use existing theory on risk ordering to derive theoretical orderings between the three portfolios in Section 5.3. We illustrate these orderings with numerical examples in Section 6.

5.1 Risk measures

In this section we look at means of assessing the effect of potential catastrophes on the riskiness of a portfolio. A risk measure is a system that allows us to quantify or compare risks (Wirch, 1999). To compare the risk levels of our catastrophe models, we will consider three risk measures used in risk management: the *Stop-Loss* premium, the *Value-at-Risk* (VaR) and the *Conditional Value at Risk* ($CVaR$) (also called conditional tail expected loss).

5.1.1 Coherent risk measures

Several different measures of riskiness have been proposed in the literature. Since different measures may lead to different risk orderings, it is preferable to restrict our class of potential risk measures to a set of risk measures that satisfy minimal requirements. Artzner et *al.*

(1998, 1999) and Wirch (1999) give five properties that are desirable for risk measures. They call risk measures that satisfy these properties *coherent risk measures.*

Definition 1 *A* coherent risk measure *has the following properties:*

- *Property 1: The risk measure must be limited above by the maximum possible net loss.*
- *Property 2: The risk measure must be subadditive.*
- *Property 3: The risk measure must be multiplicative by a scalar.*
- *Property 4: The risk measure must be independent of the size of possible gains.*
- *Property 5: The risk measure must be scalar additive.*

We examine separately three risk measures. While the $CVaR$ is coherent, the stop-loss premium and the VaR are not; we consider them nonetheless as they are widely used risk measures in practice.

5.1.2 Stop-Loss premium

The stop loss premium, defined as $\pi_S(d) = E\left[(S-d)_+\right]$, is the pure premium for a stop-loss reinsurance contract with a given retention level $d \geq 0$.

5.1.3 Value-at-Risk

The Value at Risk (VaR) is a popular risk measure in risk management and actuarial science. In the actuarial literature, it is also referred to as the maximal probable loss.

Definition 2 *The Value at Risk (VaR) with a confidence level α associated to the r.v. S is defined by*

$$VaR_\alpha(S) = \inf\left\{x \in R : F_S(x) \geq \alpha\right\},$$

where $0 \leq \alpha \leq 1$.

The VaR is a popular risk measure even though it is not a coherent risk measure. The properties 2 and 3 (Wirch, 1999) stated above are not satisfied by this risk measure. For further information on the VaR see e.g. Embrechts et al. (2002) and Hürlimann (2003).

5.1.4 Conditional Value-at-Risk

The Conditional Value-at-Risk, also called the conditional tail expectation, is a coherent risk measure proposed by Artzner et *al.* (1998, 1999) as an alternative to the VaR (see e.g.. Wirch (1999) and Hürlimann (2001)).

Definition 3 Let $0 \leq \alpha \leq 1$. The Conditional Value-at-risk $CVaR_\alpha(S)$ with a confidence level α associated to the r.v. S is defined by

$$CVaR_\alpha(S) = E\left[S \middle| S > VaR_\alpha(S)\right].$$

As mentioned e.g. in Hürlimann (2001), we have

$$\begin{aligned} CVaR_\alpha(S) &= E\left[S \middle| S > VaR_\alpha(S)\right] = \frac{\int_{VaR_\alpha(S)}^{\infty} s\, dF_S(s)}{\Pr\left(S > VaR_\alpha(S)\right)} \\ &= \frac{1}{\alpha} \int_{VaR_\alpha(S)}^{\infty} \left\{1 - F_S(s)\right\} ds + VaR_\alpha(S). \end{aligned}$$

A popular measure of the risk of death of an individual in demography and life actuarial science is the residual life expectancy

$$e_S(x) = E\left[S - x \middle| S > x\right] = \frac{\pi_S(x)}{1 - F_S(x)},$$

see e.g. Klugman et *al.* (1998) or Bowers et al. (1997). The $CVaR$ can be expressed as a function of the residual life expectancy and VaR

$$CVaR_\alpha(S) = e_{S_m}\left(VaR_\alpha(S)\right) + VaR_\alpha(S),$$

or, equivalently,

$$CVaR_{\alpha}\left(S\right) = \frac{\pi_S(VaR_{\alpha}(S))}{1 - F_S\left(VaR_{\alpha}(S)\right)} + VaR_{\alpha}(S) = \frac{\pi_S(VaR_{\alpha}(S))}{\alpha} + VaR_{\alpha}(S).$$

5.2 Construction of three comparable portfolios

We now construct three portfolios of risks, one portfolio coming from each of the three models presented in Section 3. To make the comparisons meaningful, we have to construct the portfolios so that the n random variables representing the damage ratios, $U_1^{CAT}, \ldots, U_n^{CAT}$, have the same marginal distributions in each portfolio.

The portfolios have the same structure, where we model the catastrophic claim amount for the ith risk, X_i^{CAT}, by

$$X_i^{CAT} = \begin{cases} b_i U_i^{CAT}, & M_0 = 1 \\ 0, & M_0 = 0, \end{cases}$$

where M_0 is a Bernoulli(q) random variable that takes on value 1 if, and only if, there is a catastrophe affecting the portfolio in the period of interest. We will construct the portfolios so that the joint distribution of $U_1^{CAT(j)}, \ldots, U_n^{CAT(j)}$ will be different for each portfolio. We present the portfolios in reverse order ($j = 3, 2, 1$) as it is simpler to construct portfolios with identical marginal distributions for the $U_i^{CAT(j)}$ this way.

