$W^{1,p}$ REGULARITY OF SOLUTIONS TO KOLMOGOROV EQUATION WITH GILBARG-SERRIN MATRIX

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ABSTRACT. In $\mathbb{R}^d$, $d \geq 3$, consider the divergence and the non-divergence form operators

$$-\Delta - \nabla \cdot (a-I) \cdot \nabla + b \cdot \nabla, \qquad (i)$$

$$-\Delta - (a-I) \cdot \nabla^2 + b \cdot \nabla, \qquad (ii)$$

where the second order perturbations are given by the matrix

$$a - I = c|x|^{-2} x \otimes x, \quad c > -1.$$

The vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is form-bounded with the form-bound $\delta > 0$ (this includes a sub-critical class $[L^d + L^\infty]^d$, as well as vector fields having critical-order singularities). We characterize quantitative dependence on c and δ of the $L^q \to W^{1,qd/(d-2)}$ regularity of the resolvents of the operator realizations of (i), (ii) in L^q, $q \geq 2 \vee (d-2)$ as (minus) generators of positivity preserving L^∞ contraction C_0 semigroups.

Consider in $\mathbb{R}^d$, $d \geq 3$, the operator

$$-\Delta + b \cdot \nabla \equiv -\sum_{i=1}^d \partial_{x_i}^2 + \sum_{k=1}^d b_k(x) \partial_{x_k}, \quad b \in \mathbf{F}_\delta,$$

where $\mathbf{F}_\delta \equiv \mathbf{F}_\delta(-\Delta)$, $\delta > 0$, is the class of form-bounded vector fields $\mathbb{R}^d \to \mathbb{R}^d$, that is, $|b| \in L^2_{\rm loc} \equiv L^2_{\rm loc}(\mathbb{R}^d)$ and there exist a constant $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}$$

(see examples below). By [KS, Lemma 5] (see also [KiS, Theorems 3.7-3.10]), if $\delta < 1 \wedge \left(\frac{2}{d-2}\right)^2$, then for

$$u = (\mu + \Lambda_q(b))^{-1} f, \quad f \in L^q, \quad q \in [2 \vee (d-2), \frac{2}{\sqrt{\delta}}[,$$

one has

$$\|\nabla u\|_{\frac{qd}{d-2}} \leq K \|f\|_q, \qquad (\star)$$

where $\mu > \frac{\lambda\delta}{2(q-1)}$, $K = K(\mu, \delta, q)$. Here $\Lambda_q(b)$ is an operator realization of $-\Delta + b \cdot \nabla$ as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup in L^q. The estimate $(\star)$ and the iteration method in [KS] (see also [KiS, sect. 3.6]) allow to construct a Feller process associated with $-\Delta + b \cdot \nabla$.

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In this paper we are concerned with a second order perturbation of $-\Delta$,

$$-\Delta-\nabla\cdot(a-I)\cdot\nabla, \qquad a_{ij}(x)=\delta_{ij}+c|x|^{-2}x_ix_j, \quad c>-1, c\neq 0. \tag{1}$$

This is a model example of a divergence form operator that is not accessible by classical means such as the parametrix [F], [LSU, Ch. IV]. Although the matrix a is discontinuous at $x=0$, it is uniformly elliptic, so by the De Giorgi-Nash theory, the solution $u\in W^{1,2}(\mathbb{R}^d)$ to the corresponding elliptic equation $(\mu-\nabla\cdot a\cdot\nabla)u=f$, $\mu>0$, $f\in L^p\cap L^2$, $p\in]\frac{d}{2},\infty[$, is in $C^{0,\gamma}$, where the Hölder continuity exponent $\gamma\in]0,1[$ depends only on d and c. The operator (1) and its modifications both of divergence and non-divergence type have been investigated by many authors in order to precise the connection between the regularity properties of the solution and the continuity properties of the matrix, see [GS], [M], [LU, Ch. 1.2], [ABT], [OG], [A] and references therein.

In Theorem 1 below (the principal result) we show that the perturbation $-\nabla\cdot(a-I)\cdot\nabla$ of $-\Delta$ preserves, under the appropriate assumptions on c, the essential properties of $-\Delta$ that allow to establish the estimate ($\star$) for $u=\big(\mu+\Lambda_q(a,b)\big)^{-1}f$, where $\Lambda_q(a,b)$ is an operator realization of

$$-\Delta-\nabla\cdot(a-I)\cdot\nabla+b\cdot\nabla, \qquad b\in\mathbf{F}_\delta$$

as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup in L^q (Theorem 2).

The class $\mathbf{F}_\delta$ contains a sub-critical class $|b|\in L^d+L^\infty$, with $\delta>0$ arbitrarily small, as well as vector fields having critical-order singularities, e.g. $b(x)=\frac{d-2}{2}\sqrt{\delta}|x|^{-2}x$. More generally, if $|b|$ is in $L^{d,\infty}$, the Campanato-Morrey class or the Chang-Wilson-Wolff class, then $b\in\mathbf{F}_\delta$ with δ depending on the norm of $|b|$ in these classes, see e.g. [KiS] for details. We also note that, for the class of uniformly elliptic matrices as a whole, the class $\mathbf{F}_\delta$ of first order perturbations $b\cdot\nabla$ of $-\nabla\cdot a\cdot\nabla$ destroys the $C^{0,\gamma}$ regularity of bounded solutions.

Set $(\nabla a)_k:=\sum_{i=1}^d(\partial_{x_i}a_{ik})$, $1\leq k\leq d$. Then $\nabla a=c(d-1)|x|^{-2}x$, and so $\nabla a\in\mathbf{F}_\delta$, $\delta=\frac{4c^2(d-1)^2}{(d-2)^2}$. The latter allows us to construct an operator realization of the non-divergence form operator

$$-a\cdot\nabla^2+b\cdot\nabla\equiv-\sum_{i,j=1}^d a_{ij}(x)\partial_{x_i}\partial_{x_j}+\sum_{k=1}^d b_k(x)\partial_{x_k}, \quad b\in\mathbf{F}_{\delta_1}$$

in L^q as $\Lambda_q(a,\nabla a+b)$ (formally, $-a\cdot\nabla^2+b\cdot\nabla\equiv-\nabla\cdot a\cdot\nabla+(\nabla a)\cdot\nabla+b\cdot\nabla$) and then characterize quantitative dependence of the $W^{1,p}$ regularity of $u\equiv(\mu+\Lambda_q(a,\nabla a+b))^{-1}f$, $f\in L^q$, on c,d,q,μ and δ_1 (Theorems 3 and 4). This result (including the class of first order perturbations $b\cdot\nabla$, $b\in\mathbf{F}_{\delta_1}$ of $-a\cdot\nabla^2$) can not be achieved on the basis of the Krylov-Safonov a priori estimates [Kr, Ch. 4.2]. (We note that the operator $-a\cdot\nabla^2$ with $\partial_{x_k}a_{ij}\in L^{d,\infty}$ has been studied earlier in [AT]; see also [ABT].)

The method of proof of the results of this paper admits immediate extension to

$$a_{ij}(x)=\delta_{ij}+\sum_l c_l\kappa_{ij}(x-x^l), \quad \kappa_{ij}(x)=|x|^{-2}x_ix_j, \tag{2}$$

$$c_+:=\sum_{c_l>0}c_l<\infty, \quad c_-:=\sum_{c_l<0}c_l>-1,$$

where $\{x^l\}$ is an arbitrary countable subset of $\mathbb{R}^d$.

The arguments in this paper can be transferred without significant changes from $\mathbb{R}^d$ to the ball $B(0,1)$.

Theorem 2 and the iteration method in [KS] allow to construct a Feller process associated with $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$, $b \in \mathbf{F}_\delta$. The method of this paper seems to be suited to treat classes of second-order perturbations $-\nabla \cdot (a-I) \cdot \nabla$, $-(a-I) \cdot \nabla^2$ of $-\Delta$ more general than (2), for example, given by $a - I = v \otimes v$, where (bounded) $v : \mathbb{R}^d \to \mathbb{R}^d$, $v \in W^{1,2}_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$ satisfies $\big(\sum_k (\nabla v_k)^2\big)^{\frac{1}{2}} \in \mathbf{F}_\delta$. We plan to address this matter in another paper.

1. We now state our results in full. First, consider the divergence form operator.

Theorem 1 $(-\nabla \cdot a \cdot \nabla)$**.** *Let $d \geq 3$, $a(x) = I + c|x|^{-2} x \otimes x$, $c > -1$.*

(i) The formal differential operator $-\nabla \cdot a \cdot \nabla$ has an operator realization A_q on L^q, $q \in [1, \infty[$, as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup.

Set $u := (\mu + A_q)^{-1} f$, $\mu > 0$, $f \in L^q$.

(ii) Let $d \geq 4$. Assume that $q \geq d-2$ and

$$-\frac{(q-1)(d-2)^2}{q^2 \ell_2} < c < \frac{(q-1)(d-2)^2}{q^2 \ell_1},$$

where

$$\ell_1 \equiv \ell_1(q,d) := d - 1 - d\frac{d-2}{q} + (1+\theta)\frac{(d-2)^2}{q^2}, \qquad \theta = \frac{1}{2(d-1)},$$

$$\ell_2 \equiv \ell_2(q,d) := d - 1 + (q-1)\frac{(d-2)^2}{q^2}.$$

Then $u \in \bigcap_{q \leq p \leq \frac{qd}{d-2}} W^{1,p}$, and there exist constants $K_l = K_l(d,q,c)$, $l = 1,2$, such that

$$\begin{aligned} \|\nabla u\|_q &\leq K_1 \mu^{-\frac{1}{2}} \|f\|_q, \\ \|\nabla u\|_{\frac{qd}{d-2}} &\leq K_2 \mu^{\frac{1}{q} - \frac{1}{2}} \|f\|_q. \end{aligned} \tag{$\star$}$$

The dependence on q and μ in ($\star$) is the best possible.

(iii) Let $d \geq 3$. Assume that

$$-\left(1 + \frac{4(d-1)}{(d-2)^2}\right)^{-1} < c < \frac{(d-2)^2}{4}.$$

Then $u \in W^{2,2}$ and ($\star$) holds with $q = 2$.

Of special interest are the minimal assumptions on c such that the second estimate in ($\star$) holds with *some* $q > d-2$.

Corollary 1. *For $d = 3$ and $-\frac{1}{9} < c < \frac{1}{4}$,*

$$(\mu + A_2)^{-1} L^2 \subset C^{0,\gamma}, \quad \gamma = \frac{1}{2}.$$

For all $d \geq 4$, $-\frac{1}{2 + \frac{2}{d-3}} < c < 2(d-1)(d-3)$ and $q > d-2$ sufficiently close to $d-2$,

$$(\mu + A_q)^{-1} L^q \subset C^{0,\gamma}, \quad \gamma = 1 - \frac{d-2}{q}.$$

Theorem 2 ($-\nabla\cdot a\cdot\nabla+b\cdot\nabla$)**.** *Let $d\geq 3$, $a(x)=I+c|x|^{-2}x\otimes x$, $c>-1$, $b\in\mathbf{F}_\delta$.*

(i) If $\delta_1:=[1\vee(1+c)^{-2}]\,\delta<4$, then $-\nabla\cdot a\cdot\nabla+b\cdot\nabla$ has an operator realization $\Lambda_q(a,b)$ in L^q, $q\in\big[\frac{2}{2-\sqrt{\delta_1}},\infty\big[$, as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup.

(ii) Let $d\geq 4$. Assume that $q\geq d-2$, $\delta<1\wedge\frac{4}{(d-2)^2}$ and c satisfy one of the following two conditions:

1) *$c>0$ and $1+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\geq 0$, and*

$$(q-1)\frac{(d-2)^2}{q^2}-Ł_1(c,\delta)>0,$$

2) *$-1<c<0$ and $1+c\big(1+\frac{q\sqrt{\delta}}{4}\big)\geq 0$, and*

$$(q-1)\frac{(d-2)^2}{q^2}-Ł_2(-c,\delta)>0,$$

where

$$\begin{aligned}Ł_1(c,\delta)\equiv Ł_1(c,\delta,q,d):=c\Big[1+\frac{q-2}{q}(d-2)-\Big(q-1-\frac{1}{2(d-1)}\Big)\frac{(d-2)^2}{q^2}\Big]\\+\frac{c\sqrt{\delta}}{2}\Big[\frac{(d-2)^2}{q}+(d+3)(d-2)\Big]+\Big[\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\Big]\frac{(d-2)^2}{q^2},\end{aligned}$$

$$\begin{aligned}Ł_2(c,\delta)\equiv Ł_2(c,\delta,q,d):=c\Big[-d+1+\frac{q-2}{q}(d-2)+(q-1)\frac{(d-2)^2}{q^2}\Big]\\+\frac{c\sqrt{\delta}}{2}\Big[\frac{(d-2)^2}{q}+(d+3)(d-2)\Big]+\Big[\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\Big]\frac{(d-2)^2}{q^2}.\end{aligned}$$

Then there exist constants $\mu_0=\mu_0(d,q,c,\delta)>0$ and $K_l=K_l(d,q,c,\delta)$, $l=1,2$, such that for all $\mu>\mu_0$, $u:=(\mu+\Lambda_q(a,b))^{-1}f$, $f\in L^q$, is in $W^{1,q}\cap W^{1,\frac{qd}{d-2}}$, and

$$\begin{aligned}\|\nabla u\|_q\leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q,\\ \|\nabla u\|_{\frac{qd}{d-2}}\leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q.\end{aligned}\tag{$\star\star$}$$

(iii) Let $d\geq 3$. Assume that $\delta<1\vee(1+c)^{-2}$ and

$$c>0,\qquad 1-\frac{4c}{(d-2)^2}-c\sqrt{\delta}\Big(2\frac{d+3}{d-2}+1\Big)-\delta>0,$$

or

$$-1<c<0,\qquad 1-|c|-|c|\frac{4(d-1)}{(d-2)^2}-|c|\sqrt{\delta}\Big(2\frac{d+3}{d-2}+1\Big)-\delta>0.$$

Then $u\in W^{2,2}$ and ($\star\star$) holds with $q=2$.

Corollary 2. *Let $d\geq 3$, $a(x)=I+c|x|^{-2}x\otimes x$, $c>-1$, $b\in\mathbf{F}_\delta$. If*

$$\begin{cases}-\frac{1}{2+\frac{2}{d-3}}<c<2(d-1)(d-3), & d\geq 4,\\ -\frac{1}{9}<c<\frac{1}{4}, & d=3,\end{cases}\qquad \text{and } \delta>0 \text{ is sufficiently small}$$

or

$$|c| \text{ is sufficiently small and } \delta<1\wedge\frac{4}{(d-2)^2},$$

then, for all $d \geq 4$ and $q > d-2$ sufficiently close to $d-2$,

$$(\mu + \Lambda_q(a,b))^{-1} L^q \subset C^{0,\gamma}, \gamma = 1 - \frac{d-2}{q};$$

and, for $d = 3$,

$$(\mu + \Lambda_2(a,b))^{-1} L^2 \subset C^{0,\gamma} \quad \gamma = \frac{1}{2}.$$

REMARK. In Theorem 2, if $\delta = 0$, then the assumptions on q and c coincide with the ones in Theorem 1. On the other hand, if $c = 0$, then the assumptions on δ are reduced to $\delta < 1 \wedge \frac{4}{(d-2)^2}$, so we recover the result in [KS, Lemma 5], [KiS, Theorem 3.7].

2. Next, we consider the non-divergence form operator.

Theorem 3 $(-a \cdot \nabla^2)$**.** *Let $d \geq 3$, $a(x) = I + c|x|^{-2} x \otimes x$, $c > -1$.*

(i) $-a \cdot \nabla^2$ has an operator realization $\Lambda_q(a, \nabla a)$ in L^q, $q \in \big[(1 - \frac{d-1}{d-2}\frac{c}{1+c})^{-1}, \infty\big[$ if $0 < c < d-2$, and $q \in]1, \infty[$ if $-1 < c < 0$, as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup.

Set $u := (\mu + \Lambda_q(a, \nabla a))^{-1} f$, $\mu > 0$, $f \in L^q$.

(ii) Let $d \geq 4$. Assume that $q > d-2$ and

$$-\left(1 + \frac{1}{4}\frac{q}{d-2}\frac{(q-2)^2}{(q-1)(q+d-3)}\right)^{-1} < c < \frac{d-3}{2} \wedge \frac{d-2}{q-d+2}.$$

Then $u \in W^{1,q} \cap W^{1,\frac{qd}{d-2}}$, and there exist constants $K_l = K_l(d,q,c)$, $l = 1,2$, such that $(\star)$ holds (for $u = (\mu + \Lambda_q(a, \nabla a))^{-1} f$).

