General distributional properties of discounted warranty costs under minimal repair and risk adjusted warranty costs

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Abstract

In this paper we study the distributional properties (mean, variance, characteristic function) of the discounted warranty cost (DWC) for general warranty programs including free replacement (FRW) pro rata (PRW) and FRW/PRW in the context of perfect repair. Since only failure times and types are needed in the derivation of these properties, the reliability of the systems is modeled according to a general competing risk model. Under these assumptions, our results extend those obtained by Bai and Pham (*IEEE Transactions on Reliability*, 2004). By obtaining the characteristic function of the DWC, we can consider some risk management issues, an area that has not yet been extensively studied in this context (Bai and Pham, *European Journal of Operations Research*, 2006). More precisely, we show how risk adjustment principles considered in the economics and actuarial science literature can be applied to the determination of a warranty reserve. Since some of these risk adjustment calculations require the probability mass or density function of the DWC, we show how to numerically invert the characteristic function of the DWC to obtain risk adjusted warranty costs with MATLAB in an appendix.

Index Terms

Characteristic function, competing risk model, Fast Fourier Transform, MATLAB, non homogeneous Poisson process, perfect repair, premium principle, risk adjustment, warranty reserve.

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Acronyms & Abbreviations

DWC	discounted warranty cost
EDWC	expected discounted warranty cost
EWC	expected warranty cost
FFT	fast Fourier transform
FRW	free repair warranty
i.i.d.	independent and identically distributed
NHPP	nonhomogeneous Poisson process
PRW	pro rata warranty
RAWC	risk adjusted warranty cost
r.v.	random variable

Notation

$\imath$	$\sqrt{-1}$
k	number of failure types/causes
$\lambda_j(t)$	hazard rate for failure type j at time t
$\Lambda_j(t)$	cumulative hazard function for type j at time t
$C_D(t_W)$	discounted cost of a warranty of duration t_W
$F_j(t)$	probability of failing of type j before or at time t
$\varphi_X(z)$	characteristic function of a r.v. X evaluated at z
$S(t)$	system reliability function at time t
$N_j(t)$	number of failures of type j in time interval $(0, t]$
c_j	repair cost of a failure of type j
$H(t)$	discount function at duration t
$c_j H^*(t)$	discounted cost to manufacturer of a failure of type j at time t
$\Pi_{prin}\{C_D(t_W)\}$	risk adjusted cost of a warranty under adjustment principle *prin*
TC	total discounted warranty cost for a lot of L systems

I. Introduction

When a manufacturer puts a new product on the market, several choices have to be made with respect to the warranty program that will come along with the product. Examples of such choices include the type of warranty policy, the duration of the warranty coverage, the warranty premium to be included in the price of the product, etc. (see, for instance, [11], [16] or [10], for a more thorough treatment of these and related issues). Though these choices should not be made completely independently of each other, our focus in this paper will be on warranty costs to the manufacturer. Such costs have been investigated under several approaches and assumptions for a wide range of warranty programs in the literature. Good reviews of the developments prior to 2002 can be found in [16] or [10]. More recently, Ja et al. [6], [7] have studied the properties of the DWC and total warranty program costs for non renewable warranties under minimal repair. Warranty costs have also been studied by Bai and Pham in [2], where they derive some properties of the DWC for FRW and PRW policies for repairable series systems, and in [3], where they propose a *renewable full-service warranty* policy for which they derive properties of the costs for series, parallel, series-parallel and parallel-series systems. Chien [4] derived properties of the warranty cost, optimal warranty period and optimal out-of-warranty replacement in the case of a system with two possible types of failure, with minimal repair of failures of one type and perfect repair for failures of the other type.

As Thomas and Rao [16] have pointed out, the risk and uncertainty that underlie warranties is a threat to the manufacturer. It is therefore important that the manufacturer be able to measure as best as possible the financial risk that he/she faces by issuing a warranty. Bai and Pham [3] discuss the fact that "most researchers rely solely on expected warranty cost for the purpose of warranty cost modeling and analysis". If an infinite number of items were sold, and if the probabilistic model that generates the warranty claims were known exactly, then the Law of Large Numbers implies that the (perhaps discounted) EWC would give exactly the value of the upcoming financial obligations of the manufacturer's pertaining to the warranty program. Unfortunately, many conditions for this are missing in practice–viz. only a finite number of items are sold; the best probabilistic model is, at best, an approximation to the warranty claim mechanism; all the items manufactured may not be independent; etc. For these reasons, the amount of capital to be held against the warranty program should not only be its expected cost, but rather the sum of the expected cost plus a loading for risk and uncertainty; we term the latter sum a RAWC. In the warranty cost literature, such risk adjustments to the cost are usually made during the determination of a warranty reserve. These reserves are normally calculated by making use of the Central Limit Theorem to approximate the distribution of the total cost of a large lot sale (e.g. [2], [6], [7]). If the reserve is built by embedding an equal portion of it in the price of every system sold in the lot, then this amounts to charging in the system's price the RAWC based on the *standard deviation principle* [18]. Adjustments to expected costs based on other principles have received considerable attention in the economics, financial and actuarial literature. The methods proposed in these different fields usually stem from two broad axiomatic approaches: expected utility theory and distorted expectation theory [5], and we further discuss these principles and consider their application to warranty costs in Section IV.