5.2.1 Realistic portfolio

We start by constructing a portfolio that is a relatively realistic representation of a group of risks subject to potential catastrophes. This construction is based on the third model ($j = 3$) from Section 3.2.3. We first suppose that there is at most one catastrophe affecting the portfolio in the year with probability q. The indicator of such a catastrophe is a random variable M_0 having a Bernoulli distribution with mean q. Given a catastrophe, its intensity,

I, has a distribution with c.d.f. F_I on the positive real line. Conditional on an observed intensity $I = x$, we suppose that the damage ratios $U_i^{CAT(3)}$, $i = 1, \ldots, n$ are independent random variables, with $U_i^{CAT(3)}$ having conditional c.d.f. at u given by $F_{U_i^{CAT(3)}|I}(u|x)$. Thus, given M_0, $U_1^{CAT(3)}, \ldots, U_n^{CAT(j)}$ are *dependent* random variables with marginal c.d.f.'s $F_{U_1^{CAT(3)}}, \ldots, F_{U_n^{CAT(3)}}$, where $F_{U_i^{CAT(3)}}(u) = \int_0^\infty F_{U_i^{CAT(3)}|I}(u|x)dF_I(x)$, $i = 1, \ldots, n$.

5.2.2 Portfolio based on comonotonic damage ratios

We now construct a portfolio based on the second model ($j = 2$) of Section 3.2.2. We first define the property of *comonotonicity* (see e.g. Wand and Dhaene (1998), Wang (1998), Baüerle and Müller (1998), Denuit et al. (2002)).

Definition 4 *A vector of r.v., denoted* $\underline{Z}^{cm} = (Z_1^{cm}, ..., Z_n^{cm})$*, with marginals c.d.f.* F_{Z_i}*, is said to be comonotonic if one of the three following conditions is fulfilled:*

1. *The c.d.f. of* $\underline{Z}$ *is given by*

$$F_{\underline{Z}^{cm}}(\mathbf{x}) = \min(F_{Z_1}(x_1), ..., F_{Z_n}(x_n)), \underline{x} \in R^n;$$

2. *We have*

$$\mathbf{Z}^{cm} = (F_{Z_1}^{-1}(U), ..., F_{Z_n}^{-1}(U)),$$

where $U \sim Unif(0, 1)$*;*

3. *There exists a r.v.* V *and increasing functions* $f_1, ..., f_m$ *such that* $Z_i^{cm} = f_i(V)$ *for* $i = 1, 2, ..., n$.

We make our construction so that the damage ratios have the same marginal distributions as in the realistic portfolio, but are comonotonic, i.e. $U_1^{CAT(2)}, \ldots, U_n^{CAT(2)}$ can be written as

$\psi_1(I), \ldots, \psi_n(I)$ for some r.v. I and increasing functions $\psi_1, \ldots, \psi_n$. To do so, we simply let

$$U_i^{CAT(2)} = F^{-1}_{U_i^{CAT(3)}}\left(F_I\left(x\right)\right), \quad i = 1, \ldots, n, \tag{7}$$

where $F^{-1}_{U_i^{CAT(3)}}(u) = \inf\{y : F_{U_i^{CAT(3)}}(y) = u\}$ is the inverse of the c.d.f. of $U_i^{CAT(3)}$ and F_I is the c.d.f. of I, the catastrophe intensity. It is then easy to verify that under these assumptions, $U_1^{CAT(3)}, \ldots, U_n^{CAT(3)}$ and $U_1^{CAT(2)}, \ldots, U_n^{CAT(2)}$ are vectors of random variables with identical marginal distributions but with a different joint distribution. It is also obvious that the damage ratios as defined in (7) satisfy the definition of comonotonic random variables.

This portfolio with comonotonic proportions will be shown to be the riskiest of our portfolios. It is also a portfolio that leads to relatively simple calculations. It will therefore be a convenient upper bound for stop-loss premiums and other quantities of interest for realistic portfolios. Moreover, we will see in the numerical example of Section 6 that this upper bound is surprisingly tight.

5.2.3 Portfolio with independent damage ratios

We now construct a portfolio based on the first model ($j = 1$) of Section 3.2.1. In this construction the damage ratios again have the same marginal distributions as in the realistic portfolio, but this time we let these damage ratios be independent. This is simply done by letting $V_1, \ldots, V_n$ be independent uniform random variables on [0, 1], then by setting

$$U_i^{CAT(1)} = F^{-1}_{U_i^{CAT(1)}}(V_i), \quad i = 1, \ldots, n.$$

Again, one easily sees that we have independent random variables $U_1^{CAT(1)}, \ldots, U_n^{CAT(1)}$ with marginal c.d.f.'s given by $F_{U_1^{CAT(1)}}, \ldots, F_{U_n^{CAT(1)}}$, respectively.

5.3 Ordering of risks

We compare the three portfolios on the basis of stochastic orders and dependence orders. Ordering of risks is often applied, for example, in the establishment of stochastic bounds, and can also be used to rank portfolios according to their dangerousness.

5.3.1 Basic results

We first present two definitions of stochastic orders between univariate r.v.'s which is often used in actuarial science (for details, see e.g. Bäuerle et Müller (1998), Rolski et *al.* (1999) et Kaas et *al.* (2001)).

Definition 5 *Let X and X' be two r.v.'s such that $E[X] < \infty$ and $E[X'] < \infty$. Then, X precedes X' under stochastic dominance order, denoted $X \leq_{sd} X'$,if $F_X(x) \geq F_{X'}(x)$, for all $x \in R$.*

Definition 6 *Let X and X' be two r.v.'s such that $E[X] < \infty$ and $E[X'] < \infty$.Then, X precedes X' under stop-loss order, denoted $X \leq_{sl} X'$, if $E[(X-d)_+] \leq E[(X'-d)_+]$, for all $d \in R$, where $(u)_+ = \max(u,0)$.*

Consider now two vectors of r.v.'s $\underline{X} = (X_1, ..., X_n)$ and $\underline{X}' = \left(X_1', ..., X_n'\right)$ where, for each i, X_i and X_i' have the same marginal distribution (i.e. $X_i \sim X_i'$ for $i = 1, 2, ..., n$). Define also $S = X_1 + ... + X_n$ and $S' = X_1' + ... + X_n'$. Because of the assumptions on $\underline{X}$ and $\underline{X}'$, we have $E[S] = E[S']$.