(iii) Let $d \geq 3$ and $q = 2$. Assume that $-1 < c < \frac{(d-2)^2}{2(2+(d-2)(d-3))}$. Then $u \in W^{2,2}$.

Corollary 3. *(i) Let $d \geq 4$. For all $c \in]0, \frac{d-3}{2}[$ and $q \in]d-2, d + \frac{2}{d-3}[$, or for all $c \in]-\frac{1}{1+\frac{1}{4}\frac{(d-4)^2}{(d-3)(2d-5)}}, 0[$ and $q > d-2$ sufficiently close to $d-2$,*

$$(\mu + \Lambda_q(a, \nabla a))^{-1} L^q \subset C^{0,\gamma}, \quad \gamma = 1 - \frac{d-2}{q}.$$

(ii) For $d = 3$ and all $c \in]-1, \frac{1}{3}[$,

$$(\mu + \Lambda_2(a, \nabla a))^{-1} L^2 \subset C^{0,\gamma}, \quad \gamma = \frac{1}{2}.$$

REMARK. Set $a^\varepsilon := I + c|x|_\varepsilon^{-2} x \otimes x$, $|x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}$, $\varepsilon > 0$. Let $d \geq 4$. Then, in the assumptions of Theorem 3, we have

$$(\mu + \Lambda_q(a, \nabla a))^{-1} = s\text{-}L^q\text{-}\lim_{\varepsilon \downarrow 0} (\mu + \Lambda_q(a^\varepsilon, \nabla a^\varepsilon))^{-1}. \tag{3}$$

See Theorem A.2 for details. In particular, $u := (\mu + \Lambda_q(a, \nabla a))^{-1} f$, $f \in L^q$, is a good solution of $(\mu - a \cdot \nabla^2) u = f$ in the sense of [CEF].

Theorem 4 $(-a \cdot \nabla^2 + b \cdot \nabla)$**.** *Let $d \geq 3$, $a(x) = I + c|x|^{-2} x \otimes x$, $-1 < c < d-2$, $b \in \mathbf{F}_\delta$.*

Set $\delta_1 := [1 \vee (1+c)^{-2}]\, \delta$, and

$$\sqrt{\delta_2} := \begin{cases} \sqrt{\delta_1} + 2\frac{d-1}{d-2}\frac{c}{1+c}, & 0 < c < d-2, \\ \sqrt{\delta_1}, & -1 < c < 0. \end{cases}$$

(i) Assume that $\delta_2 < 4$. *Then* $-a \cdot \nabla^2 + b \cdot \nabla$ *has an operator realization* $\Lambda_q(a, \nabla a + b)$ *in* L^q, $q \in [\frac{2}{2-\sqrt{\delta_2}}, \infty[$, *as the (minus) generator of a positivity preserving* L^∞ *contraction* C_0 *semigroup.*

(ii) Let $d \geq 4$. *Assume that* $q > d-2$, $\delta < 1 \wedge \frac{4}{(d-2)^2}$ *and* c *satisfy one of the following two conditions:*

1) $c > 0$ *and* $1 + c\big(1 - \frac{q}{d-2} - \frac{q\sqrt{\delta}}{4}\big) \geq 0$, *and*

$$(q-1)\frac{(d-2)^2}{q^2} - Ł_1^{\rm nd}(c,\delta) > 0,$$

$$Ł_1^{\rm nd}(c,\delta) \equiv Ł_1^{\rm nd}(c,\delta,q,d) := c(1+\theta)\frac{(d-2)^2}{q^2} + \frac{c\sqrt{\delta}}{2}\left[\frac{(d-2)^2}{q} + (d+3)(d-2)\right] + \left[\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}, \qquad \theta = \frac{q}{d-2}.$$

2) $-1 < c < 0$ *and* $1 + c\big(1 + \frac{q\sqrt{\delta}}{4}\big) \geq 0$, *and*

$$(q-1)\frac{(d-2)^2}{q^2} - Ł_2^{\rm nd}(-c,\delta) > 0,$$

$$Ł_2^{\rm nd}(c,\delta) \equiv Ł_2^{\rm nd}(c,\delta,q,d) := c\Big[1 + (q-2)(1+\theta)\Big]\frac{(d-2)^2}{q^2} + \frac{c\sqrt{\delta}}{2}\left[\frac{(d-2)^2}{q} + (d+3)(d-2)\right] + \left[\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}, \qquad \theta := \frac{1}{4}\frac{q}{d-2}\frac{q-2}{q+d-3}.$$

Then there exist constants $\mu_0 = \mu_0(d,q,c,\delta) > 0$ *and* $K_l = K_l(d,q,c,\delta)$, $l = 1,2$, *such that* $u := (\mu + \Lambda_q(a, \nabla a + b))^{-1} f$, $f \in L^q$, *is in* $W^{1,q} \cap W^{1,\frac{qd}{d-2}}$ *for all* $\mu > \mu_0$, *and* $(\star\star)$ *hold.*

(iii) Let $d \geq 3$ and $q = 2$. Assume that

$$c > 0, \quad \sqrt{\delta} + 2\frac{d-1}{d-2}\frac{c}{1+c} < 1, \quad 1 - \frac{4c}{(d-2)^2}\left(1 + \frac{(d-2)(d-3)}{2}\right) - c\sqrt{\delta}\left(2\frac{d+3}{d-2} + 1\right) - \delta > 0$$

or

$$-1 < c < 0, \quad \delta < (1+|c|)^{-2}, \qquad 1 - |c| - |c|\sqrt{\delta}\left(2\frac{d+3}{d-2} + 1\right) - \delta > 0.$$

Then $u \in W^{2,2}$.

Corollary 4. *Let* $d \geq 3$, $a(x) = I + c|x|^{-2} x \otimes x$, $b \in \mathbf{F}_\delta$. *Assume that*

$$\begin{cases} -\frac{1}{1+\frac{1}{4}\frac{(d-4)^2}{(d-3)(2d-5)}} < c < \frac{d-3}{2}, & d \geq 4, \\ -1 < c < \frac{1}{3}, & d = 3. \end{cases} \qquad \text{and } \delta > 0 \text{ is sufficiently small,}$$

or

$$|c| \text{ is sufficiently small and } \delta < 1 \wedge \frac{4}{(d-2)^2}.$$

Let $d \geq 4$. *Then, for all* $q \in]d-2, d + \frac{2}{d-3}[$ *in case of positive* c, *and for a* $q > d-2$ *sufficiently close to* $d-2$ *in case of negative* c, *we have*

$$(\mu + \Lambda_q(a, \nabla a + b))^{-1} L^q \subset C^{0,\gamma}, \quad \gamma = 1 - \frac{d-2}{q}$$

Let $d=3$. Then

$$(\mu+\Lambda_2(a,\nabla a+b))^{-1}L^2\subset C^{0,\gamma},\;\;\gamma=\frac{1}{2}.$$

In conclusion, we mention that in Theorems 2-4 we tried to find optimal constraints on c and δ such that ($\star$), ($\star\star$) hold. The weaker result that there exist sufficiently small c and δ such that ($\star$), ($\star\star$) are valid (still not accessible by the existing results prior to our work) can be obtained with considerably less effort.

We have included Appendix A to make the paper self-contained.

1. Proof of Theorem 1

In what follows, we use notation

$$\langle h\rangle:=\int_{\mathbb{R}^d}h(x)dx,\quad\langle h,g\rangle:=\langle h\bar{g}\rangle.$$

Define $t[u,v]:=\langle\nabla u\cdot a\cdot\nabla\bar{v}\rangle$, $D(t)=W^{1,2}$. There is a unique self-adjoint operator $A\equiv A_2\geq 0$ on L^2 associated with the form t: $D(A)\subset D(t)$, $\langle Au,v\rangle=t[u,v]$, $u\in D(A)$, $v\in D(t)$. $-A$ is the generator of a positivity preserving L^∞ contraction C_0 semigroup $T_2^t\equiv e^{-tA}$, $t\geq 0$, on L^2.

By interpolation, $T_r^t:=\big[T_2^t\upharpoonright_{L^r\cap L^2}\big]^{\text{clos}}_{L^r\to L^r}$ determines a C_0 semigroup on L^r for all $r\in[2,\infty[$ and hence, by self-adjointness, for all $r\in]1,\infty[$. The (minus) generator A_r of $T_r^t\,(\equiv e^{-tA_r})$ is the desired operator realization of $\nabla\cdot a\cdot\nabla$ on L^r, $r\in]1,\infty[$. One can furthermore show that $T_1^t:=\big[T_2^t\upharpoonright_{L^1\cap L^2}\big]^{\text{clos}}_{L^1\to L^1}$ is a C_0 semigroup. This completes the proof of the assertion (i) of the theorem.

To prove (ii), we will need the following notation and auxiliary results. Define the smoothed out matrices $a^\varepsilon=(a^\varepsilon_{ij})$, $1\leq i,j\leq d$, $\varepsilon>0$ by

$$a^\varepsilon_{ij}:=\delta_{ij}+c|x|_\varepsilon^{-2}x_ix_j,\quad |x|_\varepsilon:=\sqrt{|x|^2+\varepsilon}.$$

Set $u\equiv u^\varepsilon=(\mu+A^\varepsilon_q)^{-1}f$, $A^\varepsilon_q:=A_q(a^\varepsilon)$, $0\leq f\in C^1_c$. Clearly $a^\varepsilon\in C^\infty$ and $0\leq u^\varepsilon\in W^{3,q}$. Denote $w\equiv w^\varepsilon:=\nabla u^\varepsilon$,

$$I_q:=\sum_{r=1}^d\langle(\nabla_r w)^2|w|^{q-2}\rangle,\quad J_q:=\langle(\nabla|w|)^2|w|^{q-2}\rangle,$$

$$\bar{I}_{q,\chi}:=\langle\big(x\cdot\nabla w\big)^2\chi|x|^{-2}|w|^{q-2}\rangle,\quad \bar{J}_{q,\chi}:=\langle(x\cdot\nabla|w|)^2\chi|x|^{-2}|w|^{q-2}\rangle,\qquad \chi:=|x|^2|x|_\varepsilon^{-2},$$

$$H_{q,\chi}:=\langle\chi|x|^{-2}|w|^q\rangle,\quad H_{q,\chi^2}:=\langle\chi^2|x|^{-2}|w|^q\rangle,\quad G_{q,\chi^2}:=\langle\chi^2|x|^{-4}(x\cdot w)^2|w|^{q-2}\rangle.$$

1. The following inequality plays an important role in the proof of Theorem 1.

Lemma 1 (Hardy-type inequality)**.**

$$\frac{d^2}{4}H_{q,\chi}-(d+2)H_{q,\chi^2}+3H_{q,\chi^3}\leq\frac{q^2}{4}\bar{J}_{q,\chi}\qquad\textbf{(HI)}$$

Proof. Set $F:=|x|_\varepsilon^{-1}|w|^{\frac{q}{2}}$. Then

$$\frac{q^2}{4}\bar{J}_{q,\chi}=\big\langle\big(|x|_\varepsilon^{-1}x\cdot\nabla|w|^{\frac{q}{2}}\big)^2\big\rangle=\big\langle(x\cdot\nabla F+\chi F)^2\big\rangle=\langle(x\cdot\nabla F)^2+\chi^2F^2\rangle+2\langle x\cdot\nabla F,\chi F\rangle.$$

(**HI**) follows from the inequality $\langle(x\cdot\nabla F)^2\rangle\equiv\|x\cdot\nabla F\|_2^2\geq\frac{d^2}{4}\|F\|_2^2\equiv\frac{d^2}{4}H_{q,\chi}$ and the equalities $2\langle x\cdot\nabla F,\chi F\rangle=-d\langle\chi F^2\rangle-\langle F^2,x\cdot\nabla\chi\rangle$, $x\cdot\nabla\chi=2\big(\frac{|x|^2}{|x|^2_\varepsilon}-\frac{|x|^4}{|x|^4_\varepsilon}\big)=2\chi(1-\chi)$. □

The following equalities are crucial steps in the proof of Theorem 1.

Lemma 2 (The basic equalities)**.**

$$\mu\langle |w|^q\rangle + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\biggl(1+(q-2)\frac{d}{q}\biggr)H_{q,\chi} + 2c(d-1)G_{q,\chi^2}$$
$$+ 2c\frac{q-2}{q}H_{q,\chi^2} + 8c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle = \beta_1 + \langle f,\phi\rangle, \qquad (\mathrm{BE}_+)$$

$$\mu\langle |w|^q\rangle + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\biggl(1+(q-2)\frac{d}{q}\biggr)H_{q,\chi} + cdG_{q,\chi^2}$$
$$+ 2c\frac{q-2}{q}H_{q,\chi^2} + 4c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle = -\frac{1}{2}\beta_2 + \langle f,\phi\rangle, \qquad (\mathrm{BE}_-)$$

where $\phi = -\nabla\cdot(w|w|^{q-2})$,

$$\beta_1 := -2c\langle |x|_\varepsilon^{-4}x\cdot w, x\cdot(x\cdot\nabla w)|w|^{q-2}\rangle, \qquad \beta_2 := -2c(q-2)\langle |x|_\varepsilon^{-4}(x\cdot w)^2 x\cdot\nabla|w|, |w|^{q-3}\rangle.$$

Remark. Below we use the representation (BE_+) in case $c>0$, and the representation (BE_-) in case $c<0$. (One could still use (BE_+) for $c<0$ or (BE_-) for $c>0$, but this would lead to more restrictive constraints on c.)

Proof of Lemma 2. Set $[F,G]_- := FG-GF$. We multiply $\mu u + A_q^\varepsilon u = f$ by ϕ and integrate:

$$\mu\langle |w|^q\rangle + \langle A_q^\varepsilon w, w|w|^{q-2}\rangle + \langle [\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle = \langle f,\phi\rangle,$$
$$\mu\langle |w|^q\rangle + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) + \langle [\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle = \langle f,\phi\rangle. \qquad (4)$$

The term to evaluate: $\langle [\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle \equiv \langle [\nabla_r, A_q^\varepsilon]_- u, w_r|w|^{q-2}\rangle := \sum_{r=1}^d \langle [\nabla_r, A_q^\varepsilon]_- u, w_r|w|^{q-2}\rangle$. Note that

$$[\nabla_r, A_q^\varepsilon]_- = -\nabla\cdot(\nabla_r a^\varepsilon)\cdot\nabla, \qquad (\nabla_r a^\varepsilon)_{ik} = c|x|_\varepsilon^{-2}\delta_{ri}x_k + c(|x|_\varepsilon^{-2}\delta_{rk}x_i - 2|x|_\varepsilon^{-4}x_i x_k x_r),$$

$$\langle [\nabla_r, A_q^\varepsilon]_- u, w_r|w|^{q-2}\rangle$$
$$= -c\langle w_k\nabla_i(|x|_\varepsilon^{-2}\delta_{ri}x_k) + |x|_\varepsilon^{-2}\delta_{ri}x_k\nabla_i w_k, w_r|w|^{q-2}\rangle + c\langle (|x|_\varepsilon^{-2}\delta_{rk}x_i - 2|x|_\varepsilon^{-4}x_i x_k x_r)w_k, \nabla_i(w_r|w|^{q-2})\rangle$$
$$=: \alpha_1 + \alpha_2,$$

$$\alpha_1 = -c\langle (|x|_\varepsilon^{-2}\delta_{rk} - 2|x|_\varepsilon^{-4}\delta_{ri}x_k x_r)w_k + |x|_\varepsilon^{-2}x\cdot\nabla w_r, w_r|w|^{q-2}\rangle$$
$$= -c\langle |x|_\varepsilon^{-2}|w|^q\rangle + 2c\langle |x|_\varepsilon^{-4}(x\cdot w)^2|w|^{q-2}\rangle - c\langle |x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle.$$

Then

$$\alpha_1 = -c\biggl(1-\frac{d-2}{q}\biggr)H_{q,\chi} + 2cG_{q,\chi^2} + 2\frac{c}{q}\varepsilon\langle |x|_\varepsilon^{-4}|w|^q\rangle$$

due to $\langle |x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle = \frac{1}{q}\langle |x|_\varepsilon^{-2}x\cdot\nabla|w|^q\rangle = -\frac{1}{q}\langle |w|^q\nabla\cdot(x|x|_\varepsilon^{-2})\rangle = -\frac{d}{q}H_{q,\chi} + \frac{2}{q}\langle |x|^2|x|_\varepsilon^{-4}|w|^q\rangle = -\frac{d-2}{q}H_{q,\chi} - \frac{2}{q}\varepsilon\langle |x|_\varepsilon^{-4}|w|^q\rangle$, and

$$\alpha_2 = c\langle |x|_\varepsilon^{-2}w, x\cdot\nabla(w|w|^{q-2})\rangle - 2c\langle |x|_\varepsilon^{-4}x\cdot w, x\cdot(x\cdot\nabla(w|w|^{q-2}))\rangle.$$

Then

$$\alpha_2 = \beta_1 + \beta_2 + c\langle |x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle + c(q-2)\langle |x|_\varepsilon^{-2}x\cdot\nabla|w|, |w|^{q-1}\rangle$$
$$= \beta_1 + \beta_2 + c(q-1)\biggl(\frac{d-2}{q}H_{q,\chi} + \frac{2}{q}\varepsilon\langle |x|_\varepsilon^{-4}, |w|^q\rangle\biggr).$$

In view of

$$\beta_1 = -\frac{1}{2}\beta_2 + c(d-2)G_{q,\chi^2} + 4c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle,$$

we rewrite $\alpha_1+\alpha_2 = \langle [\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle$ in two ways:

$$\langle [\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle = -\beta_1 - c\Big(1+(q-2)\frac{d-2}{q}\Big)H_{q,\chi} + 2c(d-1)G_{q,\chi^2}$$
$$-2c\frac{q-2}{q}\varepsilon\langle |x|_\varepsilon^{-4}|w|^q\rangle + 8c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle$$

and

$$\langle [\nabla, A_q^\varepsilon]_- u, w|w|^{q-2}\rangle = \frac{1}{2}\beta_2 - c\Big(1+(q-2)\frac{d-2}{q}\Big)H_{q,\chi} + cdG_{q,\chi^2}$$
$$-2c\frac{q-2}{q}\varepsilon\langle |x|_\varepsilon^{-4}|w|^q\rangle + 4c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle.$$

The last two identities applied in (4) yield (BE_+), (BE_-). □

2. Next, we estimate from above the term $\langle f,\phi\rangle$ in the right-hand side of (BE_+), (BE_-).

Lemma 3. *For each $\varepsilon_0>0$ there exists a constant $C(\varepsilon_0)<\infty$ such that*

$$\langle f,\phi\rangle \leq \varepsilon_0(I_q+J_q+H_q) + C(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2,$$

where $H_q := \langle |x|^{-2}|w|^q\rangle$.

