$W^{1,p}$ REGULARITY OF SOLUTIONS TO KOLMOGOROV EQUATION AND ASSOCIATED FELLER SEMIGROUP

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ABSTRACT. In $\mathbb{R}^d$, $d \geq 3$, consider the divergence and the non-divergence form operators

\begin{align*}
- \nabla \cdot a \cdot \nabla + b \cdot \nabla, \\
- a \cdot \nabla^2 + b \cdot \nabla,
\end{align*}

where $a = I + cf \otimes f$, the vector fields $\nabla_i f$ ($i = 1, 2, \ldots, d$) and $b$ are form-bounded (this includes the sub-critical class $[L^d + L^\infty]^d$ as well as vector fields having critical-order singularities). We characterize quantitative dependence on $c$ and the values of the form-bounds of the $L^q \to W^{1,q+d/(d-2)}$ regularity of the resolvents of the operator realizations of $[\square, \square]$ in $L^q$, $q \geq 2 \vee (d - 2)$ as (minus) generators of positivity preserving $L^\infty$ contraction $C_0$ semigroups. The latter allows to run an iteration procedure $L^p \to L^\infty$ that yields associated with $[\square, \square]$ $L^q$-strong Feller semigroups.

1. Consider in $\mathbb{R}^d$, $d \geq 3$, the formal differential operator

\begin{equation}
- \nabla \cdot a \cdot \nabla + b \cdot \nabla \equiv - \sum_{i,j=1}^d \nabla_i a_{ij}(x) \nabla_j + \sum_{j=1}^d b_j(x) \nabla_j,
\end{equation}

where

\begin{equation}
\begin{aligned}
a &= a^* : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \text{ is } \mathcal{L}^d \text{ measurable,} \\
\sigma I &\leq a(x) \leq \xi I \text{ for } \mathcal{L}^d \text{ a.e. } x \in \mathbb{R}^d \text{ for some } 0 < \sigma \leq \xi < \infty.
\end{aligned}
\end{equation}

By the De Giorgi-Nash theory, the solution $u \in W^{1,2}(\mathbb{R}^d)$ to the corresponding elliptic equation $(\mu - \nabla \cdot a \cdot \nabla + b \cdot \nabla)u = f$, $\mu > 0$, $f \in L^p \cap L^2$, $p \in ]\frac{d}{2}, \infty[$, is in $C^{0,\gamma}$, where the Hölder continuity exponent $\gamma \in ]0, 1[$ depends only on $d$ and $\sigma, \xi$, provided that $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the Nash class $(\subset [L^p + L^\infty]^d$, $p > d)$ [1], but already the class $[L^d + L^\infty]^d$ is not admissible (e.g., it is easy to find $b \in [L^d + L^\infty]^d$ that makes the two-sided Gaussian bounds on the fundamental solution of [1] invalid). On the other hand, for $-\Delta + b \cdot \nabla$, the $C^{0,\gamma}$ regularity of solutions to the corresponding elliptic equations is known to hold for $b$ having much stronger singularities. Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the class of form-bounded vector fields $F_\delta \equiv F_\delta(-\Delta)$, $\delta > 0$ if $|b| \in L^2_{\text{loc}} \equiv L^2_{\text{loc}}(\mathbb{R}^d)$ and there exist a constant $\lambda = \lambda_\delta > 0$ such that

\begin{equation}
\|\|b)(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}.
\end{equation}

(The class $F_\delta$ contains $[L^d + L^\infty]^d$ with $\delta$ arbitrarily small, as follows by the Sobolev Embedding Theorem, as well as vector fields having critical-order singularities such as $b(x) = \frac{d - 2}{2} \sqrt{\delta} |x|^{-2} x$ (by Hardy’s inequality) or, more generally, vector fields in $[L^{d,\infty} + L^\infty]^d$, the Campanato-Morrey class or the Chang-Wilson-T. Wolff class, with $\delta$ depending on the norm of the vector field in these classes, see)