Our aims in this paper are (i) to derive the distributional properties of the DWC under general conditions in the case of minimal repair and (ii) to give expressions for the RAWC under different risk adjustment principles. Our developments will be based on the competing risk model, which models the reliability of systems that can fail due to one of k possible failure types/causes. The best known example of such a system is probably a series system of k components, see [9, Chapter 15], but the competing risk model is applicable to any context where only failure types and times are needed. The competing risk model can therefore be viewed as a compromise between exact modeling of system reliability and a "black-box" approach that considers the system as a single component [3], though closer in essence to the "black-box". Our approach is also mathematically convenient, as distributional properties of the DWC and the RAWC under many principles can be obtained with relatively straightforward extensions of the usual derivations that are needed to obtain the EWC or EDWC that can be found in the literature, e.g., [2], [3], [6], [7]. The remainder of the paper is organized as follows. We first give the details of the reliability and cost model in Section II and then derive the distributional properties of the DWC in Section III. These distributional properties are first derived under general settings (a competing risk model and a general warranty program) and then results are obtained for specific cases (FRW and FRW/PRW) and, though we focus on fixed repair costs, we show how the calculations can be generalized to random costs. We then consider RAWC calculations in Section IV. We give a numerical illustration in Section V and our concluding remarks in Section VI. Since we obtain some distributional properties of warranty costs via characteristic functions (Fourrier transforms), we give an example of the MATLAB code that can be used for numerical inversion of these transforms in the Appendix.

II. Reliability and cost model

Because our warranty cost calculations only depend on the times and types of failures, we suppose a system whose reliability is dictated by a competing risk model, i.e., a system that can experiment failure of one of k possible types/causes, where the hazard of a failure of cause j at time t is given by

$$\lambda_j(t) = \lim_{h \downarrow 0} \frac{P[T \in [t, t+h), J = j|\ T \geq t]}{h}, \qquad t > 0,\ j \in \{1, \ldots, k\}, \tag{1}$$

where T and J represent the failure time and cause random variables, respectively. Note that if $k = 2$, then we retrieve the model with type I and type II failures of [4], where the probability that a failure at time t is of type I, $q(t) = P[J = 1|T = t]$, is given by $q(t) = \lambda_1(t)/\sum_j \lambda_j(t)$. (Note that even though the reliability model considered here reduces to the reliability model in [4], our cost model discussed below does not.) While the $\lambda_j(t)$ are the key elements for cost calculations under minimal repair, results under perfect repair can also be obtained with this competing risk model and they usually involve the reliability and cumulative incidence functions. The former is the probability of no failure as a function of time, while the latter is the probability of having experienced a failure of a given cause as a function of time [8, Chapter 9]. The reliability function is thus given by

$$S(t) \equiv P[T > t] = \exp\left\{ - \int_0^t \sum_{j=1}^k \lambda_j(u)\, du \right\}, \quad t > 0 \tag{2}$$

and the cumulative incidence functions by

$$F_j(t) \equiv P[T \leq t, J = j] = \int_0^t \lambda_j(u) S(u)\, du, \quad t > 0,\ j \in \{1, \ldots, k\}. \tag{3}$$

Note that equations (2) and (3) do not require the independence of the latent failure times representing the times to failure due to each cause to be valid. Moreover, even without independence of the latent failure times, the reliability and cumulative incidence functions are estimable from the data that are usually observable from warranty data or lifetests, namely failure times and causes or censoring times [8, Chapter 9].

Suppose a non renewable warranty of total duration t_W. Using a notation inspired from [2], let $N_j(t)$ denote the number of failures due to cause j in the period $(0, t]$ and let $S_{j\ell}$, $j = 1, \ldots, k$, $\ell = 1, 2, \ldots$ represent the time of the ℓth failure of the jth type. Suppose that failures of type j

are repaired at a known, fixed cost c_j, $j = 1, \ldots, k$ (we explain how to carry out the calculations under random costs in Section III). Then the DWC is equal to

$$C_D(t_W) = \sum_{j=1}^{k} \int_0^{t_W} c_j H^*(s)\, dN_j(s) = \sum_{j=1}^{k} \sum_{\ell=1}^{N_j(t_W)} c_j H^*(S_{j\ell}), \tag{4}$$

where we assume that $H^*(\cdot)$ is a deterministic function. The form of $H^*(\cdot)$ will depend on the discount function as well as the type of warranty. For instance, for a FRW we have that $H^*(s) = H(s)I(0 < s \le t_W)$, where $H(s)$ is the discount function and $I(A)$ denotes the indicator function that takes on value 1 if A is true, 0 otherwise. Note that most of the results that we will present below readily extend to the case where the repair costs c_j are time-dependent by replacing $c_j H^*(s)$ with $H_j^*(s) \equiv c_j(s)H^*(s)$ in the formulae. For instance with linearly increasing costs we set $H_j^*(s) = (a_j + b_j s)H^*(s)$ and (4) becomes

$$C_D(t_W) = \sum_{j=1}^{k} \int_0^{t_W} H_j^*(s)\, dN_j(s) = \sum_{j=1}^{k} \sum_{\ell=1}^{N_j(t_W)} H_j^*(S_{j\ell}) = \sum_{j=1}^{k} \sum_{\ell=1}^{N_j(t_W)} (a_j + b_j S_{j\ell}) H^*(S_{j\ell}).$$

III. Distributional properties of the DWC

We now derive some distributional properties of the DWC defined by (4) under various types of warranty programs. We first obtain the characteristic function of the DWC under the general model of Section II then derive some more explicit results under a few special cases. We conclude this section by outlining the procedure to get similar results under random repair costs or perfect repair.