When we want to compare $\underline{X}$ and $\underline{X}'$, we use dependence orders (see e.g. Shaked and Shantikumar (1994) and Joe (1997) for details on dependence orders). Then, based on a given relation between $\underline{X}$ and $\underline{X}'$, we can compare S and S'. If we want to establish that S precedes S' under stop-loss order, we first need to show that $\underline{X}$ precedes $\underline{X}'$ under the so-called supermodular order. The supermodular order is a dependence order which was introduced in the actuarial context by Müller (1997) and Baüerle and Müller (1998) then examined by e.g. Denuit et al. (2002). It allows the comparison of random vectors with the same marginals. We first need to define a supermodular function.

Definition 7 *A function $g : R^m \rightarrow R$ is said supermodular if*

$$\begin{aligned} & g(x_1, ..., x_i + \varepsilon, ..., x_j + \delta, ..., x_m) - g(x_1, ..., x_i + \varepsilon, ..., x_j, ..., x_m) \\ \geq \; & g(x_1, ..., x_i, ..., x_j + \delta, ..., x_m) - g(x_1, ..., x_i, ..., x_j, ..., x_m) \end{aligned}$$

is true for all $\mathbf{x} = (x_1, ..., x_m) \in R^m$, $1 \leq i \leq j \leq m$ *and all* $\varepsilon, \delta > 0$.

This definition is an extension to the notion of convexity for a function $d : R \rightarrow R$.

Proposition 6 *If g is twice differentiable, then g is supermodular if and only if*

$$\frac{\partial^2}{\partial x_i \partial x_j} g(\mathbf{x}) \geq 0,$$

for all $\mathbf{x} \in R^m$ *and* $1 \leq i \leq j \leq m$.

Proof: See e.g. Marshall et Olkin (1988) or Bäuerle et Müller (1998). □
For example, the functions $g(x_1, ..., x_m) = \sum_{i=1}^m x_i$, $g(x_1, ..., x_m) = \sum_{i=1}^m (x_i - d)^2$ and $g(x_1, ..., x_m) = \left(\sum_{i=1}^m x_i - d\right)_+$ are supermodular.

Let us define the supermodular order as presented in Bäuerle and Müller (1998):

Definition 8 *Let $\underline{X}$ and $\underline{X}'$ be two vectors of r.v.'s $\underline{X} = (X_1, ..., X_n)$ and $\underline{X}' = \left(X_1', ..., X_n'\right)$ where, for each i, X_i and X_i' have the same marginal distributions (i.e. $X_i \sim X_i'$ for $i = 1, 2, ..., n$). Then, $\underline{X}$ precedes $\underline{X}'$ under the supermodular order, denoted $\underline{X} \leq_{sm} \underline{X}'$, if $E\left[g(\underline{X})\right] \leq E\left[g(\underline{X}')\right]$ for all supermodular functions g, provided their expectations exist.*

We precise that the supermodular ordering can be applied when we compare two vectors of r.v.'s with same marginals. The supermodular ordering is used when we want to compare r.v.'s with different degrees of dependence. The relationship between supermodular ordering and comonotonicity is explored in Baürle and Müller (1998) and Goovaerts and Dhaene (1999). We now state the following important result:

Proposition 7 *If* $\underline{X} \leq_{sm} \underline{X}'$*,then* $S \leq_{sl} S'$*.*

Proof: See e.g. Baüerle and Müller (1998). □

The following proposition corresponds to Lorentz's Inequality.

Lemma 8 *Let* $\underline{X}$ *and* $\underline{X}^{cm}$ *be vectors of r.v.'s given by* $\underline{X} = (X_1, ..., X_n)$ *and* $\underline{X}^{cm} = (X_1^{cm}, ..., X_n^{cm})$ *where, for each* i*,* X_i *and* X_i^{cm} *have the same marginal distributions (*$X_i \sim X_i^{cm}$*, for* $i = 1, 2, ..., n$*), and where* $X_1^{cm}, \ldots, X_n^{cm}$ *are comonotonic. Then,* $\underline{X}$ *precedes* $\underline{X}^{cm}$ *under the supermodular order, denoted* $\underline{X} \leq_{sm} \underline{X}^{cm}$*.*

Proof: See e.g. Bäuerle and Müller (1998). □

According to previous proposition, the comonotonicity corresponds to the strongest dependent relation under supermodular ordering.

Lemma 9 *Let* $\underline{X}$ *and* $\underline{X}^{IND}$ *be vectors of r.v. given by* $\underline{X} = (X_1, ..., X_n)$ *and* $\underline{X}^{IND} = (X_1^{IND}, ..., X_n^{IND})$ *where, for each* i*,* X_i *and* X_i^{IND} *have the same marginal distributions (*$X_i \sim X_i^{IND}$*, for* $i = 1, 2, ..., n$*) and the components of* $\underline{X}^{IND}$ *are independent. We assume that the components of* $\underline{X}$ *are positively correlated. Then,* $\underline{X}^{IND}$ *precedes* $\underline{X}$ *under the supermodular order, denoted* $\underline{X}^{IND} \leq_{sm} \underline{X}$*.*

Proof: See e.g. Müller (1997) and Dhaene and Goovaerts (1996). □

5.3.2 Application to our context

We now apply the above results on the stochastic orders and the dependence orders to compare the catastrophe models.