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e.g. [KiS] for details.) It has been established in [KS] that if \( b \in F_\delta \), \( \delta < 1 \), then for every \( q \in [2, 2/\sqrt{\delta}] \) \(-\Delta + b \cdot \nabla \) has an operator realization \( \Lambda_q(b) \) on \( L^q \) as the generator of a positivity preserving, \( L^\infty \) contraction, quasi contraction \( C_0 \) semigroup \( e^{-t\Lambda_q(b)} \) such that \( u := (\mu + \Lambda_q(b))^{-1} f, \ f \in L^q \) satisfies

\[
\|\nabla u\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{2}}\|f\|_q, \quad \|\nabla |\nabla u|^{\frac{2}{q}}\|_2 \leq K_2(\mu - \mu_0)^{-\frac{1}{2}}\|f\|_q, \quad \mu > \mu_0
\]

for some constants \( \mu_0 \equiv \mu_0(d, q, \delta) > 0 \) and \( K_j = K_j(d, q, \delta) > 0, \ i = 1, 2 \). In particular, if \( \delta < 1 \wedge \left( \frac{2}{d-2} \right)^2 \), there exists \( q > 2 \vee (d-2) \) such that \( u \in C^{0, \gamma}, \gamma = 1 - \frac{d-2}{q} \). The explicit dependence of the regularity properties of \( u \) on \( \delta \) (which effectively plays the role of a “coupling constant”) is a crucial feature of the result in [KS].

In the present paper our concern is: to find a class of matrices \( a \in (H_u) \) such that the operator (1) with \( b \in F_\delta \) admits a \( W^{1,p} \) and \( C^{0,\gamma} \) regularity theory. Below we consider

\[
a = I + cf \otimes f, \quad c > -1, \quad f \in [L^\infty \cap W^{1,2}_loc]^d, \quad \|f\|_\infty = 1,
\]

\[
\nabla_i f \in F_{\delta^i}, \delta^i > 0, \quad i = 1, 2, \ldots, d, \quad \delta_f := \sum_{i=1}^d \delta^i.
\]

(C\( \delta_f \))

The model example of such \( a \) is the matrix

\[
a(x) = I + c|x|^{-2}x \otimes x, \quad x \in \mathbb{R}^d
\]

having critical discontinuity at the origin, see [GS, GrS, KiS2, OG] and references therein. (Replacing the requirement \( \nabla_i f \in F_{\delta_i} \) by a more restrictive \( \nabla_i f \in [L^p + L^\infty]^d \), \( p > d \), forces \( a \) to be Hölder continuous. On the other hand, a weaker condition \( \nabla_i f \in [L^p + L^\infty]^d \), \( p < d \), is incompatible with the uniform ellipticity of \( a \). The condition \( (C\delta^i_f) \) \( \nabla_i f \in [L^d + L^\infty]^d \) seems to be rather natural. We also note that the operator \(-a \cdot \nabla^2 \) with \( \nabla_k a_{ij} \in L^{d,\infty} \) has been studied earlier in [AT], cf. the discussion below concerning the non-divergence form operators.)

In Theorems 1, 2 below we characterize quantitative dependence on \( c, \delta, \delta_f \) of the \( L^q \to W^{1,qd/(d-2)} \) regularity of the resolvent of the operator realization of (1) as (minus) generator of positivity preserving \( L^\infty \) contraction \( C_0 \) semigroups in \( L^q, q \geq 2 \vee (d-2) \).

Consider the non-divergence form operator

\[
-a \cdot \nabla^2 + b \cdot \nabla \equiv - \sum_{i,j=1}^d a_{ij}(x) \nabla_i \nabla_j + \sum_{j=1}^d b_j(x) \nabla_j.
\]

(3)

Write \(-a \cdot \nabla^2 + b \cdot \nabla \equiv -\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla \), where \( (\nabla a)_k := \sum_{i=1}^d (\nabla_i a_{ik}), k = 1, 2, \ldots, d \). Then

\[
\nabla a = c[(\text{div}f) + f \cdot \nabla f].
\]

It is easily seen that the condition \( (C\delta_f) \) yields \( \nabla a \in F_{\delta_a} \) with \( \delta_a \leq |c|^2(\sqrt{d} + 1)^2 \delta_f \). The latter yields an analogue of Theorem 2 for (3) (Corollary 1 below).

Theorem 2 and Corollary 1 are needed to run an iteration procedure \( L^p \to L^\infty \) that yields associated with (1), (3) Feller semigroups on \( C_\infty = C_\infty(\mathbb{R}^d) \) (the space of all continuous functions vanishing at infinity endowed with the sup-norm), see Theorem 3 and Corollary 2 below.