A. *Distribution of the DWC under minimal repair*

Let us assume that failures are minimally repaired, i.e., with negligible repair time and all k hazard functions immediately pre- and post-repair at the same values. Under the competing risk model and minimal repair, we have that the processes $\{N_j(t), 0 \le t \le t_W\}$, $j = 1, \ldots, k$ are independent counting processes with intensity function $\lambda_j(t)$ (see Appendix A), i.e., NHPP. This implies that for a fixed t, the $N_j(t)$ are independent Poisson r.v. with $E[N_j(t)] = \Lambda_j(t) \equiv \int_0^t \lambda_j(u)\, du$.

The EDWC and variance of $C_D(t_W)$ under minimal repair have been derived by several authors in the literature for specific warranty types (see [2] and references therein, for example, for a recent general calculation with discounted costs). Interestingly, since the payments of a warranty are usually a deterministic function of the times of failures, the calculations of [2] with an arbitrary discount function for the FRW and/or PRW can also be used to evaluate the mean and variance of the DWC under any type of warranty under minimal repair, provided that the function that maps the times of failures to the cost of the warranty is deterministic. Unfortunately, there is usually no simple analytical form for the density or probability mass function of $C_D(t_W)$, which is central to some risk adjustment methods. Nonetheless, there is a relatively simple form for its characteristic function in general[1]. We can first find an expression for the characteristic function of the DWC (see Appendix A):

$$\varphi_{C_D(t_W)}(z) \equiv E\left[e^{\imath z C_D(t_W)}\right] = \exp\left[\sum_{j=1}^{k} \Lambda_j(t_W)\left\{\int_0^{t_W} e^{\imath z c_j H^*(s)} \frac{\lambda_j(s)}{\Lambda_j(t_W)}\, ds - 1\right\}\right], \qquad (5)$$

where $\imath = \sqrt{-1}$. If we let C_ℓ denote the DWC for the ℓth system in a lot of L independent such systems, then the characteristic function of the total DWC, $TC = \sum_\ell C_\ell$, is given by

$$\varphi_{TC}(z) = \prod_{\ell=1}^{L} \varphi_{C_\ell}(z) = \exp\left[L \sum_{j=1}^{k} \Lambda_j(t_W)\left\{\int_0^{t_W} e^{\imath z c_j H^*(s)} \frac{\lambda_j(s)}{\Lambda_j(t_W)}\, ds - 1\right\}\right]. \qquad (6)$$

In specific cases (i.e., given warranty type, discount function and cause-specific hazards), the probability mass or density function of $C_D(t_W)$ [resp. TC] can readily be obtained from $\varphi_{C_D(t_W)}(z)$ [resp. $\varphi_{TC}(z)$] via the (inverse) FFT algorithm [12, Chapter 12]. In Appendix B, we give an example of the MATLAB code that can be used to implement the FFT. Applying standard results from counting process theory [1, Chapter 2] to (4) or differentiating (5) with respect to z, we retrieve the usual expressions (e.g., [2], [6]) for the EDWC and the variance of the DWC:

$$E[C_D(t_W)] = \sum_{j=1}^{k} \int_0^{t_W} c_j H^*(s) \lambda_j(s)\, ds \qquad (7)$$

$$Var[C_D(t_W)] = \sum_{j=1}^{k} \int_0^{t_W} c_j^2 H^{*2}(s) \lambda_j(s)\, ds. \qquad (8)$$

Equations (5)-(8) do not simplify if we leave $H^*(\cdot)$ and the $\lambda_j(\cdot)$ arbitrary, but explicit results can be obtained under specific discount functions, warranty programs and distributions. Here are some illustrations.

[1]Though the moment generating function (or Laplace transform) of the DWC r.v. usually exists, we work with its characteristic function so that the (inverse) FFT algorithm can readily be applied

1) FRW without discount: This is the simplest case since then $H^*(s) = 1$. We get

$$\varphi_{C_D(t_W)}(z) = \exp\left\{\sum_{j=1}^{k} \Lambda_j(t_W)(e^{\imath z c_j} - 1)\right\}. \tag{9}$$

2) FRW/PRW without discount and exponential failures: Let us assume an FRW/PRW warranty policy, with a free replacement period $(0, t_F]$ and a pro rata replacement period $(t_F, t_P]$, i.e.,

$$H^*(s) = \begin{cases} H(s), & 0 < s \le t_F \\ \left(1 - \frac{s - t_F}{t_P - t_F}\right) H(s), & t_F < s \le t_P, \end{cases} \tag{10}$$

where $H(s)$ is the discount function. Let us put $H(s) = 1$ (no discount) and let us suppose that $\lambda_j(t) = \lambda_j$. Then

$$\varphi_{C_D(t_W)}(z) = \exp\left\{\sum_{j=1}^{k} t_P(\xi - 1)\right\}, \tag{11}$$

where

$$\xi = \frac{t_F}{t_P} e^{\imath z c_j} + \left(1 - \frac{t_F}{t_P}\right) \frac{e^{\imath z c_j} - 1}{\imath z c_j}.$$

3) With discount: When we use the discount function $H(s) = \exp(-\delta s)$, then the integral involved in (5) cannot be solved in closed form. Fortunately, this integral needs not be solved explicitly as it can be viewed as the characteristic function of a r.v. $V_j = c_j \exp\{-\delta \; H^*(U_j)\}$, with U_j a r.v. having density $\lambda_j(s)/\Lambda_j(t_W)$ for $s \in [0, t_W]$, and we shall see how to use this fact to numerically invert $\varphi_{C_D(t_W)}(z)$ in Section V.