Proposition 10 *We have* $\underline{U}^{CAT(1)} \leq_{sm} \underline{U}^{CAT(3)} \leq_{sm} \underline{U}^{CAT(2)}$.

Proof: Clearly, the components of

$$\underline{U}^{CAT(2)} = \left(U_1^{CAT(2)}, U_2^{CAT(2)}, ..., U_n^{CAT(2)}\right)$$

are comonotonic, since they are deterministic functions of the intensity of the catastrophe. The components of $\underline{U}^{CAT(3)}$ are defined by common mixture which implies that they are positively correlated. Then, we have $\underline{U}^{CAT(1)} \leq_{sm} \underline{U}^{CAT(2)}$ and $\underline{U}^{CAT(3)} \leq_{sm} \underline{U}^{CAT(2)}$ from lemmas 8 and 9. □

Proposition 11 *We have*

$$S^{CAT(1)} \leq_{sl} S^{CAT(3)} \leq_{sl} S^{CAT(2)}.$$

Proof: First, we have $\underline{B}^{CAT(1)} \leq_{sm} \underline{B}^{CAT(3)} \leq_{sm} \underline{B}^{CAT(2)}$, since $B_i^{CAT(j)} = b_i U_i^{CAT(j)}$ and because the supermodular order is preserved under scalar multiplication (see Bäuerle and Müller (1998)). Then, it follows that

$$D^{CAT(1)} \leq_{sl} D^{CAT(3)} \leq_{sl} D^{CAT(2)} \tag{8}$$

from Proposition 10. From Kaas et al. (2001, Chapter 10), (8) leads to $S^{CAT(1)} \leq_{sl} S^{CAT(3)} \leq_{sl} S^{CAT(2)}$.□

In the last proposition, we show that the third model for catastrophe risks, the more realistic one, is bounded below by the first model and bounded above by the second model. This implies that for any retention level $d \geq 0$, we have

$$\pi_{S^{CAT(1)}}(d) \leq \pi_{S^{CAT(3)}}(d) \leq \pi_{S^{CAT(2)}}(d). \tag{9}$$

Hürlimann (2003) shows that this further implies that

$$CVaR_\alpha\left(S^{CAT(1)}\right) \leq CVaR_\alpha\left(S^{CAT(3)}\right) \leq CVaR_\alpha\left(S^{CAT(2)}\right), \tag{10}$$

for $0 < \alpha < 1$. The relations (9) and (10) are illustrated in the numeral example of Section 6. These relations are useful when one wants to establish stochastic bounds for the distribution of the third model.

Unfortunately, we cannot conclude that $F_{S^{CAT(1)}}(x) \geq F_{S^{CAT(2)}}(x) > \pi_{S^{CAT(3)}}(x)$ for all $x \geq 0$, which would have implied that $VaR_\alpha\left(S^{CAT(1)}\right) \leq VaR_\alpha\left(S^{CAT(3)}\right) \leq VaR_\alpha\left(S^{CAT(2)}\right)$ for $0 < \alpha < 1$. Stochastic bounds on $VaR_\alpha\left(S^{CAT(2)}\right)$ can still be obtained, as is explained in Denuit et al. (1999).

6 Numerical illustration of the three risk models

We keep the construction of the three portfolios as described in Section 5. We consider a portfolio of 300 insured risks divided into three classes of 100 risks each, with all the risks in a given class having the same characteristics (say, same building type). Each class contains 25 risks of insured value 1, 25 risks of insured value 2, 25 risks of insured value 3 and 25 risks of insured value 4 (here one unit could represent, say, \$US100,000). Thus, the total insured value of this portfolio is $b_{TOT} = 750$. The catastrophe occurrence indicator M_0 is a Bernoulli random variable with mean $q = 0.2$ (this value seems somewhat excessive, but it is convenient for illustrative purposes).

Within the framework of the third model, we assume that, given a catastrophe occurs, the r.v. I, representing the catastrophe intensity, takes on a value in $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ with a mass probability function given by $\Pr(I = \theta_1) = 0.2$, $\Pr(I = \theta_2) = 0.4$, $\Pr(I = \theta_3) = 0.2$, $\Pr(I = \theta_4) = 0.15$ and $\Pr(I = \theta_5) = 0.05$.

The conditional mass probability of the damage ratio r.v.'s given $I = \theta_i$ are provided in Tables 1, 2 and 3. These conditional mass probabilities are inspired from the ATC-13 report (Applied Technology Council, 1985), which uses earthquake data from California to model the distribution of the damage ratios as a function of earthquake intensity. If the property covered by the insurance contract i belongs to building type j, we let

$$\Pr\left(U_i^{CAT(3)} = u \,|I = \theta_k\right) = p_{j,\theta_k}(u), \quad u \in \{0.1, 0.2, ..., 1.0\}, \; k = 1, 2, ..., 5.$$

[Insert Table 1 here...]

[Insert Table 2 here...]

[Insert Table 3 here...]

We derive the mass probability function (see Table 4) for $U_i^{CAT(3)}$ and, consequently, of $U_i^{CAT(1)}$ and $U_i^{CAT(2)}$ for the three types of buildings.

[Insert Table 4 here...]

In Table 5, we provide the expectation and the standard deviation of the damage ratios $U^{CAT(j)}$ $(j = 1, 2, 3)$ for the three types of building.

[Insert Table 5 here...]