Proof of Lemma 3. Clearly,

$$\langle f,\phi\rangle = \langle f, (-\Delta u)|w|^{q-2}\rangle - (q-2)\langle f, |w|^{q-3}w\cdot\nabla|w|\rangle =: F_1+F_2.$$

Since $-\Delta u = \nabla\cdot(a^\varepsilon-1)\cdot w - \mu u + f$ and

$$F_1 = \langle \nabla\cdot(a^\varepsilon-1)\cdot w, |w|^{q-2}f\rangle + \langle(-\mu u+f), |w|^{q-2}f\rangle$$
(we expand the first term using $\nabla a^\varepsilon = c(d+1)x|x|_\varepsilon^{-2} - 2c|x|^2|x|_\varepsilon^{-4}x$)
$$= c(d+1)\langle |x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}f\rangle - 2c\langle \chi|x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}(-b\cdot w+f)\rangle$$
$$+ c\langle |x|_\varepsilon^{-2}x\cdot(x\cdot\nabla w), |w|^{q-2}f\rangle + \langle(-\mu u+f), |w|^{q-2}f\rangle.$$

We bound from above F_1 and F_2 by applying consecutively the following estimates:

1) $\langle |x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}f\rangle \leq H_q^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.
2) $\langle \chi|x|_\varepsilon^{-2}x\cdot w, |w|^{q-2}f\rangle \leq H_q^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.
3) $\langle |x|_\varepsilon^{-2}x\cdot(x\cdot\nabla w), |w|^{q-2}f\rangle \leq (\bar{I}_{q,\chi})^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.
4) $\langle -f, |w|^{q-2}\mu u\rangle \leq 0$.
5) $\langle f, |w|^{q-2}f\rangle \leq \|w\|_q^{q-2}\|f\|_q^2$.
6) $(q-2)\langle -f, |w|^{q-3}w\cdot\nabla|w|\rangle \leq (q-2)J_q^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.

1)-6) and the standard quadratic estimates now yield the lemma. □

We choose $\varepsilon_0 > 0$ in Lemma 3 so small that in the estimates below we can ignore $\varepsilon_0(I_q + J_q + H_q)$.

3. We will use (BE$_+$), (BE$_-$) and Lemma 3 to prove the following inequality

$$\mu\langle |w|^q\rangle + \eta J_q \leq C\|w\|_q^{q-2}\|f\|_q^2, \qquad C = C(\varepsilon_0) \tag{5}$$

for some $\eta = \eta(q, d, \varepsilon_0) > 0$.

Case $c > 0$**.** In (BE$_+$) we omit the term $8c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle$, obtaining

$$\mu\langle |w|^q\rangle + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\Big(1 + \frac{q-2}{q}d\Big)H_{q,\chi} + 2c(d-1)G_{q,\chi^2}$$
$$+ 2c\frac{q-2}{q}H_{q,\chi^2} \leq \beta_1 + \langle f, \phi\rangle.$$

Estimating β_1 from above using the standard quadratic estimates,

$$\beta_1 \leq 2c\langle |x|_\varepsilon^{-4}(x\cdot w)^2|w|^{q-2}\rangle^{\frac{1}{2}}\langle |x|_\varepsilon^{-4}|x|^2(x\cdot\nabla w)^2|w|^{q-2}\rangle^{\frac{1}{2}} \leq 2c(G_{q,\chi^2}\bar{I}_{q,\chi})^{\frac{1}{2}} \leq c\theta\bar{I}_{q,\chi} + c\theta^{-1}G_{q,\chi^2}$$

($\theta > 0$), and then applying Lemma 3, we have

$$\mu\langle |w|^q\rangle + I_q + c(1-\theta)\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\Big(1 + \frac{q-2}{q}d\Big)H_{q,\chi} + c\big(2(d-1) - \theta^{-1}\big)G_{q,\chi^2}$$
$$+ 2c\frac{q-2}{q}H_{q,\chi^2} \leq C\|w\|_q^{q-2}\|f\|_q^2.$$

Let $0 < \theta < 1$. Using the inequalities $J_q \leq I_q$, $\bar{J}_{q,\chi} \leq \bar{I}_{q,\chi}$ and $\frac{4}{q^2}\left(\frac{d^2}{4}H_{q,\chi} - (d+2)H_{q,\chi^2} + 3H_{q,\chi^3}\right) \leq \bar{J}_{q,\chi}$, see (**HI**), we have

$$\mu\langle |w|^q\rangle + \eta J_q + (-\eta + q - 1)J_q + \big[2c(d-1) - c\theta^{-1}\big]G_{q,\chi^2} + c\langle M(\chi)|x|^{-2}|w|^q\rangle \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$M(\chi) := \left[(q-1-\theta)\frac{4}{q^2}\Big(\frac{d^2}{4} - (d+2)\chi + 3\chi^2\Big) - \Big(1 + \frac{q-2}{q}d\Big) + 2\frac{q-2}{q}\chi\right]\chi,$$

i.e.

$$M(\chi) := [\mathfrak{a}\chi^2 + \mathfrak{b}\chi + \mathfrak{c}_0]\chi,$$

where

$$\mathfrak{a} = \frac{12}{q^2}(q-1-\theta), \quad \mathfrak{b} = -4(q-1-\theta)\frac{d+2}{q^2} + 2\frac{q-2}{q}, \quad \mathfrak{c}_0 = \frac{d^2}{q^2}(q-1-\theta) + \frac{2d}{q} - 1 - d.$$

Elementary arguments show that the choice $\theta := \frac{1}{2(d-1)}$ is the best possible. In particular,

$$\min_{0\leq t\leq 1} M(t) = M(1) < 0.$$

Since $-\eta + q - 1 > 0$ for all $\eta > 0$ sufficiently small, we can use $J_q \geq \frac{(d-2)^2}{q^2}H_q$, obtaining

$$\mu\langle |w|^q\rangle + \eta J_q + \left((-\eta + q - 1)\frac{(d-2)^2}{q^2} + cM(1)\right)H_q \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

Recalling the assumption $(q-1)\frac{(d-2)^2}{q^2} - c\ell_1 > 0$, $\ell_1 = -M(1)$, it is seen that there exists $\eta > 0$ such that $(-\eta + q - 1)\frac{(d-2)^2}{q^2} \geq c\ell_1$. (5) is proved for $0 < c < \frac{(q-1)(d-2)^2}{q^2\ell_1}$.

The choice of $\theta \in [1, 1 + c^{-1}]$ leads to sub-optimal constraints on c and q.

Case $-1 < c < 0$**.** Set $s := |c|$. In (BE_-), we estimate ($\theta > 0$)

$$|\beta_2| \leq 2s(q-2)\big(\theta \bar{J}_{q,\chi} + 4^{-1}\theta^{-1} G_{q,\chi^2}\big).$$

By (BE_-) and Lemma 3,

$$\mu\langle |w|^q\rangle + I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) + s\left(1 + (q-2)\frac{d}{q}\right) H_{q,\chi}$$
$$- 2s\frac{q-2}{q} H_{q,\chi^2} - sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^3} - s(q-2)\frac{1}{4\theta} G_{q,\chi^2} \leq C\|w\|_q^{q-2}\|f\|_q^2.$$

Clearly, $I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) \geq (q-1-s-s(q-2)(1+\theta))J_q$. Therefore

$$\mu\langle |w|^q\rangle + \big(q-1-s-s(q-2)(1+\theta)\big)J_q + s\left(1 + (q-2)\frac{d}{q}\right) H_{q,\chi}$$
$$- 2s\frac{q-2}{q} H_{q,\chi^2} - sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^3} - s(q-2)\frac{1}{4\theta} G_{q,\chi^2} \leq C\|w\|_q^{q-2}\|f\|_q^2.$$

Using $H_{q,\chi} \geq H_{q,\chi^2}$ and $G_{q,\chi} \leq H_{q,\chi}$, we obtain

$$\mu\langle |w|^q\rangle + \big(q-1-s-s(q-2)(1+\theta)\big)J_q + s\left(1 + (q-2)\frac{d-2}{q}\right) G_{q,\chi}$$
$$- sdG_{q,\chi^2} - 4sG_{q,\chi^2} + 4sG_{q,\chi^3} - s(q-2)\frac{1}{4\theta} G_{q,\chi^2} \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

i.e.

$$\mu\langle |w|^q\rangle + \eta J_q + \big(-\eta + q-1-s-s(q-2)(1+\theta)\big)J_q + s\langle M(\chi)|x|^{-4}(x\cdot w)^2|w|^{q-2}\rangle \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$M(\chi) := \left[1 + (q-2)\frac{d-2}{q} + \left(-d-4+4\chi - (q-2)\frac{1}{4\theta}\right)\chi\right]\chi,$$

i.e.

$$M(\chi) = [\mathfrak{a}\chi^2 + \mathfrak{b}\chi + \mathfrak{c}_0]\chi,$$

where

$$\mathfrak{a} = 4, \quad \mathfrak{b} = -d-4-\frac{1}{2}(q-2)\frac{d-2}{q}, \quad \mathfrak{c}_0 = 1 + (q-2)\frac{d-2}{q}.$$

Select $\theta := \frac{1}{2}\frac{q}{d-2}$. (Motivation: Below we estimate $I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) \geq (q-1-s-s(q-2)(1+\theta))J_q \geq (q-1-s-s(q-2)(1+\theta))\frac{(d-2)^2}{q^2}G_q$, so estimating the terms involving θ in the resulting inequality as $-s(q-2)\theta\frac{(d-2)^2}{q^2}G_q - (q-2)\frac{1}{4\theta}G_{q,\chi^2} \geq \big(-s(q-2)\theta\frac{(d-2)^2}{q^2} - (q-2)\frac{1}{4\theta}\big)G_q$, we arrive clearly at $\theta = \frac{1}{2}\frac{q}{d-2}$.)

Elementary arguments show that $\min_{0\leq t\leq 1} M(t) = M(1) < 0$. By the assumptions of the theorem,

$$(-\eta + q-1-s-s(q-2)(1+\theta))\frac{(d-2)^2}{q^2} + sM(1) \geq 0.$$

Thus, by $J_q \geq \frac{(d-2)^2}{q^2}H_q$ and $H_q \geq G_q$,

$$\mu\langle |w|^q\rangle + \eta J_q + \big[(-\eta + q-1-s-s(q-2)(1+\theta))\frac{(d-2)^2}{q^2} + sM(1)\big]G_q \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

or, setting $\ell_2 := [1+(q-2)(1+\theta)]\frac{(d-2)^2}{q^2} - M(1)$,

$$\mu\langle|w|^q\rangle + \eta J_q + \left[(-\eta+q-1)\frac{(d-2)^2}{q^2} + c\ell_2\right] G_q \leq C\|w\|_q^{q-2}\|f\|_q^2.$$

(5) is proved.

4. By (5), $\mu\|w\|_q^q \leq C\|w\|_q^{q-2}\|f\|_q^2$, $w = \nabla u^\varepsilon$, $\varepsilon > 0$, and so

$$\|\nabla u^\varepsilon\|_q \leq K_1\mu^{-\frac{1}{2}}\|f\|_q, \quad K_1 := C^{\frac{1}{2}}.$$

Again by (5), $\eta J_q \leq C\|w\|_q^{q-2}\|f\|_q^2$, $J_q = \frac{4}{q^2}\|\nabla|w|^{\frac{q}{2}}\|_2$, so in view of the previous inequality $\eta\|\nabla|\nabla u^\varepsilon|^{\frac{q}{2}}\|_2 \leq \frac{q^2}{4}CK_1^{q-2}\mu^{1-\frac{q}{2}}\|f\|_q^q$. The Sobolev Embedding Theorem now yields

$$\|\nabla u^\varepsilon\|_{qj} \leq K_2\mu^{\frac{1}{q}-\frac{1}{2}}\|f\|_q, \quad K_2 := C_S\eta^{-\frac{1}{q}}(q^2/4)^{\frac{1}{q}}C^{\frac{1}{q}}K_1^{\frac{q-2}{q}}.$$

Since the weak gradient in L^q is closed, Theorem A.2(i) (with $b = 0$) yields $\|\nabla u\|_q \leq K_1\mu^{-\frac{1}{2}}\|f\|_q$, $\|\nabla u\|_{qj} \leq K_2\mu^{\frac{1}{q}-\frac{1}{2}}\|f\|_q$ for $u = (\mu + A_q)^{-1}f$, $0 \leq f \in C_c^\infty$, and thus for all $f \in L^q$.

We have proved (ii).

Proof of (iii). Let $q = 2$, $d \geq 3$. The arguments above yield ($I_2 \equiv I_2(u^\varepsilon)$)

(a) For $c > 0$,

$$\begin{aligned}
&\langle[\nabla, A_2^\varepsilon]_- u, w\rangle = -\beta_1 - cH_{2,\chi} + 2c(d-1)G_{2,\chi^2},\\
&\mu\|w\|_2^2 + I_2 + c\bar{I}_{2,\chi} - \beta_1 - cH_{2,\chi} + 2c(d-1)G_{2,\chi^2} = \langle f, -\nabla\cdot w\rangle\\
&\beta_1 = -2c\langle|x|^{-4}x\cdot w, x\cdot(x\cdot\nabla w)\rangle.
\end{aligned}$$

By $\beta_1 \leq 2c\sqrt{G_{2,\chi^2}\bar{I}_{2,\chi}} \leq c\bar{I}_{2,\chi} + cG_{2,\chi^2}$, $G_{2,\chi^2} \leq H_{2,\chi}$ and $I_2 \geq \frac{(d-2)^2}{4}H_{2,\chi}$, it follows that $1 - c\frac{4}{(d-2)^2} > 0 \Rightarrow I_2 \leq K\|f\|_2^2$;

(b) For $c < 0$,

$$\begin{aligned}
&\langle[\nabla, A_2^\varepsilon]_- u, w\rangle = \frac{1}{2}\beta_2 - cH_{2,\chi} + cdG_{2,\chi^2}, \quad \beta_2 = 0,\\
&\mu\|w\|_2^2 + I_2 + c\bar{I}_2 - cH_{2,\chi} + cdG_{2,\chi^2} = \langle f, -\nabla\cdot w\rangle, \quad I_2 \geq \bar{I}_2,\\
&\mu\|w\|_2^2 + (1-|c|)I_2 + |c|H_{2,\chi} - |c|dG_{2,\chi^2} \leq \langle f, -\nabla\cdot w\rangle.
\end{aligned}$$

Thus, by $G_{2,\chi^2} \leq H_{2,\chi} \leq \frac{4}{(d-2)^2}I_2$, we conclude that $1 - |c| + |c|(1-d)\frac{4}{(d-2)^2} > 0 \Rightarrow I_2(u^\varepsilon) \leq K\|f\|_2^2$.

By passing to the limit $\varepsilon \downarrow 0$, using Theorem A.2, we obtain $I_2(u) \leq K\|f\|_2$. Therefore, $u \in W^{2,2}$.