B. Extensions to random costs or perfect repair

We now briefly outline how the properties of the DWC can be obtained under random costs or perfect repair.

1) Random costs: Most of the calculations done in this section can also be done when the warranty costs c_j are replaced by random costs C_j whose (joint) distribution is known. Indeed, for probability and expectation (the latter including characteristic function) calculations, all one has to do is to consider all results previously given in this section as conditional on the event $\{C_j = c_j, \; j = 1, \ldots, k\}$, then integrate the results with respect to $dF(c_1, \ldots, c_k)$, the joint distribution of the C_j. For instance, let us look at the EDWC under minimal repair. From (7) we get

$$
\begin{aligned}
E[C_D(t_W)] &= \int \cdots \int E[C_D(t_W)|C_j = c_j,\ j = 1,\ldots,k]\ dF(c_1,\ldots,c_k) \\
&= E[E[C_D(t_W)|C_j,\ j = 1,\ldots,k]] = E\left[\sum_{j=1}^{k} C_j \int_0^{t_W} H^*(s)\lambda_j(s)\ ds\right] \\
&= \sum_{j=1}^{k} \mu_j \int_0^{t_W} H^*(s)\lambda_j(s)\ ds, \qquad (12)
\end{aligned}
$$

where $\mu_j = E[C_j]$; of course if the C_j are constant, then $\mu_j = c_j$.

For variance calculations, we must use the formula

$$
Var[C_D(t_W)] = Var[E[C_D(t_W)|C_j, j = 1,\ldots,k]] + E[Var[C_D(t_W)|C_j, j = 1,\ldots,k]], \quad (13)
$$

where the conditional mean and variance in (13) are given by (7) and (8), respectively. This yields

$$
\begin{aligned}
Var[C_D(t_W)] = &\sum_{j=1}^{k} \sigma_j^2 \left\{\int_0^{t_W} H^*(s)\lambda_j(s)\ ds\right\}^2 + \sum_{j=1}^{k}(\sigma_j^2 + \mu_j^2)\left\{\int_0^{t_W} H^{*2}(s)\lambda_j(s)\ ds\right\} \\
&+ 2\sum\sum_{j<j'} \sigma_{jj'} \left\{\int_0^{t_W} H^*(s)\lambda_j(s)\ ds\right\}\left\{\int_0^{t_W} H^*(s)\lambda_{j'}(s)\ ds\right\}, \qquad (14)
\end{aligned}
$$

where $\sigma_j^2 = Var[C_j]$ and $\sigma_{jj'} = Cov(C_j, C_{j'})$. (Note that the term on the second line of (14) vanishes when the C_j are independent, and that $\sigma_j^2 = \sigma_{jj'} = 0$ when the C_j are constant.)

2) Perfect repair: We can also derive some distributional properties of the DWC under perfect repair and renewable FRW, i.e., when the system fails before time t_W, the system is replaced by a new system and the warranty starts anew. Under this type of warranty, the total duration of the policy is unknown because the warranty will last until the system finally survives t_W consecutive units of time without failure. To derive the distribution of the cost of this warranty, let us suppose that the cost of a perfect repair of a failure due to cause j is a fixed, known constant c_j. Let us denote the number of failures due to cause j during the warranty by N_j and let $N = \sum_{j=1}^{k} N_j$ be the total number of failures under warranty. In a different but closely related context, Bai and Pham [3] have studied some of the distributional properties of N and the N_j. Since N is the number of failures before the system finally survives to time t_W, N follows a geometric distribution with "success" probability $S(t_W)$ and has probability mass function

$$
P[N = n] = S(t_W)\{1 - S(t_W)\}^n, \quad n = 0, 1 \ldots \qquad (15)
$$

Given $N = n$, the joint distribution of $N_1, \ldots, N_k$ is multinomial with total n and respective success probabilities $\alpha_j(t_W)$, $j = 1, \ldots, k$, where $\alpha_j(t_W) = P[J = j | T \leq t_W] = F_j(t_W)/\{1 - S(t_W)\}$. We can now obtain (see Appendix A) the characteristic function of the undiscounted warranty cost, $C(t_W) = \sum_{j=1}^{k} c_j N_j$:

$$\varphi_{C(t_W)}(z) = \frac{S(t_W)}{1 - \sum_{j=1}^{k} F_j(t_W) e^{\imath z c_j}}. \tag{16}$$

From (16), we can derive the EWC, $E[C(t_W)] = \sum_{j=1}^{k} c_j F_j(t_W)/S(t_W)$, and the variance of the cost,

$$Var[C(t_W)] = \frac{S(t_W) \sum_{j=1}^{k} c_j^2 F_j(t_W) + \sum_{j=1}^{k} \sum_{l=1}^{k} c_j c_l F_j(t_W) F_l(t_W)}{\{S(t_W)\}^2}.$$

Once again, we can use the FFT algorithm and (16) to get the probability mass function of $C(t_W)$. We are currently investigating the distributional properties of the DWC with perfect repair and non renewable warranty.