In the context of the three models, the expected aggregate financial losses for the whole portfolio is 80.05, i.e. $E\left[S^{CAT(1)}\right] = E\left[S^{CAT(2)}\right] = E\left[S^{CAT(3)}\right] = 80.05$ and, given that a catastrophe occurs, the expected aggregate financial losses for the whole portfolio is 400.25, i.e.,

$$E\left[S^{CAT(j)}\,|M_0 = 1\right] = E\left[D^{CAT(1)}\right] = E\left[D^{CAT(2)}\right] = E\left[D^{CAT(3)}\right] = 400.25.$$

For each model, the standard deviations of the aggregate financial losses for the whole portfolio are $\sqrt{Var\left(S^{CAT(1)}\right)} = 160.17$, $\sqrt{Var\left(S^{CAT(2)}\right)} = 176.75$, and $\sqrt{Var\left(S^{CAT(3)}\right)} = 172.66$, and we also have $\sqrt{Var\left(D^{CAT(1)}\right)} = 358.11$, $\sqrt{Var\left(D^{CAT(2)}\right)} = 395.19$ and $\sqrt{Var\left(D^{CAT(3)}\right)} = 386.04$.

For the three models, we have provided in Tables 6 to 9 the c.d.f., the stop-loss premiums, the VaR and the $CVaR$, respectively, of total amount of financial losses for the portfolio. As we have shown in Section 5, we observe in Tables 7 and 8 that

$$\pi_{S^{CAT(1)}}\left(d\right) \leq \pi_{S^{CAT(3)}}\left(d\right) \leq \pi_{S^{CAT(2)}}\left(d\right)$$

for $d \in [0, 750]$ and

$$CVaR_{\alpha}\left(S^{CAT(1)}\right) \leq CVaR_{\alpha}\left(S^{CAT(3)}\right) \leq CVaR_{\alpha}\left(S^{CAT(2)}\right)$$

for $\alpha = 0.80, 0.85, 0.90, 0.95, 0.99$. These orderings are not observed for the c.d.f. and the VaR. However, it seems that after a given point, the c.d.f. and the VaR are ordered in the same way.

[Insert Table 6 here...]

[Insert Table 7 here...]

[Insert Table 8 here...]

[Insert Table 9 here...]

Suppose that the insurance company requires for each contract a loaded premium equal to 125% of the expectation of eventual losses covered by the contract. Then, the insurance company has a total amount of 100.04 that can be applied to finance the losses due to a catastrophe. This amount is far below the conditional expectation of the aggregate financial losses for the whole portfolio given that a catastrophe occurs, which is 400.25. In the context of the more realistic Model 3, the probability that the company will not have enough money to pay the losses is 20%, which is the assumed probability that a catastrophe occurs over the next year.

Let us also mention that the values given in the Tables 6 to 9 are exact and that no simulation methods have been used. It is also interesting to notice that the stop-loss premiums and the conditional values at risk obtained with Model 2 are safe (conservative) and close upper bounds to those calculated with the more realistic Model 3. The computation of stop-loss premiums and conditional VaR of a sum of comonotonic r.v.'s is relatively straightforward compared to the same computations under Model 3. The computation of stop-loss premiums for sums of comonotonic r.v.'s is treated in detail in, for example, Dhaene et al. (2002ab) and Kaas et al. (2001).

7 Conclusion

We have proposed a realistic individual catastrophe risk model. While realistic enough to include random dependence of damage ratios on catastrophe intensities, the model is sufficiently tractable to allow calculations of quantities such as premiums or limiting ruin probabilities, and the derivation of stochastic orderings. Furthermore, the model is flexible and general and can be applied to other types of insurance where "catastrophes" are possible (e.g., epidemics in health insurance, other natural catastrophes). We also derived some interesting results from this model, such as a formal illustration of the non diversifiability of the catastrophic risk.

Future work can be done in this area. In this paper we have modelled the risk over a single fixed time period. It would be interesting to extend the model proposed to a dynamic model in continuous time. To our knowledge a survey of the specific forms of the functions ψ

that link the catastrophe intensities to the damage ratios for the main types of catastrophes does not exist in a form ready to be used by actuaries. Such a survey would benefit both practitioners and researchers in actuarial science.

A Proofs of results from Section 5

Most of the proofs below require the use of Chebychev's inequality:

Lemma 12 *(Chebychev's inequality) Let U be a r.v. with mean $E(U)$ and variance $Var(U)$. Then $\forall k>0$, we have*

$$P\left(|U-E[U]|>k\sqrt{Var(U)}\right)\leq\frac{1}{k^{2}}.$$

A.1 Proof of Proposition 1:

We have

$$\begin{aligned}\zeta_{n} &= \Pr\left(T_{n}>\sum_{i=1}^{n}(1+\eta)E[Y_{i}]\right)\\ &= \Pr\left(T_{n}>(1+\eta)E[T_{n}]\right)\\ &= \Pr\left(T_{n}-E[T_{n}]>\eta E[T_{n}]\right)\\ &\leq \Pr\left(|T_{n}-E[T_{n}]|>\eta E[T_{n}]\right). \end{aligned}\tag{A.1}$$

Applying the Chebychev Inequality, (A.1) becomes

$$\begin{aligned}\zeta_{n} &\leq \Pr\left(|T_{n}-E[T_{n}]|>\eta E[T_{n}]\right)=\Pr\left(|T_{n}-E[T_{n}]|>\frac{\eta E[T_{n}]}{\sqrt{Var[T_{n}]}}\sqrt{Var[T_{n}]}\right)\\ &\leq \frac{\left(\sqrt{Var[T_{n}]}\right)^{2}}{(\eta E[T_{n}])^{2}}.\end{aligned}$$

From (5) and (6), we have

$$\zeta_n \leq \frac{\left(\sqrt{Var\left[T_n\right]}\right)^2}{\left(\eta E\left[T_n\right]\right)^2} = \frac{\sum_{i=1}^n Var\left[Y_i\right]}{\left(\eta \sum_{i=1}^n E\left[Y_i\right]\right)^2} \leq \frac{\sum_{i=1}^n b_2}{\left(\eta \sum_{i=1}^n a_1\right)^2} = \frac{n \times b_2}{\left(\eta \times n \times a_1\right)^2}.$$