The proof of Theorem 1 is completed. □

2. Proof of Theorem 2

Proof of (i). A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2$), if $b_a^2 := b\cdot a^{-1}\cdot b \in L^1_{\rm loc}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that

$$\|b_a(\lambda + A)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta_1}.$$

It is easily seen that if $b \in \mathbf{F}_\delta$, then $b \in \mathbf{F}_{\delta_1}(A)$, where $\delta_1 := \delta$ if $c > 0$, and $\delta_1 := \delta(1+c)^{-2}$ if $-1 < c < 0$. By our assumption, $\delta_1 < 4$. Therefore, by [KiS, Theorem 3.2], $-\nabla\cdot a\cdot\nabla + b\cdot\nabla$ has an operator realization $\Lambda_q(a,b)$ in L^q, $q \in \left[\frac{2}{2-\sqrt{\delta_1}}, \infty\right[$, as the (minus) generator of a positivity preserving

L^∞ contraction quasi contraction C_0 semigroup. Moreover, $(\mu+\Lambda_q(a,b))^{-1}$ is well defined on L^q for all $\mu>\frac{\lambda\delta}{2(q-1)}$. This completes the proof of (i).

Proof of (ii). Set $a^\varepsilon := I + c|x|_\varepsilon^{-2} x\otimes x$, $|x|_\varepsilon := \sqrt{|x|^2+\varepsilon}$, $\varepsilon>0$. Put $A^\varepsilon = A(a^\varepsilon)$. It is clear that $b\in \mathbf{F}_{\delta_1}(A^\varepsilon)$ for all $\varepsilon>0$.

Let $\mathbf{1}_n$ denote the indicator of $\{x\in\mathbb{R}^d \mid |x|\leq n, |b(x)|\leq n\}$, and set $b_n := \gamma_{\epsilon_n} * \mathbf{1}_n b \in C^\infty$, where γ_ϵ is the K. Friedrichs mollifier, $\epsilon_n\downarrow 0$. Since our assumptions on δ and thus δ_1 involve strict inequalities only, we can select $\epsilon_n \downarrow 0$ so that $b_n\in\mathbf{F}_{\delta_1}(A^\varepsilon)$, $\varepsilon>0$, $n\geq 1$. Therefore, in view of the previous discussion, $(\mu+\Lambda_q(a^\varepsilon,b_n))^{-1}$ is well defined on L^q, $\mu>\frac{\lambda\delta}{2(q-1)}$, $\varepsilon>0$, $n\geq 1$. Here $\Lambda_q(a^\varepsilon,b_n) = -\nabla\cdot a^\varepsilon\cdot\nabla + b_n\cdot\nabla$, $D(\Lambda_q(a^\varepsilon,b_n)) = W^{2,q}$.

Define $0\leq u\equiv u^{\varepsilon,n} := (\mu+\Lambda_q(a^\varepsilon,b_n))^{-1} f$, $0\leq f\in C_c^1$. Then $u\in W^{3,q}$. Set $w\equiv w^{\varepsilon,n} := \nabla u^{\varepsilon,n}$ and

$$I_q := \langle (\nabla_r w)^2 |w|^{q-2}\rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2}\rangle,$$

$$\bar I_{q,\chi} := \langle \big(x\cdot\nabla w\big)^2 \chi |x|^{-2}|w|^{q-2}\rangle, \quad \bar J_{q,\chi} := \langle (x\cdot\nabla|w|)^2\chi|x|^{-2}|w|^{q-2}\rangle, \quad \chi = |x|^2|x|_\varepsilon^{-2},$$

$$H_{q,\chi} := \langle \chi|x|^{-2}|w|^q\rangle, \quad H_{q,\chi^2} := \langle \chi^2|x|^{-2}|w|^q\rangle, \quad G_{q,\chi^2} := \langle \chi^2|x|^{-4}(x\cdot w)^2|w|^{q-2}\rangle.$$

Below we follow closely the proof of Theorem 1.

1. We repeat the proof of Lemma 2, where in the right-hand side of (BE_+), (BE_-) we now get an extra term $\langle -b_n\cdot w,\phi\rangle$:

$$\begin{aligned}&\mu\langle |w|^q\rangle + I_q + c\bar I_{q,\chi} + (q-2)(J_q + c\bar J_{q,\chi}) - c\bigg(1+(q-2)\frac{d}{q}\bigg)H_{q,\chi} + 2c(d-1)G_{q,\chi^2}\\ &+2c\frac{q-2}{q}H_{q,\chi^2} + 8c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle = \beta_1 + \langle -b_n\cdot w,\phi\rangle + \langle f,\phi\rangle, \end{aligned} \qquad (\text{BE}_{+,b})$$

$$\begin{aligned}&\mu\langle |w|^q\rangle + I_q + c\bar I_{q,\chi} + (q-2)(J_q + c\bar J_{q,\chi}) - c\bigg(1+(q-2)\frac{d}{q}\bigg)H_{q,\chi} + cdG_{q,\chi^2}\\ &+2c\frac{q-2}{q}H_{q,\chi^2} + 4c\varepsilon\langle |x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle = -\frac{1}{2}\beta_2 + \langle -b_n\cdot w,\phi\rangle + \langle f,\phi\rangle, \end{aligned} \qquad (\text{BE}_{-,b})$$

where, recall,

$$\beta_1 := -2c\langle |x|_\varepsilon^{-4} x\cdot w, x\cdot(x\cdot\nabla w)|w|^{q-2}\rangle, \qquad \beta_2 := -2c(q-2)\langle |x|_\varepsilon^{-4}(x\cdot w)^2 x\cdot\nabla|w|, |w|^{q-3}\rangle.$$

We estimate $\langle -b_n\cdot w,\phi\rangle$ as follows.

Lemma 4. *There exist constants C_i $(i=1,2)$ such that*

$$\begin{aligned}&\langle -b_n\cdot w,\phi\rangle\\ &\leq |c|(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac12}J_q^{\frac12} + |c|\frac{q\sqrt{\delta}}{2}\bar I_{q,\chi}^{\frac12}J_q^{\frac12} + \bigg(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\bigg)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Proof of Lemma 4. For brevity, below $b\equiv b_n$. We have:

$$\begin{aligned}\langle -b\cdot w,\phi\rangle &= \langle -\Delta u, |w|^{q-2}(-b\cdot w)\rangle + (q-2)\langle |w|^{q-3}w\cdot\nabla|w|, -b\cdot w\rangle\\ &=: F_1 + F_2.\end{aligned}$$

Set $B_q := \langle |b\cdot w|^2|w|^{q-2}\rangle$. We have

$$F_2 \leq (q-2)B_q^{\frac12}J_q^{\frac12}.$$

Next, we bound F_1. We represent $-\Delta u = \nabla \cdot (a^\varepsilon - 1) \cdot w - \lambda u - b \cdot w + f$, and evaluate

$$
\begin{aligned}
F_1 &= \langle \nabla \cdot (a^\varepsilon - 1) \cdot w, |w|^{q-2}(-b \cdot w)\rangle + \langle (-\lambda u - b \cdot w + f), |w|^{q-2}(-b \cdot w)\rangle \\
&\quad (\text{we expand the first term using } \nabla a^\varepsilon = c(d+1)|x|_\varepsilon^{-2}x - 2c|x|^2|x|_\varepsilon^{-4}x) \\
&= c(d+1)\langle |x|_\varepsilon^{-2}x \cdot w, |w|^{q-2}(-b \cdot w)\rangle \\
&\quad - 2c\langle \chi |x|_\varepsilon^{-2} x \cdot w, |w|^{q-2}(-b \cdot w)\rangle \\
&\quad + c\langle |x|_\varepsilon^{-2} x \cdot (x \cdot \nabla w), |w|^{q-2}(-b \cdot w)\rangle \\
&\quad + \langle (-\lambda u - b \cdot w + f), |w|^{q-2}(-b \cdot w)\rangle.
\end{aligned}
$$

We bound F_1 from above by applying consecutively the following estimates:

1°) $\langle |x|_\varepsilon^{-2} x \cdot w, |w|^{q-2}(-b \cdot w)\rangle \leq G_{q,\chi^2}^{\frac{1}{2}} B_q^{\frac{1}{2}}$.

2°) $\langle \chi |x|_\varepsilon^{-2} x \cdot w, |w|^{q-2}(-b \cdot w)\rangle \leq G_{q,\chi^4}^{\frac{1}{2}} B_q^{\frac{1}{2}} \leq G_{q,\chi^2}^{\frac{1}{2}} B_q^{\frac{1}{2}}$.

3°) $\langle |x|_\varepsilon^{-2} x \cdot (x \cdot \nabla w), |w|^{q-2}(-b \cdot w)\rangle \leq \bar{I}_{q,\chi}^{\frac{1}{2}} B_q^{\frac{1}{2}}$.

4°) $\langle (-\lambda u), |w|^{q-2}(-b \cdot w)\rangle \leq \frac{\lambda}{\lambda - \omega_q} B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$ $\big(\text{here } \frac{2}{2-\sqrt{\delta}} < q \Rightarrow \|u_n\|_q \leq (\lambda - \omega_q)^{-1}\|f\|_q\big)$.

5°) $\langle b \cdot w, |w|^{q-2} b \cdot w\rangle = B_q$.

6°) $\langle f, |w|^{q-2}(-b \cdot w)\rangle| \leq B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q$.

In 4°) and 6° we estimate $B_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q \leq \varepsilon_0 B_q + \frac{1}{4\varepsilon_0}\|w\|_q^{q-2}\|f\|_q^2$ ($\varepsilon_0 > 0$).

Therefore,

$$
\begin{aligned}
&\langle -b \cdot w, \phi\rangle \\
&\leq |c|(d+3) G_{q,\chi^2}^{\frac{1}{2}} B_q^{\frac{1}{2}} + |c| \bar{I}_q^{\frac{1}{2}} B_q^{\frac{1}{2}} + B_q + (q-2) B_q^{\frac{1}{2}} J_q^{\frac{1}{2}} + \varepsilon_0 B_q + C_2(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2.
\end{aligned}
$$

It is easily seen that $b \in \mathbf{F}_\delta$ is equivalent to the inequality

$$
\langle b^2 |\varphi|^2\rangle \leq \delta\langle |\nabla \varphi|^2\rangle + \lambda\delta\langle |\varphi|^2\rangle, \quad \varphi \in W^{1,2}.
$$

Thus,

$$
B_q \leq \|b|w|^{\frac{q}{2}}\|_2^2 \leq \delta\|\nabla |w|^{\frac{q}{2}}\|_2^2 + \lambda\delta\|w\|_q^q = \frac{q^2\delta}{4} J_q + \lambda\delta\|w\|_q^q,
$$

and then selecting $\varepsilon_0 > 0$ sufficiently small, and noticing that the assumption on δ in the theorem is a strict inequality, we can and will ignore below the terms multiplied by ε_0. The proof of Lemma 4 is completed. □

2. We estimate $\langle f, \phi\rangle$ in $(\mathrm{BE}_{+,b})$, $(\mathrm{BE}_{-,b})$ by an evident analogue of Lemma 3:

$$
\langle f, \phi\rangle \leq \varepsilon_0(I_q + J_q + H_q + \|w\|_q^q) + C(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2
$$

(selecting $\varepsilon_0 > 0$ sufficiently small so that we will ignore below the terms multiplied by ε_0).

Applying Lemma 4 and the last inequality in $(\mathrm{BE}_{+,b})$, $(\mathrm{BE}_{-,b})$, and using $\beta_1 \leq c\theta\bar{I}_{q,\chi} + c\theta^{-1}G_{q,\chi^2}$, $|\beta_2| \leq 2|c|(q-2)\big(\theta\bar{J}_{q,\chi} + 4^{-1}\theta^{-1}G_{q,\chi^2}\big)$, $\theta > 0$, we obtain:

If $c>0$, then

$$\begin{aligned}&\mu\langle|w|^q\rangle+I_q+c(1-\theta)\bar I_{q,\chi}+(q-2)(J_q+c\bar J_{q,\chi})-c\Big(1+\frac{q-2}{q}d\Big)H_{q,\chi}+c\big(2(d-1)-\theta^{-1}\big)G_{q,\chi^2}\\&+2c\frac{q-2}{q}H_{q,\chi^2}\\&\leq c(d+3)\frac{q\sqrt\delta}{2}G^{\frac12}_{q,\chi^2}J_q^{\frac12}+c\frac{q\sqrt\delta}{2}\bar I^{\frac12}_{q,\chi}J_q^{\frac12}+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt\delta}{2}\right)J_q+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}\tag{6}$$

If $-1<c<0$, then (set $s:=|c|$)

$$\begin{aligned}&\mu\langle|w|^q\rangle+I_q-s\bar I_{q,\chi}+(q-2)(J_q-s(1+\theta)\bar J_{q,\chi})+s\left(1+(q-2)\frac{d}{q}\right)H_{q,\chi}\\&-2s\frac{q-2}{q}H_{q,\chi^2}-sdG_{q,\chi^2}-4sG_{q,\chi^2}+4sG_{q,\chi^3}-s(q-2)\frac{1}{4\theta}G_{q,\chi^2}\\&\leq s(d+3)\frac{q\sqrt\delta}{2}G^{\frac12}_{q,\chi^2}J_q^{\frac12}+s\frac{q\sqrt\delta}{2}\bar I^{\frac12}_{q,\chi}J_q^{\frac12}+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt\delta}{2}\right)J_q+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}\tag{7}$$

3. Employing (6), (7) we will prove the following inequality

$$\mu\langle|w|^q\rangle+\eta J_q\leq C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2,\qquad C_i=C_i(\varepsilon_0),\quad i=1,2,\tag{8}$$

for some $\eta=\eta(q,d,\varepsilon_0)>0$.

Case $c>0$. In the LHS of (6) we select $\theta:=\frac{1}{2(d-1)}$ (<1). Consider two subcases:

C1) $1-\frac{cq\sqrt\delta}{4}\geq 0$. Arguing as in the proof of Theorem 1, using $H_q\geq G_{q,\chi^2}$, we obtain from (6):

$$\begin{aligned}&\mu\langle|w|^q\rangle+I_q+(q-2)J_q+cM(1)H_q\\&\leq c(d+3)\frac{q\sqrt\delta}{2}H_q^{\frac12}J_q^{\frac12}+c\frac{q\sqrt\delta}{2}\bar I^{\frac12}_{q,\chi}J_q^{\frac12}+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt\delta}{2}\right)J_q+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2,\end{aligned}$$

where $M(1):=\big(q-1-\frac{1}{2(d-1)}\big)\frac{(d-2)^2}{q^2}-\big(1+\frac{q-2}{q}(d-2)\big)<0$.

Using the quadratic estimates, we obtain $(\theta_2,\theta_3>0)$

$$\begin{aligned}&\mu\langle|w|^q\rangle+I_q+(q-2)J_q+cM(1)H_q\\&\leq c(d+3)\frac{q\sqrt\delta}{4}(\theta_2J_q+\theta_2^{-1}H_q)+c\frac{q\sqrt\delta}{4}(\theta_3\bar I_{q,\chi}+\theta_3^{-1}J_q)+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt\delta}{2}\right)J_q\\&+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

We select $\theta_2=\frac{q}{d-2}$, $\theta_3=1$, so

$$\begin{aligned}&\mu\langle|w|^q\rangle+I_q+(q-2)J_q+cM(1)H_q\\&\leq c(d+3)\frac{q\sqrt\delta}{4}\left(\frac{q}{d-2}J_q+\frac{d-2}{q}H_q\right)+c\frac{q\sqrt\delta}{4}(\bar I_{q,\chi}+J_q)+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt\delta}{2}\right)J_q\\&+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Since $1-\frac{cq\sqrt\delta}{4}\geq 0$, we have $I_q-\frac{cq\sqrt\delta}{4}\bar I_{q,\chi}\geq\big(1-\frac{cq\sqrt\delta}{4}\big)J_q$, so using $J_q\geq\frac{(d-2)^2}{q^2}H_q$ we obtain

$$\mu\langle|w|^q\rangle+\eta J_q+\left[\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2}-Ł_1(c,\delta)\right]H_q\leq C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$L_1(c,\delta) = \left[c\frac{q\sqrt{\delta}}{4} + c(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2} + c\frac{q\sqrt{\delta}}{4} + \frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}$$
$$+ c\left[-M(1) + (d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\right].$$

By the assumptions of the theorem, $\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2} - L_1(c,\delta) \geq 0$ for all $\eta>0$ sufficiently small. (8) is proved.