IV. Risk adjusted warranty cost

As argued by Thomas and Rao [16, Section 4.4], risk and uncertainty should be taken into account when making warranty economic decisions. Unfortunately, as remarked by Bai and Pham [3], most studies of warranty costs focus on the EWC or EDWC. In this section we illustrate how one can compute a *risk adjusted warranty cost* (RAWC). This RAWC can then be embedded in the price of the system and used to build a warranty reserve. This problem of building a warranty reserve has received considerable attention in the warranty literature (see for example [6], [7], or [2] and references therein), where the problem is usually approached by making use of the Central Limit Theorem to find the RAWC such that the probability that the total warranty cost exceeds the warranty reserve is a (small) fixed value. In economics, finance and insurance, this approach is referred to as the *standard deviation principle*. However in these research areas there are other principles available to determine the amount of money one would need to agree to take on a financial risk [18]. All of these principles, including the standard deviation principle, have in common that if one is risk neutral, one will take on the risk X for an amount of $E[X]$ and the more risk averse one gets, the more one will want an amount of money greater than $E[X]$ in exchange of the risk. These principles, referred to as premium principles in the insurance literature, usually relate to two broad axiomatic approaches. The first approach is zero-utility theory: the premium charged in exchange of the risk is such that the expected utility of the premium $\Pi(X)$ minus the expected value of the risk is the same as the utility of a wealth of 0, i.e., $u(0) = E[u\{\Pi(X) - X\}]$. A second approach consists in charging a premium $\Pi(X)$ that is the expected value of the risk under a distorted distribution that gives more weight to high values of the risk. Young [18] gives a nice review of premium principles and their properties. In this section we consider three of these principles which, we feel, possess a certain mathematical convenience and/or theoretical properties that make them useful under the warranty cost model considered, and see how we can use these principles to calculate the RAWC.

A. *Standard deviation principle*

The premium in exchange of a risk X under this principle is $\Pi_{stdev}(X) = E[X] + b\sqrt{Var[X]}$. This is the principle that is used in [6], [7], [2] where the value of b is chosen as follows. Let L be the total number of systems sold and let C_ℓ be the DWC r.v. for the ℓth system. Assuming that the L systems are independent and follow the same reliability model, by the Central Limit Theorem one has that

$$P\left[\frac{TC - L\ \mu}{\sqrt{L\ \sigma^2}} \leq z_{1-\alpha}\right] \approx 1 - \alpha,$$

where $z_{1-\alpha}$ is the $100(1-\alpha)$th percentile of a standard normal distribution, $\mu = E[C_\ell]$ and $\sigma^2 = Var[C_\ell]$. Hence with a reserve of $L(E[C_\ell] + b\sqrt{Var[C_\ell]})$ with $b = z_{1-\alpha}/\sqrt{L}$, the manufacturer has probability $1 - \alpha$ that the total DWC will not exceed the warranty reserve. For our warranty model of Section II, substitution of (7) and (8) in Π_{stdev} yields

$$\Pi_{stdev}\{C_D(t_W)\} = \sum_{j=1}^{k} \int_0^{t_W} c_j H^*(s)\lambda_j(s)\ ds + b\sqrt{\sum_{j=1}^{k} \int_0^{t_W} c_j^2 H^{*2}(s)\lambda_j(s)\ ds}. \qquad (17)$$

B. *PH transform principle*

The proportional hazard (PH) transform principle has received considerable attention in the literature because it possesses several desirable properties sought in a risk adjustment principle [17]. It is defined as $\Pi_{PH}(X) = \int_0^\infty S^b(x)\ dx$, where $S(x) = P[X > x]$ and $b \in (0, 1)$. Thus

$$\Pi_{PH}\{C_D(t_W)\} = \int_0^\infty \{P[C_D(t_W) > s]\}^b\ ds. \qquad (18)$$

C. Esscher principle

This principle also has nice properties, albeit not as many as the PH transform. However, in the case of discounted warranty costs, it is certainly more mathematically convenient than the PH transform, as no numerical inversion of the characteristic function is needed. Indeed, the premium for risk X under this principle is $\Pi_{Esscher}(X) = d/dz \ln M_X(z)|_{z=b}$, where $M_X(z) = E[\exp(zX)]$ is the moment generating function of X and $b > 0$ is the risk aversion parameter. Thus, from (5), we have that the Esscher RAWC in our warranty model is given by

$$\Pi_{Esscher}\{C_D(t_W)\} = \sum_{j=1}^{k} \int_0^{t_W} c_j H^*(s) e^{bc_j H^*(s)} \lambda_j(s)\, ds. \tag{19}$$

Besides its mathematical convenience, this adjustment principle also has nice interpretations. It can be viewed as a zero-utility premium when the utility function is given by $u(x) = \{1 - \exp(-bx)\}/b$. It also corresponds to the premium that minimizes the loss function $E[\{X - \Pi(X)\}^2 e^{bX}]$, i.e., a weighted square loss with weight growing exponentially such that underestimation of the risk is penalized more harshly.

D. General properties

Young [18] discusses several properties that might be desirable in risk adjustment principles. Here are four properties that seem to be particularly relevant in the case of RAWC.