In the same manner as it was done in [KiS3] for the operator \(-\Delta + b \cdot \nabla \), the Feller process constructed in Corollary 2 admits a characterization as a weak solution to the stochastic differential equation

\[
dX_t = -b(X_t)dt + \sqrt{2a(X_t)}dW_t, \quad X_0 = x_0 \in \mathbb{R}^d.
\]
We plan to address this matter in another paper.

All the proofs below work for

\[ a = I + \sum_{j=1}^{\infty} c_j f_j \otimes f_j, \quad \| f_j \|_{L^\infty} = 1, \]  

(4)

with \( f_j \) satisfying (C\( \delta_t \)), and \( c_+ := \sum_{c_j > 0} c_j < \infty, c_- := \sum_{c_j < 0} c_j > -1 \). (A decomposition (1) can be obtained from the spectral decomposition of a general uniformly elliptic \( a \).

2. We now state our results in full.

**Theorem 1** (\(-\nabla \cdot a \cdot \nabla\)). Let \( d \geq 3 \). Let \( a = I + cf \otimes f \) be given by (2).

(i) The formal differential expression \(-\nabla \cdot a \cdot \nabla\) has an operator realization \( A_q \) in \( L^q \) for all \( q \in [1, \infty[ \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup.

(ii) Assume that (C\( \delta_t \)) holds with \( \delta_t, c \) and \( q \geq 2 \vee (d-2) \) satisfying the following constraint:

\[-(1 + q\sqrt{\delta_t})^{-1} < c < \begin{cases} 
16(8 + q\sqrt{\delta_t})^{-1} & \text{if } q\sqrt{\delta_t} \leq 4, \\
(q\sqrt{\delta_t} - 1)^{-1} & \text{if } q\sqrt{\delta_t} \geq 4.
\end{cases}\]

Then, for each \( \mu > 0 \) and \( f \in L^q \), \( u := (\mu + A_q)^{-1} f \) belongs to \( W^{1,q} \cap W^{1,\frac{qe}{d-2}} \). Moreover, there exist constants \( \mu_0 = \mu_0(d,q,c,\delta_t) > 0 \) and \( K_i = K_i(d,q,c,\delta_t) \), \( l = 1,2 \), such that, for all \( \mu > \mu_0 \),

\[ \| \nabla u \|_{L^q} \leq K_1(\mu - \mu_0)^{-\frac{1}{2}} \| f \|_{L^q}, \]

(\(*\*)

\[ \| \nabla u \|_{L^q} \leq K_2(\mu - \mu_0)^{\frac{1}{2}} \| f \|_{L^q}. \]

**Remarks.** 1. \( \delta_t \) effectively estimates from above the “size” of the discontinuities of \( a \).

2. For the matrix (2), the constraints on \( c \) in Theorem 1 (and in other results below) can be substantially relaxed, see [KiS2].

**Theorem 2** (\(-\nabla \cdot a \cdot \nabla + b \cdot \nabla\)). Let \( d \geq 3 \). Let \( a = I + cf \otimes f \) be given by (2). Let \( b \in F_\delta \).

(i) If \( \delta_1 := [1 \vee (1 + c)^{-2}] \delta < 4 \), then \(-\nabla \cdot a \cdot \nabla + b \cdot \nabla\) has an operator realization \( \Lambda_q(a,b) \) in \( L^q \) for all \( q \in \left[ \frac{2}{2-\sqrt{\delta_1}}, \infty \right] \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup.

(ii) Assume that (C\( \delta_t \)) holds, \( \nabla a \in F_\delta, \delta < 1 \land \left( \frac{2}{d-2} \right)^2, \delta_1, \delta_t, c \) and \( q \geq 2 \vee (d-2) \) satisfy the constraints:

\[ 0 < c < (q-1 - Q) \begin{cases} 
[(q-1)q\sqrt{\delta_t} + q^2(\sqrt{\delta_t} + \sqrt{\delta})^2]^{-1} + (q-2)q\sqrt{\delta_t}]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta_t}}{4} - \frac{2\sqrt{\delta}}{4} \geq 0, \\
(q-1)q\sqrt{\delta_t} + q^2(\sqrt{\delta_t} + \sqrt{\delta})^2]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta_t}}{4} < \frac{2\sqrt{\delta}}{4}, \\
[(q-1)q\sqrt{\delta_t} - q\sqrt{\delta}]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta_t}}{4} < 0,
\end{cases}\]