C2) $1-\frac{cq\sqrt{\delta}}{4}<0$. Arguing as above, we obtain

$$\mu\langle|w|^q\rangle + I_q + (q-2)J_q + c(1-\theta)\bar{I}_{q,\chi} + cM(1)H_q$$
$$\leq c(d+3)\frac{q\sqrt{\delta}}{4}\left(\frac{q}{d-2}J_q + \frac{d-2}{q}H_q\right) + c\frac{q\sqrt{\delta}}{4}(\bar{I}_{q,\chi}+J_q) + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q$$
$$+ C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.$$

where $M(1) := \big(q-2\big)\frac{(d-2)^2}{q^2} - \big(1+\frac{q-2}{q}(d-2)\big) < 0$.

If $1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}<0$, then clearly $c\big(1-\frac{1}{2(d-1)}\big)\bar{I}_{q,\chi} - c\frac{q\sqrt{\delta}}{4}\bar{I}_{q,\chi} \geq c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)I_q$. By the assumption of the theorem, $1+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\geq 0$, so in the previous estimate $\big[1+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\big]I_q \geq \big[1+c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\big]J_q$, and thus

$$\mu\langle|w|^q\rangle + \eta J_q + \left[\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2} - L_1(c,\delta)\right]H_q \leq C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2, \tag{9}$$

where

$$L_1(c,\delta) = \left[-c\left(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\right) + c\frac{q\sqrt{\delta}}{4} + c(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2} + \frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}$$
$$+ c\left[-M(1) + (d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\right]$$

(another representation for $L_1(c,\delta)$). By the assumptions of the theorem, $\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2} - L_1(c,\delta)\geq 0$ for all $\eta>0$ sufficiently small. (8) is proved.

If $1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\geq 0$, then clearly $I_q + c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\bar{I}_{q,\chi} \geq J_q + c\big(1-\frac{1}{2(d-1)}-\frac{q\sqrt{\delta}}{4}\big)\bar{J}_{q,\chi}$. Arguing as above, we obtain (9) and therefore (8).

Case $-1<c<0$. In the LHS of (7) we select $\theta = \frac{1}{2}\frac{q}{d-2}$. Arguing as in the proof of Theorem 1, we obtain from (7):

$$\mu\langle|w|^q\rangle + I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) + sM(1)G_q \tag{10}$$
$$\leq s(d+3)\frac{q\sqrt{\delta}}{2}G_q^{\frac{1}{2}}J_q^{\frac{1}{2}} + s\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C\|w\|_q^{q-2}\|f\|_q^2,$$

where $M(1) := -d+1+\frac{1}{2}\frac{(q-2)(d-2)}{q} < 0$. In the RHS of (7) we have used $G_q \geq G_{q,\chi^2}$.

Further (θ_2, $\theta_3>0$),

$$\begin{aligned}&\mu\langle|w|^q\rangle+I_q-s\bar{I}_{q,\chi}+(q-2)(J_q-s(1+\theta)\bar{J}_{q,\chi})+sM(1)G_q\\&\leq s(d+3)\frac{q\sqrt{\delta}}{4}(\theta_2J_q+\theta_2^{-1}G_q)+s\frac{q\sqrt{\delta}}{4}(\theta_3\bar{I}_{q,\chi}+\theta_3^{-1}J_q)+\Big(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\Big)J_q\\&+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Selecting $\theta_2=\frac{q}{d-2}$, $\theta_3=1$ and using the inequalities $I_q\geq\bar{I}_{q,\chi}$, $J_q\geq\bar{J}_{q,\chi}$, we obtain

$$\begin{aligned}&\mu\langle|w|^q\rangle+\Big[1-s\Big(1+\frac{q\sqrt{\delta}}{4}\Big)\Big]I_q\\&+\Big[(q-2)\Big(1-s\Big(1+\frac{1}{2}\frac{q}{d-2}\Big)\Big)-s\frac{q\sqrt{\delta}}{4}-s(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2}-\frac{q^2\delta}{4}-(q-2)\frac{q\sqrt{\delta}}{2}\Big]J_q\\&+s\Big[M(1)-(d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\Big]G_q\leq C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

By the assumptions of the theorem, $1-s\big(1+\frac{q\sqrt{\delta}}{4}\big)\geq 0$. So, using the inequalities $J_q\leq I_q$, $J_q\geq\frac{(d-2)^2}{q^2}G_q$, we arrive at

$$\mu\langle|w|^q\rangle+\eta J_q+\Big[\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2}-Ł_2(s,\delta)\Big]G_q\leq C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$\begin{aligned}&Ł_2(s,\delta)\\&=\Big[s\Big(1+\frac{q\sqrt{\delta}}{4}\Big)+(q-2)s\Big(1+\frac{1}{2}\frac{q}{d-2}\Big)+s\frac{q\sqrt{\delta}}{4}+s(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2}+\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\Big]\frac{(d-2)^2}{q^2}\\&+s\Big[-M(1)+(d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\Big].\end{aligned}$$

By the assumptions of the theorem, $\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2}-Ł_2(s,\delta)\geq 0$ for all $\eta>0$ sufficiently small. (8) is proved.

4. The Sobolev Embedding Theorem and Theorem A.2(i) now yield ($\star\star$) (cf. the proof of Theorem 1, step 4).

Proof of (iii). Let $q=2$, $d\geq 3$. Recall that since $b\in\mathbf{F}_\delta$, then $b\in\mathbf{F}_{\delta_1}(A)$, where $\delta_1:=\delta$ if $c>0$, and $\delta_1:=\delta(1+c)^{-2}$ if $-1<c<0$ By our assumptions, $\delta_1<1$, so $\Lambda_2(a^\varepsilon,b_n)=A_2^\varepsilon+b_n\cdot\nabla$ are well defined on L^2. Following the proof of Theorem 1(iii), we obtain: for $c>0$

$$\begin{aligned}&\langle[\nabla,A_2^\varepsilon]_-u,w\rangle=-\beta_1-cH_{2,\chi}+2c(d-1)G_{2,\chi^2},\\&\mu\|w\|_2^2+I_2+c\bar{I}_{2,\chi}-\beta_1-cH_{2,\chi}+2c(d-1)G_{2,\chi^2}=\langle-b_n\cdot w,-\nabla\cdot w\rangle+\langle f,-\nabla\cdot w\rangle\\&\beta_1=-2c\langle|x|^{-4}x\cdot w,x\cdot(x\cdot\nabla w)\rangle;\end{aligned}$$

for $c<0$

$$\begin{aligned}&\langle[\nabla, A_2^\varepsilon]_- u, w\rangle = \frac{1}{2}\beta_2 - cH_{2,\chi} + cdG_{2,\chi^2}, \quad \beta_2 = 0,\\ &\mu\|w\|_2^2 + I_2 + c\bar{I}_2 - cH_{2,\chi} + cdG_{2,\chi^2} = \langle -b_n\cdot w, -\nabla\cdot w\rangle + \langle f, -\nabla\cdot w\rangle, \quad I_2 \geq \bar{I}_2,\\ &\mu\|w\|_2^2 + (1-|c|)I_2 + |c|H_{2,\chi} - |c|dG_{2,\chi^2} \leq \langle -b_n\cdot w, -\nabla\cdot w\rangle + \langle f, -\nabla\cdot w\rangle.\end{aligned}$$

Now, applying Lemma 4 and arguing as in step 3 above, we arrive at $\sup_{\varepsilon>0,n\geq 1} I_2(u^{\varepsilon,n}) \leq K\|f\|_2$, and so (by passing to the limit $\varepsilon\downarrow 0$, using Theorem A.2, we arrive at $I_2(u)\leq K\|f\|_2 \Rightarrow u\in W^{2,2}$.

The proof of Theorem 2 is completed. □

3. Proof of Theorem 3

Recall that a vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1>0$, the class of form-bounded vector fields (with respect to $A\equiv A_2$), if $b_a^2 := b\cdot a^{-1}\cdot b \in L^1_{\rm loc}$ and there exists a constant $\lambda=\lambda_{\delta_1}>0$ such that $\|b_a(\lambda+A)^{-\frac{1}{2}}\|_{2\to 2}\leq\sqrt{\delta_1}$.

We will need the following auxiliary results. Recall: $(\nabla a)_k = \sum_{i=1}^d(\partial_{x_i}a_{ik})$, $1\leq k\leq d$.

Lemma 5. *$\nabla a = c(d-1)|x|^{-2}x \in \mathbf{F}_\delta(A)$, where $\delta := 4\big(\frac{d-1}{d-2}\frac{c}{1+c}\big)^2$.*

Proof. It is easy to see that $b:=\nabla a = c(d-1)|x|^{-2}x$, $a^{-1} = I - \frac{c}{c+1}|x|^{-2}x\otimes x$ and $b_a^2 := b\cdot a^{-1}\cdot b = \frac{[(d-1)c]^2}{c+1}|x|^{-2}$. Now, that $b\in\mathbf{F}_\delta(A)$ is immediate from the following Hardy-type inequality:

$$(c+1)\frac{(d-2)^2}{4}\||x|^{-1}h\|_2^2 \leq \langle\nabla h\cdot a\cdot\nabla\bar{h}\rangle, \qquad h\in W^{1,2}(\mathbb{R}^d). \tag{$\star$}$$

It remains to prove ($\star$). Since $\langle\phi, x\cdot\nabla\phi\rangle = -\frac{d}{2}\langle\phi,\phi\rangle$, $\phi\in C_c^\infty$, we have

$$\langle\phi, -\nabla\cdot(a-1)\cdot\nabla\phi\rangle = c\big(\|x\cdot\nabla(|x|^{-1}\phi)\|_2^2 - (d-1)\||x|^{-1}\phi\|_2^2\big) \tag{11}$$

Next, the following inequality (with the sharp constant) is valid:

$$\|x\cdot\nabla f\|_2 \geq \frac{d}{2}\|f\|_2, \qquad (f\in D(\mathcal{D})), \tag{12}$$

where the operator $\mathcal{D} = (\mathcal{D}\upharpoonright C_c^\infty)^{clos}_{L^2\to L^2}$, $\mathcal{D}\upharpoonright C_c^\infty = \frac{\sqrt{-1}}{2}(x\cdot\nabla + \nabla\cdot x)$, is selfadjoint.

Indeed, by the Spectral Theorem, $\|(\mathcal{D}-\zeta)^{-1}\|_{2\to 2} = \frac{1}{|\mathrm{Im}\zeta|}$ for $\mathrm{Re}\zeta = 0$, and hence

$$\|x\cdot\nabla f\|_2 = \|\frac{1}{2}(x\cdot\nabla+\nabla\cdot x - d)f\|_2 = \|(\mathcal{D}-\sqrt{-1}\,\frac{d}{2})f\|_2 \geq \frac{d}{2}\|f\|_2, \qquad (f\in C_c^\infty).$$

(12) is proved.

Let $c>0$. By (11) and (12),

$$\langle\phi, -\nabla\cdot(a-1)\cdot\nabla\phi\rangle \geq c\frac{(d-2)^2}{4}\||x|^{-1}\phi\|_2^2 \qquad (\phi\in C_c^\infty).$$

($\star$) follows now from the equality $\langle\phi,-\nabla\cdot a\cdot\nabla\phi\rangle = \langle\phi,-\nabla\cdot(a-1)\cdot\nabla\phi\rangle + \langle\phi,-\Delta\phi\rangle$ and Hardy's inequality $\langle\phi,-\Delta\phi\rangle\geq\frac{(d-2)^2}{4}\||x|^{-1}\phi\|_2^2$. Finally, the obvious inequality $(1+c)\langle\phi,-\Delta\phi\rangle\geq\langle\phi,-\nabla\cdot a\cdot\nabla\phi\rangle$ shows that the constant in ($\star$) is sharp.

If $-1<c<0$, ($\star$) is a trivial consequence of Hardy's inequality. □

If $0 < c < d-2$, then $\nabla a \in \mathbf{F}_\delta(A)$ with $\delta < 4$ by Lemma 5, so $\Lambda_q(a, \nabla a)$ is well defined for all $q \in \big](1-\frac{d-1}{d-2}\frac{c}{1+c})^{-1}, \infty\big[$, see [KiS, Theorem 3.2].

If $-1 < c < 0$, then $\Lambda_q(a,\nabla a)$ is well defined for all $q \in \big]1,\infty\big[$ by Theorem A.1 (there take $b=0$). We have proved assertion (i) of the theorem.

In order to prove assertion (ii), we will need the following result. Set

$$|x|_\varepsilon := \sqrt{|x|^2+\varepsilon},\ \varepsilon > 0, \quad \chi := |x|^2|x|_\varepsilon^{-2}.$$

Lemma 6. *Set $a^\varepsilon(x) := I + c|x|_\varepsilon^{-2}x^t \cdot x$, $A^\varepsilon \equiv [-\nabla \cdot a^\varepsilon \cdot \nabla \upharpoonright C_c^\infty]_{2\to 2}^{\text{clos}}$. If $d \geq 4$, $-1 < c \leq \frac{d-3}{2}$, or if $d=3$, $-1<c<0$, then*

$$\nabla a^\varepsilon \in \mathbf{F}_\delta(A^\varepsilon) \quad \text{with } \delta := 4\bigg(\frac{d-1}{d-2}\frac{c}{1+c}\bigg)^2.$$

Proof. 1. First, let $c>0$. Note that $(a^\varepsilon)^{-1}(x) = I - \frac{c\chi}{1+c\chi}|x|^{-2}x\otimes x$, $(\nabla a^\varepsilon) = c\chi(d+1-2\chi)|x|^{-2}x$ and

(a) $(\nabla a^\varepsilon)\cdot(a^\varepsilon)^{-1}\cdot(\nabla a^\varepsilon) = \frac{(c\chi)^2[(d+1-2\chi)]^2}{1+c\chi}|x|^{-2}$.

(b) $\langle -\nabla\cdot a^\varepsilon\cdot\nabla h, h\rangle = \langle (\nabla h)^2\rangle + c\langle |x|_\varepsilon^{-2}(x\cdot\nabla h)^2\rangle \geq \frac{(d-2)^2}{4}\langle |x|^{-2}h^2\rangle + c\big\langle\big(\frac{d^2}{4}-(d+2)\chi+3\chi^2\big)\chi|x|^{-2}h^2\big\rangle$ $h\in C_c^\infty$, see (**HI**).

Combining (a) and (b), we obtain that $\nabla a^\varepsilon \in \mathbf{F}_{\delta^\varepsilon}(A^\varepsilon)$ for any δ^ε such that

$$\delta^\varepsilon\bigg\langle\bigg[\frac{(d-2)^2}{4} + c\bigg(\frac{d^2}{4}-(d+2)\chi+3\chi^2\bigg)\chi\bigg]|x|^{-2}h^2\bigg\rangle \geq \bigg\langle \frac{(c\chi)^2[(d+1-2\chi)]^2}{1+c\chi}|x|^{-2}h^2\bigg\rangle, \quad h\in C_c^\infty;$$

we can take

$$\delta^\varepsilon := \sup_{0\leq t\leq 1}\frac{(ct)^2[d+1-2t]^2}{(1+ct)\bigg[\frac{(d-2)^2}{4}+ct\bigg(\frac{d^2}{4}-(d+2)t+3t^2\bigg)\bigg]}$$

Let us show that

$$\delta^\varepsilon = 4\bigg(\frac{d-1}{d-2}\frac{c}{1+c}\bigg)^2 \equiv \delta,$$

which would imply that $\nabla a^\varepsilon \in \mathbf{F}_\delta(A^\varepsilon)$, as claimed.