- **Positive risk loading**, $\Pi(X) \geq E[X]$: This means that the RAWC should be greater than or equal to the expected cost. All principles considered here have this property.
- **No rip-off**, $\Pi(X) \leq \max\ \text{supp}(X)$: This condition states that the RAWC should not exceed the largest possible value of the cost. The standard deviation principle does **not** fulfill this condition, but the PH transform and Esscher principles do.
- **Homegeneity**, $\Pi(aX) = a\Pi(X)$: Also referred to as scale invariance, this property states that a multiplicative change in the measurement units of the costs (e.g., change of currency) should result in the same change in the RAWC. In the list above, only the standard deviation and PH transform principles exhibit this property.
- **Monotonicity**, $P[X \leq Y] = 1 \Rightarrow \Pi(X) \leq \Pi(Y)$: If it is certain that the realized value of risk X cannot exceed that of risk Y, then the risk adjusted cost of X should not exceed the risk adjusted cost of Y. The only principle in the list above with this property is the PH transform.

E. Choice of α or b

In order to apply these principles in practice, the use of the standard deviation principle requires fixing the value of the probability α that the total cost will exceed the reserve, while the PH transform or Esscher principles need a value for the risk aversion parameter b. Though it is beyond the scope of this paper to give a thorough treatment of this issue, we can point to a few options found in the literature. A first option is to find the value of b or α by eliciting the manufacturer's utility function (e.g., [14]). Game theoretical approaches (e.g., [13], [2]) can also be used to set the values of α or b as functions of the competition that the manufacturer faces. Another method that would be applicable to any principle, as long as a model for sales as a function of price is available, consists in finding the warranty duration and cost that maximize the expected profit (e.g, [15], [7], [4]).

V. Numerical illustration

In this section we give the EDWC and the RAWC under the three principles considered above. For the sake of comparison, we use the setup from the numerical study in [2]. Thus we consider a FRW policy (i.e., $H^*(s) = \exp(-\delta s)$ for $s \in (0, t_W])$ on a lot of 1,000 independent systems that can suffer failures of three different types. The failure times and types follow the competing risk model with cause-specific failure rates (with time measured in, say, years) given by

$$\lambda_1(t) = 0.0611; \qquad \lambda_2(t) = 0.0423t; \qquad \lambda_3(t) = 0.0187t^2.$$

The repair costs are $c_1 = 100$, $c_2 = 150$ and $c_3 = 200$ and the discount rate is $\delta = 0.05$.

To derive all the quantities of interest, we first numerically obtain the distribution of $C_D(t_W)$ using the FFT. Once this distribution is available, then it is straightforward to calculate the EDWC and the RAWC under the various risk adjustment principles. Since we work with the FFT, we must approximate the density of $C_D(t_W)$ by a probability mass function over the discrete domain $\{0, h, 2h, 3h, \dots, (2^n - 1)h\}$, where h is small and n is large (we used $h = 0.01$ and $n = 16$ in our calculations). Let U_j, $j = 1, 2, 3$, represent r.v. with cumulative distribution functions

$$F_{U_j}(u) = P[U_j \le u] = \begin{cases} 0, & u < 0 \\ \Lambda_j(u)/\Lambda_j(t_W), & 0 \le u < t_W \\ 1, & u \ge t_W, \end{cases} \qquad j = 1, 2, 3$$

and put $V_j = c_j \exp(-\delta U_j)$. The cumulative distribution function of V_j is

$$F_{V_j}(u) = P[V_j \leq v] = \begin{cases} 0, & v < c_j \exp(-\delta t_W) \\ 1 - \Lambda_j\{-\ln(v/c_j)/\delta\}/\Lambda_j(t_W), & c_j \exp(-\delta t_W) \leq v < c_j \\ 1, & v \geq c_j. \end{cases}$$

The probability mass function of V_j is then defined as $p_{V_j}(sh) = F_{V_j}((s+1)h) - F_{V_j}(sh)$ for $s \in \{0, 1, \ldots, 2^n - 1\}$. The probability mass function of $C_D(t_W)$ is obtained via this algorithm:

1) Compute φ_{V_j}, the FFT of p_{V_j} for $j = 1, \ldots, k$.
2) Compute $\varphi_{C_D(t_W)}$, the FFT of the probability mass function of $C_D(t_W)$, which is given by $\varphi_{C_D(t_W)} = \exp\{\sum_{j=1}^{k} \Lambda(t_W)(\varphi_{V_j} - 1)\}$.
3) Invert $\varphi_{C_D(t_W)}$ to obtain the probability mass function of $C_D(t_W)$.

We give an example of the implementation of this algorithm in MATLAB in Appendix B.

Using the above algorithm, we obtain results that are, up to numerical error, the same[2] as [2]. Besides the mean and variance of the DWC, in Table I we also give the values of the RAWC under the three risk adjustment principles considered in this paper. We give the RAWC for a few values of α or b to give an idea of the sensitivity of the RAWC to the parameter (α or b) values. As we can see, changing these values has a relatively large impact on the RAWC. Another interesting observation coming out of Table I is that the RAWC computed under the Esscher principle grows quicker as a function of t_W than the RAWC under the PH transform which, itself, increases quicker with t_W than the RAWC under the standard deviation principle. Since a longer warranty is riskier, it is not surprising to see the conservative Esscher RAWC increasing quickly in t_W.

VI. Conclusion

This paper has given a broad treatment of the distributional properties of the DWC under minimal repair, for which the characteristic function under a general competing risk model and warranty program has been derived. From this characteristic function, we have shown how the EDWC and RAWC under three risk adjustment principles can be derived. We have also highlighted how some of the results presented can be extended to random and/or time-varying repair costs.

[2]Actually, we obtain the same EDWC as [2], but we only obtain their variance if we replace c_j^2 in (8) by c_j.