Hence,

$$\lim_{n\to\infty}\zeta_n \leq \lim_{n\to\infty}\frac{n \times b_2}{\left(\eta \times n \times a_1\right)^2} \to 0,$$

implying that the ruin probability ζ_n goes to 0 when $n \to \infty$. $\square$

A.2 Proof of Proposition 2:

Let $\xi = |\eta| = -\eta$. We have

$$\begin{aligned}
\zeta_n &= \Pr\left(T_n > \left(1+\eta\right) E\left[T_n\right]\right) = \Pr\left(T_n - E\left[T_n\right] > -\xi E\left[T_n\right]\right) \\
&= 1 - \Pr\left(T_n - E\left[T_n\right] < -\xi E\left[T_n\right]\right) \\
&\geq 1 - \Pr\left(\left|T_n - E\left[T_n\right]\right| > \xi E\left[T_n\right]\right). \qquad \text{(A.2)}
\end{aligned}$$

Applying the Chebychev Inequality, (A.2) becomes

$$\begin{aligned}
\zeta_n &\geq 1 - \Pr\left(\left|T_n - E\left[T_n\right]\right| > \xi E\left[T_n\right]\right) \\
&= 1 - \Pr\left(\left|T_n - E\left[T_n\right]\right| > \frac{\xi E\left[T_n\right]}{\sqrt{Var\left[T_n\right]}}\sqrt{Var\left[T_n\right]}\right) \\
&\geq 1 - \frac{\left(\sqrt{Var\left[T_n\right]}\right)^2}{\left(\eta E\left[T_n\right]\right)^2}. \qquad \text{(A.3)}
\end{aligned}$$

Substituting (5) and (6) into (A.3) we get

$$\zeta_n \geq 1 - \frac{\left(\sqrt{Var\left[T_n\right]}\right)^2}{\left(\xi E\left[T_n\right]\right)^2} = 1 - \frac{\sum_{i=1}^n Var\left[Y_i\right]}{\left(\xi \sum_{i=1}^n E\left[Y_i\right]\right)^2} \geq 1 - \frac{\sum_{i=1}^n b_2}{\left(\xi \sum_{i=1}^n a_1\right)^2} = 1 - \frac{n \times b_2}{\left(\xi \times n \times a_1\right)^2}.$$

Then,

$$\lim_{n\to\infty}\zeta_n \geq 1 - \lim_{n\to\infty}\frac{n \times b_2}{\left(\eta \times n \times a_1\right)^2} \to 1$$

implying that the probability ζ_n goes to 1 when $n \to \infty$. $\square$

A.3 Proof of Proposition 3:

When we condition on M_0, the ruin probability $\zeta_n^{CAT(1)}$ can be written in those terms

$$\begin{aligned}
\zeta_n^{CAT(1)} &= \Pr\left(S_n^{CAT(1)} > \sum_{i=1}^{n} \pi_i^{CAT(1)}\right) \\
&= \sum_{k=0}^{1} Pr(M_0 = k) Pr\left(S_n^{CAT(1)} > \sum_{i=1}^{n} \pi_i^{CAT(1)} \middle| M_0 = k\right) \\
&= qPr\left(D_n^{CAT(1)} > \sum_{i=1}^{n} \pi_i^{CAT(1)}\right)
\end{aligned}$$

Then, we get

$$\lim_{n\to\infty} \zeta_n^{CAT(1)} = q \lim_{n\to\infty} Pr\left(D_n^{CAT(1)} > \sum_{i=1}^{n} \pi_i^{CAT(1)}\right)$$

Following Proposition 3, we have

$$\lim_{n\to\infty} Pr\left(D_n^{CAT(1)} > \sum_{i=1}^{n} \pi_i^{CAT(1)}\right) = 1$$

and consequently

$$\lim_{n\to\infty} \zeta_n^{c} = q \times 1 = q = 1 - \Pr\left(M = 0\right),$$

which is the desired result. □

A.4 Proof of Proposition 4:

We have

$$\begin{aligned}
\zeta_n^{CAT(2)} &= \Pr\left(S_n^{CAT(2)} > \sum_{i=1}^{n} \pi_i^{CAT(2)}\right) \\
&= \sum_{k=0}^{1} \Pr\left(M_0 = k\right) \Pr\left(S_n^{CAT(2)} > \sum_{i=1}^{n} \pi_i^{CAT(2)} | M = k\right) \\
&= q \Pr\left(S_n^{CAT(2)} > \sum_{i=1}^{n} \pi_i^{CAT(2)} | M = 1\right)
\end{aligned}$$

$$
\begin{aligned}
&= q\Pr\left(\sum_{i=1}^{n} b_i U_i^{CAT(2)} > (1+\eta)\times E\left[M_0\right]\times\sum_{i=1}^{n} b_i E\left[U_i^{CAT(2)}\right]\right)\\
&= q\Pr\left(\sum_{i=1}^{n} b_i \psi_i\left(I\right) > (1+\eta)\times E\left[M_0\right]\times\sum_{i=1}^{n} b_i E\left[\psi_i\left(I\right)\right]\right). \qquad \text{(A.4)}
\end{aligned}
$$

Since, under our assumptions on η, $(1+\eta)\,E\left[M_0\right] < 1$, it is clear that

$$
(1+\eta)\times E\left[M_0\right]\times\sum_{i=1}^{n} b_i E\left[\psi_i\left(I\right)\right] < \sum_{i=1}^{n} b_i E\left[\psi_i\left(I\right)\right].
$$