Note that $\frac{d^2}{4}-(d+2)t+3t^2 \geq \frac{(d-2)^2}{4}$ for all $0\leq t\leq 1$ and $d\geq 4$. Thus

$$\delta \leq \delta^\varepsilon \leq 4\sup_{0\leq t\leq 1}\frac{c^2[d+1-2t]^2t^2}{(d-2)^2(1+ct)^2} = 4\bigg[\frac{c}{d-2}\sup_{0\leq t\leq 1}\frac{(d+1-2t)t}{1+ct}\bigg]^2 = \delta.$$

2. Let $-1<c<0$. Then $\langle -\nabla\cdot a^\varepsilon\cdot\nabla h, h\rangle \geq \frac{(d-2)^2}{4}(1+c)\langle |x|^{-2}h^2\rangle$, so by (a) above $\nabla a^\varepsilon \in \mathbf{F}_{\delta^\varepsilon}(A^\varepsilon)$ for any δ^ε such that

$$\delta^\varepsilon\frac{(d-2)^2}{4}(1+c)\langle |x|^{-2}h^2\rangle \geq \bigg\langle\frac{c^2[(d+1-2\chi)]^2}{1+c\chi}\chi^2|x|^{-2}h^2\bigg\rangle, \quad h\in C_c^\infty;$$

we can take

$$\delta^\varepsilon := \sup_{0\leq t\leq 1}\frac{c^2(d+1-2t)^2t^2}{\frac{(d-2)^2}{4}(1+c)(1+ct)}.$$

Finally, since $1+c\leq 1+ct$,

$$\delta\leq\delta_\varepsilon\leq 4\Big[\frac{c}{(1+c)(d-2)}\sup_{0<t<1}(d+1-2t)t\Big]^2=\delta.$$

$\square$

1. We start the proof of assertion (*ii*) of the theorem. Let $d\geq 4$. We follow closely the proof of Theorem 1. Set $u^\varepsilon=(\mu+\Lambda_q(a^\varepsilon,\nabla a^\varepsilon))^{-1}f$, $0\leq f\in C_c^1$. Since $a^\varepsilon\in C^\infty$, we have $0\leq u^\varepsilon\in W^{3,q}$. Below

$$w\equiv w^\varepsilon:=\nabla u^\varepsilon,\quad \phi:=-\nabla\cdot(w|w|^{q-2}),$$

$$I_q:=\langle(\nabla_r w)^2|w|^{q-2}\rangle,\quad J_q:=\langle(\nabla|w|)^2|w|^{q-2}\rangle,$$

$$\bar I_{q,\chi}:=\langle\big(x\cdot\nabla w\big)^2\chi|x|^{-2}|w|^{q-2}\rangle,\quad \bar J_{q,\chi}:=\langle(x\cdot\nabla|w|)^2\chi|x|^{-2}|w|^{q-2}\rangle,$$

$$H_{q,\chi}:=\langle\chi|x|^{-2}|w|^q\rangle,\qquad G_{q,\chi^2}:=\langle\chi^2|x|^{-4}(x\cdot w)^2|w|^{q-2}\rangle,$$

where $\chi=|x|^2|x|_\varepsilon^{-2}$. We will need

Lemma 7 (The basic equalities, non-divergence form)**.**

$$\mu\langle|w|^q\rangle+I_q+c\bar I_{q,\chi}+(q-2)(J_q+c\bar J_{q,\chi})-c\frac{d(d-1)}{q}H_{q,\chi}$$
$$+2c\frac{2d+1}{q}H_{q,\chi^2}-\frac{8c}{q}H_{q,\chi^3}=\beta_1+\langle f,\phi\rangle, \tag{BE$^{\rm nd}_+$}$$

$$\mu\langle|w|^q\rangle+I_q+c\bar I_{q,\chi}+(q-2)(J_q+c\bar J_{q,\chi})-c\frac{d(d-1)}{q}H_{q,\chi}-c(d-2)G_{q,\chi^2}$$
$$+2c\frac{2d+1}{q}H_{q,\chi^2}-\frac{8c}{q}H_{q,\chi^3}-4c\varepsilon\langle|x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle=-\frac{1}{2}\beta_2+\langle f,\phi\rangle, \tag{BE$^{\rm nd}_-$}$$

where

$$\beta_1:=-2c\langle|x|_\varepsilon^{-4}x\cdot w,x\cdot(x\cdot\nabla w)|w|^{q-2}\rangle,\qquad \beta_2:=-2c(q-2)\langle|x|_\varepsilon^{-4}(x\cdot w)^2x\cdot\nabla|w|,|w|^{q-3}\rangle.$$

Proof of Lemma 7. We modify the proof of Lemma 2. In the left-hand side of (BE$_+$), (BE$_-$) we have the extra term $\langle(\nabla a^\varepsilon)\cdot w,-\nabla\cdot(w|w|^{q-2})\rangle$, which we evaluate as follows:

$$\begin{aligned}&\langle(\nabla a^\varepsilon)\cdot w,-\nabla\cdot(w|w|^{q-2})\rangle\\&\text{(we integrate by parts)}\\&=c(d-1)\big(H_{q,\chi}+\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|,|w|^{q-1}\rangle-2G_{q,\chi^2}\big)\\&+2c\varepsilon\big(\langle|x|_\varepsilon^{-4}|w|^q\rangle+\langle|x|_\varepsilon^{-4}x\cdot\nabla|w|,|w|^{q-1}\rangle-4\langle|x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\rangle\big).\end{aligned}$$

Note that

$$\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|,|w|^{q-1}\rangle=\frac{1}{q}\langle|x|_\varepsilon^{-2}x\cdot\nabla|w|^q\rangle=-\frac{d-2}{q}H_{q,\chi}-\frac{2}{q}\varepsilon\langle|x|_\varepsilon^{-4}|w|^q\rangle,$$

$$\langle|x|_\varepsilon^{-4}x\cdot\nabla|w|,|w|^{q-1}\rangle=\frac{1}{q}\langle|x|_\varepsilon^{-4}x\cdot\nabla|w|^q\rangle=-\frac{1}{q}\langle|w|^q\nabla\cdot(x|x|_\varepsilon^{-4})\rangle=-\frac{d-4}{q}\langle|x|_\varepsilon^{-4}|w|^q\rangle-\frac{4}{q}\varepsilon\langle|x|_\varepsilon^{-6}|w|^q\rangle.$$

Thus,

$$\begin{aligned}&\big\langle(\nabla a^\varepsilon)\cdot w, -\nabla\cdot(w|w|^{q-2})\big\rangle\\&= c(d+1)\left(1-\frac{d}{q}\right)H_{q,\chi} + c\Big(-2+\frac{2}{q}(2d+3)\Big)H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3}\\&\quad - 2c(d-1)G_{q,\chi^2} - 8c\varepsilon\big\langle|x|_\varepsilon^{-6}(x\cdot w)^2|w|^{q-2}\big\rangle.\end{aligned}$$

The latter, added to the left-hand side of (BE_+), (BE_-) yields (BE_+^{nd}), (BE_-^{nd}). □

2. We estimate from above the term $\langle f,\phi\rangle$ in the right-hand side of (BE_+^{nd}), (BE_-^{nd}) employing an evident analogue of Lemma 3:

$$\langle f,\phi\rangle \le \varepsilon_0(I_q+J_q+H_q) + C(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2. \tag{13}$$

Again we choose $\varepsilon_0>0$ so small that in the estimates below we can ignore the terms multiplied by ε_0.

3. We will use (BE_+^{nd}), (BE_-^{nd}) and (13) to establish the inequality

$$\mu\langle|w|^q\rangle + \eta J_q \le C\|w\|_q^{q-2}\|f\|_q^2, \qquad C=C(\varepsilon_0), \qquad \eta=\eta(q,d,\varepsilon_0)>0. \tag{14}$$

Case $c>0$. By the assumptions of the theorem, $c<\frac{d-3}{2}\wedge\frac{d-2}{q-d+2}$. In ($\text{BE}_+^{\text{nd}}$), we estimate

$$\beta_1 \le c\theta\bar{I}_{q,\chi} + c\theta^{-1}G_{q,\chi^2}, \;\; \theta>0,$$

and then apply (13) to obtain

$$\begin{aligned}&\mu\langle|w|^q\rangle + I_q + c(1-\theta)\bar{I}_{q,\chi} + (q-2)J_q + c(q-2)\bar{J}_{q,\chi} - c\frac{d(d-1)}{q}H_{q,\chi}\\&+2c\frac{2d+1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}G_{q,\chi^2} \le C\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

We exclude the case $0<\theta\le1$ by noting that $\bar{I}_{q,\chi}\ge\frac{(d-2)^2}{q^2}G_{q,\chi^2}$ and $f(\theta)=(1-\theta)\frac{(d-2)^2}{q^2}-\frac{1}{\theta}$ achieves its maximum at $\theta=\frac{q}{d-2}>1$.

Let $\theta>1$. Clearly we have to assume now that $1+c(1-\theta)>0$. Since $I_q+c(1-\theta)\bar{I}_{q,\chi}\ge(1+c(1-\theta))I_q\ge(1+c(1-\theta))J_q$ and $H_{q,\chi^2}\ge G_{q,\chi^2}$ we have

$$\begin{aligned}&\mu\langle|w|^q\rangle + (q-1+c(1-\theta))J_q + c(q-2)\bar{J}_{q,\chi} - c\frac{d(d-1)}{q}H_{q,\chi}\\&+2c\frac{2d+1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}H_{q,\chi^2} \le C\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Using $\bar{J}_{q,\chi}\ge\frac{4}{q^2}\left(\frac{d^2}{4}H_{q,\chi}-(d+2)H_{q,\chi^2}+3H_{q,\chi^3}\right)$, see (**HI**), we obtain

$$\begin{aligned}&\mu\langle|w|^q\rangle + (q-1+c(1-\theta))J_q + c(q-2)\frac{4}{q^2}\left(\frac{d^2}{4}H_{q,\chi}-(d+2)H_{q,\chi^2}+3H_{q,\chi^3}\right) - c\frac{d(d-1)}{q}H_{q,\chi}\\&+2c\frac{2d+1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}H_{q,\chi^2} \le C\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Thus, by $J_q\ge\frac{(d-2)^2}{q^2}\langle|x|^{-2}|w|^q\rangle$, for all $\eta>0$ sufficiently small,

$$\mu\langle|w|^q\rangle + \eta J_q + \left\langle\left[(-\eta+q-1+c(1-\theta))\frac{(d-2)^2}{q^2} + cM(\chi)\right]|x|^{-2}|w|^q\right\rangle \le C\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$M(\chi) := \left[(q-2)\frac{4}{q^2}\left(\frac{d^2}{4}-(d+2)\chi+3\chi^2\right)-\frac{d(d-1)}{q}+2\frac{2d+1}{q}\chi-\frac{8}{q}\chi^2-\frac{1}{\theta}\chi\right]\chi.$$

Select $\theta := \frac{q}{d-2}$. (Motivation: estimating the terms involving θ from below by $\big[-c\theta\frac{(d-2)^2}{q^2}-\frac{c}{\theta}\big]H_q$ and maximizing the latter in θ, we arrive at $\theta = \frac{q}{d-2}$.) Then, since $c < \frac{d-2}{q-d+2}$, we have $1+c(1-\theta) > 0$. Elementary arguments show that

$$\min_{0\leq t\leq 1} M(t) = M(1) < 0,$$

and so

$$\mu\langle|w|^q\rangle + \eta J_q + \left[(-\eta+q-1)\frac{(d-2)^2}{q^2} - c\ell_1^{\rm nd}\right]H_q \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

where $\ell_1^{\rm nd} := \frac{q+d-2}{q^2}(d-2)$. By the assumption $c < \frac{d-3}{2}$ of the theorem, there exists $\eta > 0$ such that $(-\eta+q-1)\frac{(d-2)^2}{q^2} - c\ell_1^{\rm nd} \geq 0$. Thus (14) is proved.

Case $-1 < c < 0$**.** By the assumptions of the theorem, $-\big(1+\frac{1}{4}\frac{q}{d-2}\frac{q-2}{q-1}\frac{q-2}{q+d-3}\big)^{-1} < c < 0$. Set $s := |c|$. In ($\text{BE}_-^{\rm nd}$) we estimate ($\theta > 0$)

$$|\beta_2^\varepsilon| \leq 2s(q-2)\big(\theta\bar{J}_q + 4^{-1}\theta^{-1}G_{q,\chi^2}\big),$$

obtaining

$$\mu\langle|w|^q\rangle + I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) + s\frac{d(d-1)}{q}H_{q,\chi}$$
$$- 2s\frac{2d+1}{q}H_{q,\chi^2} + \frac{8s}{q}H_{q,\chi^3} + s\left(d+2-(q-2)\frac{1}{4\theta}\right)G_{q,\chi^2} - 4sG_{q,\chi^3} \leq C\|w\|_q^{q-2}\|f\|_q^2.$$

Then by the obvious inequalities $I_q - s\bar{I}_{q,\chi} \geq (1-s)J_q$ and $J_q - s(1+\theta)\bar{J}_{q,\chi} \geq (1-s(1+\theta))J_q$,

$$\mu\langle|w|^q\rangle + \big(q-1-s-s(q-2)(1+\theta)\big)J_q + s\frac{d(d-1)}{q}H_{q,\chi} \tag{$\star$}$$
$$- 2s\frac{2d+1}{q}H_{q,\chi^2} + \frac{8s}{q}H_{q,\chi^3} + s\left(d+2-(q-2)\frac{1}{4\theta}\right)G_{q,\chi^2} - 4sG_{q,\chi^3} \leq C\|w\|_q^{q-2}\|f\|_q^2.$$

Note that $d(d-1) - 2(2d+1)t + 8t^2 \geq 0,\ (d\geq 3,\ 0\leq t\leq 1)$, and so

$$\frac{d(d-1)}{q}H_{q,\chi} - 2\frac{2d+1}{q}H_{q,\chi^2} + \frac{8}{q}H_{q,\chi^3} \geq \frac{d(d-1)}{q}G_{q,\chi} - 2\frac{2d+1}{q}G_{q,\chi^2} + \frac{8}{q}G_{q,\chi^3}.$$

Therefore, we obtain from ($\star$)

$$\mu\langle|w|^q\rangle + \big(q-1-s-s(q-2)(1+\theta)\big)J_q + s\frac{d(d-1)}{q}G_{q,\chi}$$
$$- 2s\frac{2d+1}{q}G_{q,\chi^2} + \frac{8s}{q}G_{q,\chi^3} + s\left(d+2-(q-2)\frac{1}{4\theta}\right)G_{q,\chi^2} - 4sG_{q,\chi^3} \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

i.e.

$$\mu\langle|w|^q\rangle + \big[q-1-s-s(q-2)(1+\theta)\big]J_q - s\big\langle M(\chi)|x|^{-4}(x\cdot w)^2|w|^{q-2}\big\rangle \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$M(\chi) := q^{-1}\big[\mathfrak{a}\chi^2 + \mathfrak{b}\chi + \mathfrak{c}_0\big]\chi.$$

$$\mathfrak{a} := 4(q-2), \quad \mathfrak{b} := 2(2d+1) - q\biggl(d+2-(q-2)\frac{1}{4\theta}\biggr), \quad \mathfrak{c}_0 := -d(d-1).$$

Select $\theta := \frac{1}{4}\frac{q}{d-2}\frac{q-2}{q+d-3}$ if $q > 2$. Then $M(0) = M(1) = \max_{0 \leq t \leq 1} M(t) = 0$. This is the best possible choice of θ. (Selecting a larger θ, so that $\max_{0\leq t\leq 1} M(t) < 0$, decreases the term $[\dots]J_q$. On the other hand, selecting a smaller θ, so that $\max_{0\leq t\leq 1} M(t) > 0$, leads to constraints on c which are sub-optimal, i.e. which can be improved by selecting a larger θ.)

Note that $q-1-s-s(q-2)(1+\theta) > 0$ by the assumptions of the theorem. Thus,

$$\mu\langle |w|^q\rangle + \bigl(q-1-s-s(q-2)(1+\theta)\bigr)J_q \leq C\|w\|_q^{q-2}\|f\|_q^2,$$

and hence (14) is proved for $q > 2$.

We are left to treat the case $d = 4$ and $q = 2$. Note that the proof above still works. See also a proof of (iii) below.

4. For $d \geq 4$, the Sobolev Embedding Theorem and Theorem A.2(ii) (with $\delta = 0$) now yield estimates $(\star)$ and convergence (3). The proof of (ii) is completed.

Proof of (iii). Let $q = 2$, $d \geq 3$. If $c < 0$, then we can argue as in steps 1-3 obtaining

$$\sup_{\varepsilon>0} I_2(u^\varepsilon) \leq K\|f\|_2, \qquad \text{and so } I_2(u) \leq K\|f\|_2 \quad \Rightarrow u \in W^{2,2}.$$

Now, let $c > 0$. By Lemma 5, $\nabla a \in \mathbf{F}_\delta(A)$, $\delta = 4\bigl(\frac{d-1}{d-2}\frac{c}{1+c}\bigr)^2$. Since $c < \frac{d-2}{d}$, we have $\delta < 1$, and so $\Lambda_2(a, \nabla a)$ is well defined. By the Miyadera Perturbation Theorem and Theorem 1, $D(\Lambda_2(a,\nabla a)) = D(A_2) \subset W^{2,2}$, and $u := (\mu + \Lambda_2(a,\nabla a))^{-1}f$, $\mu > 0$, $f \in L^2$, belongs to $W^{2,2}$. Multiplying $(\mu + A_2 + \nabla a \cdot \nabla)u = f$ by $\phi_m := -E_m \nabla \cdot w$, where $w := \nabla u$, $E_m = (1 - m^{-1}\Delta)^{-1}$, $m \geq 1$, and integrating by parts we have (omitting the summation sign in the repeated indices):

$$\mu\langle |w|^2\rangle + \langle a\cdot\nabla w_r, E_m\nabla w_r\rangle + \langle -(\nabla_r a)\cdot w, E_m \nabla w_r)\rangle + \langle \nabla a \cdot w, \phi_m\rangle = \langle f, \phi_m\rangle, \tag{$\star$}$$

Now we pass in $(\star)$ to the limit $m \to \infty$.