TABLE I

RESULTS OBTAINED USING THE SETTINGS OF THE NUMERICAL STUDY OF [2] FOR A FRW OF VARIOUS DURATIONS t_W. ALL RESULTS ARE BASED ON THE DISCRETIZED DISTRIBUTION OF $C_D(t_W)$ OBTAINED WITH THE FFT.

t_W	$E[C_D(t_W)]$	$\sqrt{Var[C_D(t_W)]}$	$\Pi_{stdev}^{\alpha=0.1}$	$\Pi_{stdev}^{\alpha=0.05}$	$\Pi_{Esscher}^{b=1e-04}$	$\Pi_{Esscher}^{b=1e-03}$	$\Pi_{PH}^{b=0.975}$	$\Pi_{PH}^{b=0.95}$
1	10.3	35.6	11.7	12.1	10.4	11.6	11.0	11.8
2	33.0	67.3	35.7	36.5	33.5	37.9	34.6	36.3
3	73.3	102	77.5	78.7	74.6	84.9	75.9	78.5

It would be interesting to see how the methods proposed in this paper could be extended to other types of repair, or when different types of failures require repairs of different types, as in [4]. It would also be interesting to see how the risk adjustment methods proposed here would apply to other types of warranties for different types of systems, such as that considered in [3]. Since our focus was on the statistical/probabilistic aspects of the problem, a more thorough treatment of economics and management issues, such as elicitation of the risk aversion parameter or the derivation of an optimal duration in a competitive environment, would be of interest.

APPENDIX A
SOME TECHNICAL DETAILS

Proof of independence of $N_j(t)$, $j = 1, \dots, k$, competing risks with minimal repair

We sketch the proof for $k = 2$ for notational convenience, but the proof for a general k is similar. We must show that $P[N_1(t) = m, N_2(t) = n] = P[N_1(t) = m]P[N_2(t) = n]$. We have that

$$
\begin{aligned}
P[N_1(t)=m, N_2(t)=n] &= \int_{t_{11},\ldots,t_{1m},t_{21},\ldots,t_{2n}} \cdots \int P\big[N_1(t)=m, N_2(t)=n \text{ and} \\
&\qquad \{N_1 \text{ jumps at } t_{11},\ldots,t_{1m}, N_2 \text{ jumps at } t_{21},\ldots,t_{2n}\}\big]\, d\mathbf{t} \\
&= \int_{t_{11},\ldots,t_{1m},t_{21},\ldots,t_{2n}} \cdots \int \left\{\prod_{\ell=1}^{m} \lambda_1(t_{1\ell})\right\} \left\{\prod_{\ell=2}^{n} \lambda_2(t_{2\ell})\right\} e^{-\{\Lambda_1(t)+\Lambda_2(t)\}}\, d\mathbf{t} \\
&= \int_{t_{11},\ldots,t_{1m}} \cdots \int \left\{\prod_{\ell=1}^{m} \lambda_1(t_{1\ell})\right\} e^{-\Lambda_1(t)}\, d\mathbf{t}_1 \\
&\qquad \times \int_{t_{21},\ldots,t_{2n}} \cdots \int \left\{\prod_{\ell=1}^{n} \lambda_2(t_{2\ell})\right\} e^{-\Lambda_2(t)}\, d\mathbf{t}_2 \\
&= P[N_1(t)=m] \times P[N_2(t)=n].
\end{aligned}
\tag{20}
$$

(Remark: If we consider k latent failure times $T_1,\ldots,T_k$ and set $T=\min(T_1,\ldots,T_k)$ and $J=\{j : T_j=T\}$, then no assumption of independence on $T_1,\ldots,T_k$ is necessary for (20) to hold [8, Section 11.4]. The assumption of independence is required only when one needs the joint distribution of $T_1,\ldots,T_k$, which is not the case under our warranty cost model.)

Characteristic function of cost under minimal repair

We want to compute

$$
\varphi_{C_D(t_W)}(z) = E\left[\exp\left\{\imath z \sum_{j=1}^{k} \sum_{\ell=1}^{N_j(t_W)} c_j H^*(S_{j\ell})\right\}\right].
$$

We will use two facts here: (i) under minimal repair, the $\{N_j(t)\}$ are independent NHPP and (ii) for an NHPP with intensity $\lambda_j(t)$, the conditional joint distribution of the jump times given the number of jumps in $(0,t_W]$ is that of the order statistics of a sample of i.i.d. random variables with density $\lambda_j(t)/\Lambda_j(t_W)$ [2]. Fact (i) implies that

$$
\varphi_{C_D(t_W)}(z) = \prod_{j=1}^{k} E\left[\exp\left\{\imath z \sum_{\ell=1}^{N_j(t_W)} c_j H^*(S_{j\ell})\right\}\right].
$$