There exists a strictly positive number c such that

$$
\Pr\left(\sum_{i=1}^{n} b_i \psi_i\left(I\right) > (1+\eta)\times E\left[M_0\right]\times\sum_{i=1}^{n} b_i E\left[\psi_i\left(I\right)\right]\right) > c > 0,
$$

for all $n > 0$. Then (A.4) becomes $q \leq \zeta_n^{CAT(2)} \geq q \times c$. $\square$

A.5 Proof of Proposition 5:

We have

$$
\begin{aligned}
\zeta_n^{CAT(3)} &= \Pr\left(S_n^{CAT(3)} > \sum_{i=1}^{n} \pi_i^{CAT(3)}\right)\\
&= \sum_{k=0}^{1} \Pr\left(M_0 = k\right)\Pr\left(S_n^{CAT(3)} > \sum_{i=1}^{n} \pi_i^{CAT(3)} \,|M_0 = k\right)\\
&= q\Pr\left(D_n^{CAT(3)} > \sum_{i=1}^{n} \pi_i^{CAT(3)}\right).
\end{aligned}
$$

Then, we condition on the intensity of the catastrophe,

$$
\begin{aligned}
\zeta_n^{CAT(3)} &= q\Pr\left(D_n^{CAT(3)} > \sum_{i=1}^{n} \pi_i^{CAT(3)}\right)\\
&= q\int_{\theta\in\Omega} \Pr\left(\sum_{i=1}^{n} b_i U_i^{CAT(3)} > \sum_{i=1}^{n} \pi_i^{CAT(3)} \,|I = \theta\right) dF_I\left(\theta\right)\\
&= q\int_{\theta\in\Omega} \Pr\left(\sum_{i=1}^{n} b_i U_i^{CAT(3)} > (1+\eta)\times E\left[M_0\right]\times\sum_{i=1}^{n} b_i E\left[U_i^{CAT(3)}\right] |I = \theta\right) dF_I\left(\theta\right).
\end{aligned}
$$

There exists a $\theta_0 \in \Omega$ such that

$$(1+\eta) \times E[M_0] \times \sum_{i=1}^{n} b_i E\left[U_i^{CAT(3)}\right] < \sum_{i=1}^{n} b_i E\left[U_i^{CAT(3)} | I = \theta\right]$$

for all $\theta \geq \theta_0$. We recall that $\left(U_i^{CAT(3)} | I = \theta\right)$ are independent by assumption. From Propositions 2 and 3, we have

$$\lim_{n\to\infty} \Pr\left(\sum_{i=1}^{n} b_i U_i^{CAT(3)} > (1+\eta) \times E[M_0] \times \sum_{i=1}^{n} b_i E\left[U_i^{CAT(3)}\right] | I = \theta\right) \to 0$$

for $\theta < \theta_0$ and

$$\lim_{n\to\infty} \Pr\left(\sum_{i=1}^{n} b_i U_i^{CAT(3)} > (1+\eta) \times E[M_0] \times \sum_{i=1}^{n} b_i E\left[U_i^{CAT(3)}\right] | I = \theta\right) \to 1$$

for $\theta \geq \theta_0$. Then, the dominated convergence theorem allows us to pass the limit inside the integral, which gives

$$\begin{aligned}
\lim_{n\to\infty} \zeta_n^{CAT(3)} &= q \int_{\theta \in I} \left\{ \lim_{n\to\infty} \zeta_n^{CAT(3)} \Pr\left(\sum_{i=1}^{n} b_i U_i^{CAT(3)} > (1+\eta)\right.\right. \\
&\quad \left.\left. \times E[M_0] \times \sum_{i=1}^{n} b_i E\left[U_i^{CAT(3)}\right] | I = \theta\right)\right\} dF_I(\theta) \\
&= q\left(1 - F_I(\theta_0)\right)
\end{aligned}$$

and, letting $c = 1 - F_I(\theta_0)$, we obtain the desired result. □

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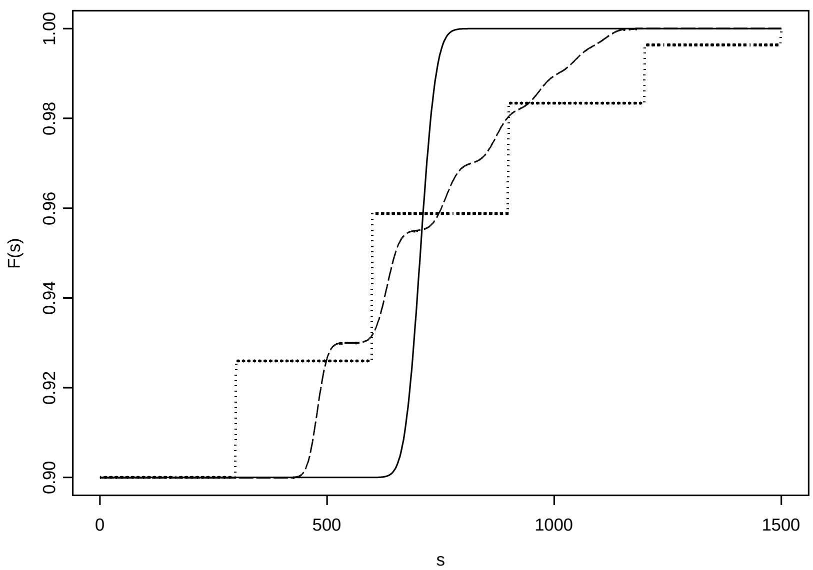

Figure 1: Cumulative distribution function of the aggregate catastrophic loss for the three models. Solid line: independent damage ratios ($j = 1$), dotted line: damage ratios deterministic ($j = 2$), dashed line: damage ratios random ($j = 3$)

	θ_1	θ_2	θ_3	θ_4	θ_5
u	$p_{1,\theta_1}(u)$	$p_{1,\theta_2}(u)$	$p_{1,\theta_3}(u)$	$p_{1,\theta_4}(u)$	$p_{1,\theta_5}(u)$
0.1	0.2	0	0	0	0
0.2	0.4	0.05	0	0	0
0.3	0.2	0.25	0	0	0
0.4	0.1	0.20	0.05	0	0
0.5	0.1	0.15	0.25	0.05	0
0.6	0	0.10	0.25	0.25	0
0.7	0	0	0.20	0.25	0.1
0.8	0	0	0.15	0.20	0.2
0.9	0	0	0.10	0.15	0.4
1.0	0	0	0	0.10	0.3

Table 1: Conditional probabilities of damage ratios for building type 1.