Then following closely the proof of $(\mathrm{BE}^{\mathrm{nd}}_+)$ for $q = 2$ we obtain:

$$\mu\langle |w|^2\rangle + I_2 + c\bar{I}_2 - \frac{c}{2}(d-2)(d-3)H_2 = \beta + \langle f, \phi\rangle,$$

where $I_2 := \langle \nabla w_r, \nabla w_r\rangle$, $\bar{I}_2 := \langle \bigl(x\cdot\nabla w\bigr)^2|x|^{-2}\rangle$, $H_2 := \langle |x|^{-2}|w|^2\rangle$, $\beta := -2c\langle |x|^{-4}x\cdot w, x\cdot(x\cdot\nabla w)\rangle$.

Using the inequalities $\beta \leq c\bar{I}_2 + cH_2$, $\frac{(d-2)^2}{4}H_2 \leq I_2$, we have

$$\mu\langle |w|^2\rangle + \biggl[1 - \frac{4}{(d-2)^2}\biggl(1 + \frac{(d-2)(d-3)}{2}\biggr)c\biggr]I_2 \leq \langle f, \phi\rangle.$$

The proof of (iii) follows.

The proof of Theorem 3 is completed. □

4. Proof of Theorem 4

We follow closely the proofs of Theorems 2, 3.

Proof of (i). It is easily seen that if $b \in \mathbf{F}_\delta$, then $b \in \mathbf{F}_{\delta_1}(A)$, where $\delta_1 := \delta$ if $c > 0$, and $\delta_1 := \delta(1+c)^{-2}$ if $-1 < c < 0$. Further, by Lemma 5, $\nabla a = c(d-1)|x|^{-2}x \in \mathbf{F}_{\delta_0}(A)$, where $\delta_0 := 4\bigl(\frac{d-1}{d-2}\frac{c}{1+c}\bigr)^2$. Now, set as in the formulation of assertion (i),

$$\sqrt{\delta_2} := \begin{cases} \sqrt{\delta_1} + \sqrt{\delta_0}, & 0 < c < d-2, \\ \sqrt{\delta_1}, & -1 < c < 0. \end{cases}$$

For $c > 0$, we have by our assumption $\delta_2 < 4$ if $c > 0$, so by [KiS, Theorem 3.2] the formal differential expression $-a \cdot \nabla^2 + b \cdot \nabla$ ($\equiv -\nabla \cdot a \cdot \nabla + (\nabla a) \cdot \nabla + b \cdot \nabla$) has an operator realization $\Lambda_q(a, \nabla a + b)$ in L^q, $q \in \left[\frac{2}{2-\sqrt{\delta_2}}, \infty\right[$, as the (minus) generator of a positivity preserving L^∞ contraction quasi contraction C_0 semigroup; moreover, $(\mu + \Lambda_q(a,b))^{-1}$ is well defined on L^q for all $\mu > \frac{\lambda\delta_2}{2(q-1)}$. In case $c < 0$, we apply Theorem A.1. This completes the proof of (i).

Proof of (ii). Let $d \geq 4$. Set $a^\varepsilon := I + c|x|_\varepsilon^{-2} x \otimes x$, $|x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}$, $\varepsilon > 0$. Put $A^\varepsilon = A(a^\varepsilon)$. It is clear that $b \in \mathbf{F}_{\delta_1}(A^\varepsilon)$ for all $\varepsilon > 0$.

Let $\mathbf{1}_n$ denote the indicator of $\{x \in \mathbb{R}^d \mid \ |x| \leq n, |b(x)| \leq n\}$, and set $b_n := \gamma_{\epsilon_n} * \mathbf{1}_n b \in C^\infty$, where γ_ϵ is the K. Friedrichs mollifier, $\epsilon_n \downarrow 0$. Since our assumptions on δ and thus δ_1 involve strict inequalities only, we can select $\epsilon_n \downarrow 0$ so that $b_n \in \mathbf{F}_{\delta_1}(A^\varepsilon)$, $\varepsilon > 0$, $n \geq 1$. Next, note that by the assumptions of the theorem $-1 < c < \frac{d-3}{2}$, and hence by Lemma 6, $\nabla a^\varepsilon \in \mathbf{F}_{\delta_0}(A^\varepsilon)$, $\varepsilon > 0$. Thus, in view of the discussion above, $(\mu + \Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n))^{-1}$ is well defined on L^q, $\mu > \frac{\lambda\delta_2}{2(q-1)}$, $\varepsilon > 0$, $n \geq 1$. Here $\Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n) = -\nabla \cdot a^\varepsilon \cdot \nabla + (\nabla a^\varepsilon) \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n)) = W^{2,q}$.

Set $u \equiv u^{\varepsilon,n} = (\mu + \Lambda_q(a^\varepsilon, \nabla a^\varepsilon + b_n))^{-1} f$, $0 \leq f \in C_c^1$. Then $u \in W^{3,q}$. Below

$$w \equiv w^{\varepsilon,n} := \nabla u^{\varepsilon,n}, \quad \phi := -\nabla \cdot (w|w|^{q-2}),$$

$$I_q := \langle (\nabla_r w)^2 |w|^{q-2} \rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2} \rangle,$$

$$\bar{I}_{q,\chi} := \langle \big(x \cdot \nabla w\big)^2 \chi |x|^{-2} |w|^{q-2} \rangle, \quad \bar{J}_{q,\chi} := \langle (x \cdot \nabla |w|)^2 \chi |x|^{-2} |w|^{q-2} \rangle,$$

$$H_{q,\chi} := \langle \chi |x|^{-2} |w|^q \rangle, \qquad G_{q,\chi^2} := \langle \chi^2 |x|^{-4} (x \cdot w)^2 |w|^{q-2} \rangle,$$

where $\chi = |x|^2 |x|_\varepsilon^{-2}$.

1. We repeat the proof of Lemma 7, where in the right-hand side of ($\mathrm{BE}_+^{\mathrm{nd}}$), ($\mathrm{BE}_-^{\mathrm{nd}}$) we now get an extra term $\langle -b_n \cdot w, \phi \rangle$:

$$\begin{aligned} &\mu \langle |w|^q \rangle + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\frac{d(d-1)}{q} H_{q,\chi} \\ &+ 2c\frac{2d+1}{q} H_{q,\chi^2} - \frac{8c}{q} H_{q,\chi^3} = \beta_1 + \langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle, \end{aligned} \tag{$\mathrm{BE}_{+,b}^{\mathrm{nd}}$}$$

$$\begin{aligned} &\mu \langle |w|^q \rangle + I_q + c\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\frac{d(d-1)}{q} H_{q,\chi} - c(d-2) G_{q,\chi^2} \\ &+ 2c\frac{2d+1}{q} H_{q,\chi^2} - \frac{8c}{q} H_{q,\chi^3} - 4c\varepsilon \langle |x|_\varepsilon^{-6} (x \cdot w)^2 |w|^{q-2} \rangle = -\frac{1}{2}\beta_2 + \langle -b_n \cdot w, \phi \rangle + \langle f, \phi \rangle, \end{aligned} \tag{$\mathrm{BE}_{-,b}^{\mathrm{nd}}$}$$

where

$$\beta_1 := -2c \langle |x|_\varepsilon^{-4} x \cdot w, x \cdot (x \cdot \nabla w) |w|^{q-2} \rangle, \qquad \beta_2 := -2c(q-2) \langle |x|_\varepsilon^{-4} (x \cdot w)^2 x \cdot \nabla |w|, |w|^{q-3} \rangle.$$

2. By Lemma 4,

$$\langle -b_n \cdot w, \phi\rangle$$
$$\leq |c|(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + |c|\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.$$

Next, by an evident analogue of Lemma 3,

$$\langle f, \phi\rangle \leq \varepsilon_0(I_q + J_q + H_q + \|w\|_q^q) + C(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2.$$

Again we choose $\varepsilon_0 > 0$ so small that in the estimates below we can ignore the terms multiplied by ε_0.

Applying the last two inequalities in ($\text{BE}^{\text{nd}}_{+,b}$), ($\text{BE}^{\text{nd}}_{-,b}$), and using $\beta_1 \leq c\theta\bar{I}_{q,\chi} + c\theta^{-1}G_{q,\chi^2}$, $|\beta_2| \leq 2|c|(q-2)\big(\theta\bar{J}_{q,\chi} + 4^{-1}\theta^{-1}G_{q,\chi^2}\big)$, we obtain:

If $c > 0$, then:

$$\begin{aligned}&\mu\langle|w|^q\rangle + I_q + c(1-\theta)\bar{I}_{q,\chi} + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\frac{d(d-1)}{q}H_{q,\chi}\\ &+ 2c\frac{2d+1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}G_{q,\chi^2}\\ &\leq c(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + c\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned} \tag{15}$$

If $-1 < c < 0$, then (set $s := |c|$):

$$\begin{aligned}&\mu\langle|w|^q\rangle + I_q - s\bar{I}_{q,\chi} + (q-2)(J_q - s(1+\theta)\bar{J}_{q,\chi}) + s\frac{d(d-1)}{q}H_{q,\chi}\\ &- 2s\frac{2d+1}{q}H_{q,\chi^2} + \frac{8s}{q}H_{q,\chi^3} + s\left(d+2-(q-2)\frac{1}{4\theta}\right)G_{q,\chi^2} - 4sG_{q,\chi^3}\\ &\leq s(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + s\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned} \tag{16}$$

3. We will use (15), (16) to prove the following inequality

$$\mu\langle|w|^q\rangle + \eta J_q \leq C_1\|w\|_q^{q-2} + C_2\|w\|_q^{q-2}\|f\|_q^2, \qquad C_i = C_i(\varepsilon_0), \quad i = 1, 2, \tag{17}$$

for some $\eta = \eta(q, d, \varepsilon_0) > 0$.

Case $c > 0$. In (15), select $\theta = \frac{q}{d-2} > 1$. By the assumptions of the theorem, $1 + c(1-\theta) > 0$. Since $I_q + c(1-\theta)\bar{I}_{q,\chi} \geq (1 + c(1-\theta))I_q$ and $H_{q,\chi^2} \geq G_{q,\chi^2}$, we have

$$\begin{aligned}&\mu\langle|w|^q\rangle + (1+c(1-\theta))I_q + (q-2)(J_q + c\bar{J}_{q,\chi}) - c\frac{d(d-1)}{q}H_{q,\chi}\\ &+ 2c\frac{2d+1}{q}H_{q,\chi^2} - \frac{8c}{q}H_{q,\chi^3} - \frac{c}{\theta}H_{q,\chi^2}\\ &\leq c(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + c\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Arguing as in the proof of Theorem 3, we arrive at

$$\begin{aligned}&\mu\langle|w|^q\rangle + (1+c(1-\theta))I_q + (q-2)J_q + cM(1)H_q\\ &\leq c(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac{1}{2}}J_q^{\frac{1}{2}} + c\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac{1}{2}}J_q^{\frac{1}{2}} + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2,\end{aligned}$$

where $M(1)=-2\frac{(d-2)^2}{q^2}$. Using $I_q \geq \bar{I}_{q,\chi}$, $H_q \geq G_{q,\chi^2}$ in the RHS, and applying the standard quadratic estimates, we obtain $(\theta_2,\theta_3>0)$,

$$\begin{aligned}&\mu\langle|w|^q\rangle+(1+c(1-\theta))I_q+(q-2)J_q+cM(1)H_q\\&\leq c(d+3)\frac{q\sqrt{\delta}}{4}(\theta_2 J_q+\theta_2^{-1}H_q)+c\frac{q\sqrt{\delta}}{4}(\theta_3 I_q+\theta_3^{-1}J_q)+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right)J_q\\&+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

We select $\theta_2=\frac{q}{d-2}$, $\theta_3=1$. By the assumptions of the theorem, $1+c\big(1-\theta-\frac{q\sqrt{\delta}}{4}\big)\geq 0$, so that $\big(1+c(1-\theta-\frac{q\sqrt{\delta}}{4})\big)I_q \geq \big(1+c(1-\theta-\frac{q\sqrt{\delta}}{4})\big)J_q$. Thus, we arrive at

$$\begin{aligned}&\mu\langle|w|^q\rangle+\left[q-1+c\left(1-\theta-\frac{q\sqrt{\delta}}{2}\right)-c(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2}-\frac{q^2\delta}{4}-(q-2)\frac{q\sqrt{\delta}}{2}\right]J_q\\&+\left[cM(1)-c(d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\right]H_q\leq C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

So, by $J_q\geq\frac{(d-2)^2}{q^2}H_q$,

$$\mu\langle|w|^q\rangle+\eta J_q+\left[(-\eta+q-1)\frac{(d-2)^2}{q^2}-Ł_1^{\mathrm{nd}}(c,\delta)\right]H_q\leq C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2,$$

where

$$\begin{aligned}Ł_1^{\mathrm{nd}}(c,\delta)&=\left[-c\left(1-\theta-\frac{q\sqrt{\delta}}{2}\right)+c(d+3)\frac{q\sqrt{\delta}}{4}\frac{q}{d-2}+\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right]\frac{(d-2)^2}{q^2}\\&+c\left[-M(1)+(d+3)\frac{q\sqrt{\delta}}{4}\frac{d-2}{q}\right].\end{aligned}$$

By the assumptions of the theorem, $\big(-\eta+q-1\big)\frac{(d-2)^2}{q^2}-Ł_1(c,\delta)\geq 0$ for all $\eta>0$ sufficiently small. (17) is proved.

Case $-1<c<0$**.** Following the proof of Theorem 3, we obtain from (16)

$$\begin{aligned}&\mu\langle|w|^q\rangle+(1-s)I_q+(q-2)\big(1-s(1+\theta)\big)J_q+s\frac{d(d-1)}{q}G_{q,\chi}\\&-2s\frac{2d+1}{q}G_{q,\chi^2}+\frac{8s}{q}G_{q,\chi^3}+s\left(d+2-(q-2)\frac{1}{4\theta}\right)G_{q,\chi^2}-4sG_{q,\chi^3}\\&\leq s(d+3)\frac{q\sqrt{\delta}}{2}G_{q,\chi^2}^{\frac12}J_q^{\frac12}+s\frac{q\sqrt{\delta}}{2}\bar{I}_{q,\chi}^{\frac12}J_q^{\frac12}+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right)J_q+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2,\end{aligned}$$

In the RHS, we use $\frac{q^2}{(d-2)^2}J_q\geq G_{q,\chi^2}$, $I_q\geq\bar{I}_{q,\chi}$, $\frac12(I_q+J_q)\geq I_q^{\frac12}J_q^{\frac12}$

$$\begin{aligned}&\mu\langle|w|^q\rangle+(1-s)I_q+(q-2)\big(1-s(1+\theta)\big)J_q+s\frac{d(d-1)}{q}G_{q,\chi}\\&-2s\frac{2d+1}{q}G_{q,\chi^2}+\frac{8s}{q}G_{q,\chi^3}+s\left(d+2-(q-2)\frac{1}{4\theta}\right)G_{q,\chi^2}-4sG_{q,\chi^3}\\&\leq s(d+3)\frac{q\sqrt{\delta}}{2}\frac{q}{d-2}J_q+s\frac{q\sqrt{\delta}}{4}(I_q+J_q)+\left(\frac{q^2\delta}{4}+(q-2)\frac{q\sqrt{\delta}}{2}\right)J_q+C_1\|w\|_q^q+C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

Arguing as in the proof of Theorem 3, and selecting $\theta := \frac{1}{4}\frac{q}{d-2}\frac{q-2}{q+d-3}$, we arrive at

$$\begin{aligned}&\mu\langle |w|^q\rangle + (1-s)I_q + (q-2)\big(1-s(1+\theta)\big)J_q \\ &\leq s(d+3)\frac{q\sqrt{\delta}}{2}\frac{q}{d-2}J_q + s\frac{q\sqrt{\delta}}{4}(I_q+J_q) + \left(\frac{q^2\delta}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right)J_q + C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

By the assumptions of the theorem, $1 - s\big(1+\frac{q\sqrt{\delta}}{4}\big) \geq 0$. Therefore, since $J_q \leq I_q$,

$$\begin{aligned}&\mu\langle |w|^q\rangle + \left[q-1-s-s(q-2)(1+\theta) - s(d+3)\frac{q\sqrt{\delta}}{2}\frac{q}{d-2} - s\frac{q\sqrt{\delta}}{2} - \frac{q^2\delta}{4} - (q-2)\frac{q\sqrt{\delta}}{2}\right]J_q \\ &\leq C_1\|w\|_q^q + C_2\|w\|_q^{q-2}\|f\|_q^2.\end{aligned}$$

By the assumptions of the theorem, $q-1-s-s(q-2)(1+\theta)-s(d+3)\frac{q\sqrt{\delta}}{2}\frac{q}{d-2}-s\frac{q\sqrt{\delta}}{2}-\frac{q^2\delta}{4}-(q-2)\frac{q\sqrt{\delta}}{2} > 0$. Hence (17) is proved.