Then from fact (ii) we have that

$$E\left[\exp\left\{\imath z\sum_{\ell=1}^{N_j(t_W)} c_j H^*(S_{j\ell})\right\}\middle| N_j(t_W)=n\right] = E\left[\exp\left\{\imath z c_j\sum_{\ell=1}^{n} H^*(U_{j(\ell)})\right\}\right]$$

$$\left(\text{since } \sum_\ell H^*(U_{j(\ell)})=\sum_\ell H^*(U_{j\ell})\ \right) \qquad = E\left[\exp\left\{\imath z c_j\sum_{\ell=1}^{n} H^*(U_{j\ell})\right\}\right]$$

$$= \left(E\left[\exp\left\{\imath z c_j H^*(U_{j1})\right\}\right]\right)^n = \left\{\varphi_{H^*}(zc_j)\right\}^n,$$

where $\varphi_{H^*}(zc_j)=\int_0^{t_W} e^{\imath z c_j H^*(u)}\lambda_j(u)/\Lambda_j(t_W)\,du$. Thus,

$$E\left[\exp\left\{\imath z\sum_{\ell=1}^{N_j(t_W)} H^*(S_{j\ell})\right\}\right] = \varphi_{N_j(t_W)}\{\ln\varphi_{H^*}(zc_j)\},$$

with $\varphi_{N_j(t_W)}(\cdot)$ the characteristic function of the r.v. $N_j(t_W)$. The latter being Poisson with mean $\Lambda_j(t_W)$, we get

$$E\left[\exp\left\{\imath z\sum_{\ell=1}^{N_j(t_W)} H^*(S_{j\ell})\right\}\right] = \exp\left[\Lambda_j(t_W)\left\{e^{\ln\varphi_{H^*}(zc_j)}-1\right\}\right],$$

which, after simplifying and taking product over $j=1,\ldots,k$, yields (5).

Characteristic function of DWC under renewable FRW with perfect repair

$$\begin{aligned}\varphi_{C_D(t_W)}(z) &= E\left[E\left[e^{\imath z C(t_W)}\middle|\ N\right]\right] = E\left[E\left[e^{\imath z\sum_j c_j N_j}\middle|\ N\right]\right]\\ &= E\left[\varphi_{N_1,\ldots,N_k|N}(zc_1,\ldots,zc_k)\right] = E\left[\left\{\frac{\sum_{j=1}^k F_j(t_W)e^{\imath z c_j}}{1-S(t_W)}\right\}^N\right]\\ &= \frac{S(t_W)}{1-\sum_{j=1}^k F_j(t_W)e^{\imath z c_j}}.\end{aligned}$$

Appendix B

MATLAB code for Fast Fourier Transform

The FFT and its inverse are programmed in various software. We give an illustration of how the algorithm described in Section V can be implemented in MATLAB. The complete MATLAB code for the numerical study is available from the first author upon request.

```
clear all
alpha  = [0.0611,0.0432,0.0187]; % hazards = alpha * t^beta
beta = [0,1,2];
delta = 0.05; % discount rate
ks  = [100,150,200]; % repair costs
tW = 1; % warranty duration
h = 0.01; % discretization step
n = 16; % length of vector for FFT
epsilon = 10^-10 ;% pour "nettoyer" le vecteur final des proba
% calculation of Lambda_j(t_W)
lambda = alpha .* tW .^ (beta+1) ./ (beta+1);
% CDFs of V_1, V_2 and V_3
function x = cdf1(v, alpha, beta, delta, c, TW)
    if(v<c(1)*exp(-delta*TW)) x =0;
    elseif(v>=c(1)) x=1;
    elseif((v>=c(1)*exp(-delta*TW))&(v<c(1)))
        x=(1-(-log(v/c(1))/(delta*TW))^(beta(1)+1));
end
function y = cdf2(v, alpha, beta, delta, c, TW)
    if(v<c(2)*exp(-delta*TW)) y =0;
    elseif(v>=c(2)) y=1;
    elseif((v>=c(2)*exp(-delta*TW))&(v<c(2)))
        y=(1-(-log(v/c(2))/(delta*TW))^(beta(2)+1));
end
function z = cdf3(v, alpha, beta, delta, c, TW)
    if(v<c(3)*exp(-delta*TW)) z =0;
    elseif(v>=c(3)) z=1;
    elseif((v>=c(3)*exp(-delta*TW))&(v<c(3)))
        z=(1-(-log(v/c(3))/(delta*TW))^(beta(3)+1));
end
% probability mass functions of the V_j
fv1 =  zeros(1,2^n);
fv2 =  zeros(1,2^n);
fv3 =  zeros(1,2^n);
r1 = floor(ks(1)/h);
```

```
r2 = floor(ks(2)/h);
r3 = floor(ks(3)/h);
for k = 1:r1
    fv1(k+1) =  cdf1(k*h, alpha, beta, delta, ks, tW)...
      - cdf1((k-1)*h, alpha, beta, delta, ks, tW);
end
for k = 1:r2
    fv2(k+1) =  cdf2(k*h, alpha, beta, delta, ks, tW)...
      - cdf2((k-1)*h, alpha, beta, delta, ks, tW);
end
for k = 1:r3
    fv3(k+1) =  cdf3(k*h, alpha, beta, delta, ks, tW)...
      - cdf3((k-1)*h, alpha, beta, delta, ks, tW);
end
%FFT of the PMFs of the V_j
phi_fv1 = fft(fv1);
phi_fv2 = fft(fv2);
phi_fv3 = fft(fv3);
%FFT of the PMF of C_D(t_W)
phi_ctw = exp(lambda(1)*(phi_fv1-1)+lambda(2)*(phi_fv2-1)
  +lambda(3)*(phi_fv3-1));
%Inversion of the FFT of C to get the PMF of C
fctw = real(ifft(phi_ctw));
x = h*(0:(2^n-1));
r=1;
for k = 1: length(fctw)
    xPx(r,1) = x(k);
    xPx(r,2) = fctw(k);
    r = r+1;
  end
end
%  xPx is an array with the DWC in column 1 and the PMF of the DWC in column 2
```

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