	θ_1	θ_2	θ_3	θ_4	θ_5
u	$p_{2,\theta_1}(u)$	$p_{2,\theta_2}(u)$	$p_{2,\theta_3}(u)$	$p_{2,\theta_4}(u)$	$p_{2,\theta_5}(u)$
0.1	0.3	0	0	0	0
0.2	0.5	0	0	0	0
0.3	0.2	0	0	0	0
0.4	0	0.3	0	0	0
0.5	0	0.4	0.3	0	0
0.6	0	0.2	0.4	0	0
0.7	0	0.1	0.2	0.1	0
0.8	0	0	0.1	0.2	0.1
0.9	0	0	0	0.4	0.5
1.0	0	0	0	0.3	0.4

Table 2: Conditional probabilities of damage ratios for building type 2.

	θ_1	θ_2	θ_3	θ_4	θ_5
u	$p_{3,\theta_1}(u)$	$p_{3,\theta_2}(u)$	$p_{3,\theta_3}(u)$	$p_{3,\theta_4}(u)$	$p_{3,\theta_5}(u)$
0.1	0.1	0	0	0	0
0.2	0.2	0	0	0	0
0.3	0.4	0.1	0	0	0
0.4	0.2	0.2	0.1	0	0
0.5	0.1	0.4	0.2	0	0
0.6	0	0.2	0.4	0.1	0
0.7	0	0.1	0.2	0.2	0.15
0.8	0	0	0.1	0.4	0.20
0.9	0	0	0	0.2	0.40
1.0	0	0	0	0.1	0.25

Table 3: Conditional probabilities of damage ratios for building type 3.

	Building Type 1	Building Type 2	Building Type 3
u	$\Pr\left(U_i^{CAT(j)} = u\right)$	$\Pr\left(U_i^{CAT(j)} = u\right)$	$\Pr\left(U_i^{CAT(j)} = u\right)$
0.1	0.0400	0.060	0.0200
0.2	0.1000	0.100	0.0400
0.3	0.1400	0.040	0.1200
0.4	0.1300	0.120	0.1400
0.5	0.1575	0.220	0.2200
0.6	0.1475	0.160	0.1750
0.7	0.1225	0.095	0.1175
0.8	0.0700	0.055	0.0900
0.9	0.0625	0.085	0.0500
1.0	0.0300	0.065	0.0275

Table 4: Mass probability function for $U^{CAT(j)}$ for $j = 1, 2, 3$.

	Building Type 1	Building Type 2	Building Type 3
$E\left[U^{CAT(j)}\right]$	0.51325	0.54400	0.54375
$\sqrt{Var\left[U^{CAT(j)}\right]}$	0.31573	0.24426	0.20127

Table 5: Expectation and standard deviation of $U^{CAT(j)}$.

x	Model 1	Model 2	Model3
0	0.8	0.8	0.8
50	0.8	0.8	0.8
100	0.8	0.808	0.8
150	0.8	0.812	0.8
200	0.8	0.832	0.840
250	0.8	0.840	0.840
300	0.8	0.864	0.840
350	0.8	0.882	0.840
360	0.8	0.882	0.860
370	0.8	0.882	0.910
380	0.806	0.908	0.920
390	0.834	0.908	0.920
400	0.899	0.908	0.920
450	1	0.940	0.920
500	1	0.943	0.960
550	1	0.967	0.960
600	1	0.970	0.962
650	1	0.985	0.990
700	1	0.994	1
750	1	1	1

Table 6: The c.d.f. $F_{S^{CAT(1)}}$, $F_{S^{CAT(2)}}$ and $F_{S^{CAT(3)}}$ at different values x.

d	Model 1	Model 2	Model3
0	80.0300	80.0300	80.0300
50	70.0300	70.0300	70.0300
100	60.0300	60.1308	60.0300
150	50.0300	50.6312	50.0300
200	40.0300	41.6312	40.6341
250	30.0300	33.3340	32.6340
300	20.0300	25.7364	24.6340
350	10.0300	18.9382	16.6349
400	0.8684	13.6908	11.5420
450	0	9.2315	7.5424
500	0	6.3068	4.9960
550	0	3.8592	2.9960
600	0	2.2095	1.0001
650	0	0.9985	0.2365
700	0	0.2869	0
750	0	0	0

Table 7: Stop-loss premium for the 3 models and for different values of retention d.

α	Model 1	Model 2	Model3
0.8	318.8	75.	149.4
0.85	393.1	275.	357.5
0.90	400.3	375.	367.6
0.95	407.6	525.	467.6
0.99	418.0	700.	645.9

Table 8: Value at Risk for the 3 models and for different values of α.

α	Model 1	Model 2	Model3
0.8	400.2500	400.2500	400.2500
0.85	404.7880	478.8194	460.7416
0.90	408.7942	548.9134	510.2741
0.95	413.9292	644.2073	594.7062
0.99	422.4290	747.9167	673.7500

Table 9: Conditional Value at Risk for the 3 models and for different values of α.