4. For $d \geq 4$, the Sobolev Embedding Theorem and Theorem A.2(*ii*) now yield ($\star\star$). We have proved (*ii*).

Proof of (iii). Let $q=2$, $d \geq 3$. If $c<0$, then we can argue as in steps 1-3 obtaining

$$\sup_{\varepsilon>0,n} I_2(u^{\varepsilon,n}) \leq K\|f\|_2, \qquad \text{and so } I_2(u) \leq K\|f\|_2 \quad \Rightarrow u \in W^{2,2}.$$

Now, let $c>0$. We have $b+\nabla a \in \mathbf{F}_{\delta_2}(A)$, $\sqrt{\delta_2} := \sqrt{\delta} + 2\frac{d-1}{d-2}\frac{c}{1+c}$ (cf. beginning of the proof). By the assumptions of the theorem, $\delta_2<1$, and so $\Lambda_2(a,\nabla a)$ is well defined. By the Miyadera Perturbation Theorem and Theorem 1, $D(\Lambda_2(a,\nabla a+b)) = D(A_2) \subset W^{2,2}$, and $u := (\mu+\Lambda_2(a,\nabla a+b))^{-1}f$, $\mu>0$, $f\in L^2$, belongs to $W^{2,2}$. Multiplying $(\mu + A_2 + (\nabla a + b)\cdot\nabla)u = f$ by $\phi_m := -E_m\nabla\cdot w$, $E_m = (1-m^{-1}\Delta)^{-1}$, $m\geq 1$ and integrating by parts we have (omitting the summation sign in the repeated indices):

$$\mu\langle |w|^2\rangle + \langle a\cdot\nabla w_r, E_m\nabla w_r\rangle + \langle -(\nabla_r a)\cdot w, E_m\nabla w_r)\rangle + \langle \nabla a\cdot w,\phi_m\rangle = \langle -b\cdot w,\phi_m\rangle + \langle f,\phi_m\rangle, \quad (\star)$$

Now we pass in ($\star$) to the limit $m\to\infty$. We obtain an analogue of ($\text{BE}^{\text{nd}}_{+,b}$) for $q=2$:

$$\mu\langle |w|^2\rangle + I_2 + c\bar{I}_2 - \frac{c}{2}(d-2)(d-3)H_2 = \beta + \langle -b\cdot w,\phi\rangle + \langle f, -\nabla\cdot w\rangle$$

where $I_2 := \langle \nabla w_r,\nabla w_r\rangle$, $\bar{I}_2 := \langle (x\cdot\nabla w)^2|x|^{-2}\rangle$, $H_2 := \langle |x|^{-2}|w|^2\rangle$, $\beta := -2c\langle |x|^{-4}x\cdot w, x\cdot(x\cdot\nabla w)\rangle$.

Using the inequalities $\beta \leq c\bar{I}_2 + cH_2$, $\frac{(d-2)^2}{4}H_2 \leq I_2$, we have

$$\mu\langle |w|^2\rangle + \left[1 - \frac{4}{(d-2)^2}\left(1+\frac{(d-2)(d-3)}{2}\right)c\right]I_2 \leq \langle -b\cdot w,\phi\rangle + \langle f,\phi\rangle,$$

so by

$$\begin{aligned}|\langle b\cdot w,\phi\rangle| &\leq c(d+3)\frac{\sqrt{\delta}}{2}\left(\frac{2}{d-2}J_2 + \frac{d-2}{2}H_q\right) + c\frac{\sqrt{\delta}}{2}(I_2+J_2) + \delta J_2 \qquad (J_2\leq I_2) \\ &\leq \left(2c\sqrt{\delta}\frac{d+3}{d-2} + c\sqrt{\delta} + \delta\right)I_2.\end{aligned}$$

Therefore,

$$\mu\langle |w|^2\rangle + \left[1 - \frac{4}{(d-2)^2}\left(1+\frac{(d-2)(d-3)}{2}\right)c - c\sqrt{\delta}\left(2\frac{d+3}{d-2}+1\right) - \delta\right]I_2 \leq \langle f,\phi\rangle,$$

where the coefficient of I_2 is positive by the assumptions of the theorem. The proof of (iii) follows.

The proof of Theorem 4 is completed. □

Appendix A.

Theorem A.1. *Let $d \geq 3$. Let $a = I + c|x|^{-2}x \otimes x$, $-1 < c < 0$, $a^\varepsilon := I + c|x|_\varepsilon^{-2}x \otimes x$, $|x|_\varepsilon^2 := |x|^2 + \varepsilon$, $\varepsilon > 0$. Set $a_n := a^{\varepsilon_n}$, $\varepsilon_n \downarrow 0$. Let $b \in \mathbf{F}_\delta(A)$, $0 < \delta < 4$. Let $\mathbf{1}_n$ denote the indicator of $\{x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n\}$, and set $b_n := \gamma_{\epsilon_n} * \mathbf{1}_n b$, where γ_ϵ is the K. Friedrichs mollifier, $\epsilon_n \downarrow 0$. Then $-\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla$ has an operator realization $\Lambda_r(a, b)$ in L^r, $r \geq \frac{2}{2-\sqrt{\delta}}$, as the (minus) generator of a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup on L^r, and, for each $r > \frac{2}{2-\sqrt{\delta}}$,*

$$e^{-t\Lambda_r(a,\nabla a+b)} = s\text{-}L^r\text{-}\lim_{n\to\infty} e^{-t\Lambda_r(a,\nabla a_n+b_n)}.$$

Proof. We modify the proof of [KiS, Theorem 3.2] replacing b_n there with $\nabla a_n + b_n$.

In comparison with [KiS, Theorem 3.2], essentially we have an extra term $\langle \nabla a_n \cdot \nabla v, v^{r-1}\rangle$ to deal with. Since $\nabla a_n = c(d-1)|x|_{\varepsilon_n}^{-2}x + 2c\varepsilon_n|x|_{\varepsilon_n}^{-4}x$, we have (write $\varepsilon = \varepsilon_n$)

$$\langle \nabla a_n \cdot \nabla v, v^{r-1}\rangle = c(d-1)\langle |x|_\varepsilon^{-2}x \cdot \nabla v, v^{r-1}\rangle + 2c\varepsilon\langle |x|_\varepsilon^{-4}x \cdot \nabla v, v^{r-1}\rangle,$$

and

$$\frac{r}{2}\langle |x|_\varepsilon^{-2}x \cdot \nabla v, v^{r-1}\rangle = \langle |x|_\varepsilon^{-2}x \cdot \nabla v^{\frac{r}{2}}, v^{\frac{r}{2}}\rangle = -\frac{d-2}{2}\langle |x|_\varepsilon^{-2}v^r\rangle - \varepsilon\langle |x|_\varepsilon^{-4}v^r\rangle,$$

$$\frac{r}{2}\varepsilon\langle |x|_\varepsilon^{-4}x \cdot \nabla v, v^{r-1}\rangle = \varepsilon\langle |x|_\varepsilon^{-4}x \cdot \nabla v^{\frac{r}{2}}, v^{\frac{r}{2}}\rangle = -\frac{d-4}{2}\varepsilon\langle |x|_\varepsilon^{-4}v^r\rangle - 2\varepsilon^2\langle |x|_\varepsilon^{-6}v^r\rangle.$$

Thus, since $c < 0$ and $v > 0$, we have $\langle \nabla a_n \cdot \nabla v, v^{r-1}\rangle \geq 0$ for all $d \geqslant 3$. This inequality allows us to discard the extra term.

The rest of the proof repeats [KiS, Theorem 3.2]. □

Theorem A.2. *Let $d \geq 3$. Let $a = I + c|x|^{-2}x \otimes x$, $a^\varepsilon := I + c|x|_\varepsilon^{-2}x \otimes x$, $|x|_\varepsilon^2 := |x|^2 + \varepsilon$, $\varepsilon > 0$. Set $a_n := a^{\varepsilon_n}$, $\varepsilon_n \downarrow 0$. Let $b \in \mathbf{F}_\delta$, let b_n's be as in Theorem A.1.*

(i) Assume that $q > d-2$, $d \geq 4$, c, δ satisfy the assumptions of Theorem 2(ii), or $q = 2$, $d \geq 3$, c, δ satisfy the assumptions of Theorem 2(iii). Then $(\mu + \Lambda_q(a, b))^{-1}$, $(\mu + \Lambda_q(a_n, b_n))^{-1}$, $\mu > \omega_q$, are well defined, and

$$(\mu + \Lambda_q(a, b))^{-1} = s\text{-}L^q\text{-}\lim_n(\mu + \Lambda_q(a_n, b_n))^{-1}.$$

Here $\Lambda_q(a_n, b_n) = -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_q(a_n, b_n)) = W^{2,q}$.

(ii) Assume that $q > d-2$, $d \geq 4$, c, δ satisfy the assumptions of Theorem 4(ii), or $q = 2$, $d \geq 3$, $c < 0$, δ satisfy the assumptions of Theorem 4(iii). Then $(\mu+\Lambda_q(a, \nabla a+b))^{-1}$, $(\mu+\Lambda_q(a_n, \nabla a_n+b_n))^{-1}$, $\mu > \omega_q$, are well defined, and

$$(\mu + \Lambda_q(a, \nabla a + b))^{-1} = s\text{-}L^q\text{-}\lim_n(\mu + \Lambda_q(a_n, \nabla a_n + b_n))^{-1}.$$

Proof. We modify the proof of [KiS, Theorem 3.2] but will work with resolvents instead of semigroups.

To prove (i), set $\delta_* := [1 \vee (1+c)^{-2}]\delta$, $\tilde{b} := b$, $\tilde{b}_n := b_n$.

To prove (ii), set

$$\sqrt{\delta_*} := \begin{cases} \sqrt{\delta} + 2\frac{d-1}{d-2}\frac{c}{1+c}, & 0 < c < d-2, \\ (1+c)^{-1}\sqrt{\delta}, & -1 < c < 0. \end{cases}$$

and $\tilde{b} := b + \nabla a$, $\tilde{b}_n := b_n + \nabla a_n$.

1. First, prove (i) and (ii) for $c > 0$. Set $A^n \equiv [-\nabla \cdot a_n \cdot \nabla \upharpoonright C_c^\infty]_{2\to 2}^{\text{clos}}$. Then $\tilde{b} \in \mathbf{F}_{\delta_*}(A)$, $\tilde{b}_n \in \mathbf{F}_{\delta_*}(A^n)$ (for details see the proofs of Theorems 2 and 4, respectively), where, by our assumptions, $\delta_* < 4$. Therefore, by [KiS, Theorem 3.2] $(\mu + \Lambda_q(a,\tilde{b}))^{-1}$, $(\mu + \Lambda_q(a_n,\tilde{b}_n))^{-1}$, $q > \frac{2}{2-\sqrt{\delta_*}}$, $\mu > \omega_q$, are well defined, and $\lim_n \|(\mu + \Lambda_q(a,\tilde{b}))^{-1} f - (\mu + \Lambda_q(a,\tilde{b}_n))^{-1} f\|_q = 0$, $f \in L^q$. Thus, it suffices to show that

$$\lim_n \|(\mu + \Lambda_q(a,\tilde{b}_n))^{-1} f - (\mu + \Lambda_q(a_n,\tilde{b}_n))^{-1} f\|_q = 0.$$

(a) Fix $f \in L^\infty \cap L^2_+$. Set $u_n := (\mu + \Lambda_q(a,\tilde{b}_n))^{-1} f \geq 0$, $\tilde{u}_n := (\mu + \Lambda_q(a_n,\tilde{b}_n))^{-1} f \geq 0$. Let $v := \zeta u_n \geq 0$, where $\zeta = \zeta(R)$, $0 \leq \zeta \leq 1$, $\zeta \equiv 0$ on an open ball of radius $\simeq R$, $\zeta \equiv 1$ on the complement of an open ball of radius $\simeq 2R$, is defined in [KiS, proof of Theorem 3.2, step 1]. Note that $\langle \zeta(\mu + \Lambda_q(a,\tilde{b}_n))u_n, v^{q-1}\rangle = \langle \zeta(\mu + A + \tilde{b}_n \cdot \nabla)u_n, v^{q-1}\rangle$ according to [KiS, proof of Theorem 3.2, step 1], and hence

$$\Big\langle \zeta(\mu + A + \tilde{b}_n \cdot \nabla)u_n, v^{q-1}\Big\rangle = \langle \zeta f, v^{q-1}\rangle.$$

Now, proceeding as in [KiS, proof of Theorem 3.2, step 1], we arrive at the following. For every $\varepsilon > 0$ there exists $R > 0$ such that

$$\|\zeta u_n\|_q \leq \varepsilon, \quad n \geq 1, \quad \mu > \omega_q.$$

Similarly,

$$\|\zeta \tilde{u}_n\|_q \leq \varepsilon \quad n \geq 1, \quad \mu > \omega_q.$$

(b) Set $g_n := u_n - \tilde{u}_n$. For the R determined above, set $v := \zeta g_n$, where $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on an open ball of radius $\simeq R$, $\zeta \equiv 0$ on the complement of an open ball of radius $\simeq 2R$, is defined in [KiS, proof of Theorem 3.2, step 2]. Subtracting the equations for u_n and $\tilde{u}_n$, we have

$$\Big\langle \zeta\big(\mu + A + \tilde{b}_n \cdot \nabla\big)g - \zeta \nabla \cdot (a - a_n) \cdot \nabla \tilde{u}_n, v|v|^{q-2}\Big\rangle = 0.$$

Arguing as in [KiS], we arrive at the inequality

$$\mu \|v\|_q^q \leq M(q,d,\delta_*)R^{-\gamma}\|f\|_\infty^q + |\langle \zeta \nabla \cdot (a - a_n) \cdot \nabla \tilde{u}_n, v|v|^{q-2}\rangle|.$$

To show that $\zeta(u_n - \tilde{u}_n) \to 0$ strongly in L^q as $n \to 0$, it remains to prove that $\lim_n |Z| = 0$, where $Z := \langle (a - a_n) \cdot \nabla \tilde{u}_n, \nabla(\zeta v|v|^{q-2})\rangle$. The latter is possible due to the bounds $\|\nabla u_n\|_{\frac{qd}{d-2}} \leq K\|f\|_q$, $\|\nabla \tilde{u}_n\|_{\frac{qd}{d-2}} \leq K\|f\|_q$ (cf. the proof of Theorem 3 (steps 1-3) for (i), the proof of Theorem 4 (steps 1-3) for (ii)). Indeed,

$$\begin{aligned} Z &= q\langle \zeta^{q-1}\nabla\zeta \cdot (a - a_n) \cdot \nabla \tilde{u}_n, g_n|g_n|^{q-2}\rangle + \langle \nabla g_n \cdot (a - a_n) \cdot \nabla \tilde{u}_n, \zeta^q |g_n|^{q-2}\rangle \\ &+ (q-3)\langle \nabla|g_n| \cdot (a - a_n) \cdot \nabla \tilde{u}_n, \zeta^q g_n|g_n|^{q-3}\rangle \\ &\equiv qZ_1 + Z_2 + (q-3)Z_3, \end{aligned}$$

and ($q_* := \frac{qd}{d-2} > 2$)

$$\begin{aligned} |Z_1| &\leq q\|\nabla\zeta \cdot (a - a_n)\|_{q_*'}\|\nabla \tilde{u}_n\|_{q_*}\|g_n\|_\infty^{q-1}, \\ |Z_2| &\leq \|\nabla g_n \cdot \zeta(a - a_n) \cdot \nabla \tilde{u}_n\|_1 \|g_n\|_\infty^{q-2}, \\ |Z_3| &\leq \|\nabla|g_n| \cdot \zeta(a - a_n) \cdot \nabla \tilde{u}_n\|_1 \|g_n\|_\infty^{q-2}, \\ \|\nabla|g_n|\|_{q_*} &\leq \|\nabla g_n\|_{q_*} \leq 2K\|f\|_q, \ \sup_{i,j}\|\zeta(a_{ij} - a_{ij}^{\varepsilon_n})\|_{\frac{q_*}{q_*-2}} \to 0 \text{ as } n \to 0. \end{aligned}$$

Thus, by Hölder's inequality and $|a_{ij} - a^\varepsilon_{ij}| \leq c\varepsilon|x|_\varepsilon^{-2} \downarrow 0$ a.e. as $n \to \infty$,

$$\lim_n |Z| = 0.$$

It follows that $\zeta(u_n - \tilde{u}_n) \to 0$ in L^q.

Combining the results of (a) and (b), we obtain the required.

2. To prove (ii) with $c < 0$, we repeat the proof above but taking into account the proof of Theorem A.1. □

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