where \( Q := \frac{\sqrt{\delta}}{2}[q - 2 - (\sqrt{\delta_1} + \sqrt{\delta})\frac{q}{2}] \), or

\[-(q-1 - Q) \begin{cases} 
[(q-1)(q\sqrt{\delta_t} + q\sqrt{\delta})]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta_t}}{4} - \frac{2\sqrt{\delta}}{4} \geq 0, \\
(q-1)(q\sqrt{\delta_t} - q\sqrt{\delta}]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta_t}}{4} < \frac{2\sqrt{\delta}}{4}, \\
[(q-1)(q\sqrt{\delta_t} - q\sqrt{\delta})]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta_t}}{4} < 0.
\end{cases}\]

Then there exist constants \( \mu_0 = \mu_0(d,q,c,\delta,\delta_1,\delta_t) > 0 \) and \( K_i = K_i(d,q,c,\delta,\delta_1,\delta_t) \), \( l = 1,2 \), such that (\(*\*) \) hold for \( u := (\mu + \Lambda_q(a,b))^{-1} f, \mu > \mu_0, f \in L^q \).
Remarks. 1. Taking $c = 0$ (then $\delta_a = 0$), we recover in Theorem 2(ii) the result of [KS] Lemma 5: $\delta < 1 \land \left(\frac{2}{d-2}\right)^2$.

2. Theorem 2(i) is an immediate consequence of the following general result. Let $a$ be an $L^d$ measurable uniformly elliptic matrix on $\mathbb{R}^d$. Set $A := A_2 := [-\nabla \cdot a \cdot \nabla | C^\infty_{c,2}]_{L^2}$. A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $F_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A$), if $b^2 := a \cdot b^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that
\[ \|b_a(\lambda + A)^{-\frac{1}{2}}\|_{L^2} \leq \sqrt{\delta_1}. \]
If $b \in F_{\delta_1}(A)$, $\delta_1 < 4$, then $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $\Lambda_q(a, b)$ in $L^q$ for all $q \in \left[\frac{2}{d-2}, \infty\right]$ as the (minus) generator of a positivity preserving $L^\infty$ contraction $C_0$ semigroup, see [KiS, Theorem 3.2].

Corollary 1 ($-a \cdot \nabla^2 + b \cdot \nabla$). Let $d \geq 3$. Let $a = I + cf \otimes f$ be given by (1). Let $b \in F_{\delta}$, $\nabla a \in F_{\delta_a}$. Then $\nabla a + b \in F_{\delta_2}$, $\sqrt{\delta_2} := \sqrt{\delta_a} + \sqrt{\delta}$.

(i) If $\delta_1 := \left(1 \lor (1+c)^{-2}\right)\delta_2 > 4$, then $-a \cdot \nabla^2 + b \cdot \nabla$ has an operator realization $\Lambda_q(a, \nabla a + b)$ in $L^q$ for all $q \in \left[\frac{2}{d-2}, \infty\right]$ as the (minus) generator of a positivity preserving $L^\infty$ contraction $C_0$ semigroup.

(ii) Assume that $[C_{\delta}]$ holds, and $\delta_2 < 1 \land \left(\frac{2}{d-2}\right)^2$, $\delta_a$, $\delta_f$, $c$, $q \geq 2 \lor (d-2)$ satisfy the constraints:

\[
0 < c < (q-1 - Q) \left\{ \begin{array}{ll}
(q-1)\frac{2\sqrt{\delta}}{2} + \frac{q^2(\sqrt{\delta} + \sqrt{\delta_f})^2}{10} + (q-2)\frac{q^2\delta_f}{10} & \text{if } 1 - \frac{2q\sqrt{\delta}}{4} - \frac{q^2\delta_f}{4} \geq 0, \\
\frac{q^2\sqrt{\delta}}{10} + (q-2)\frac{q^2\delta_f}{10} + \frac{q\sqrt{\delta_f} - 1}{10} & \text{if } 0 \leq 1 - \frac{2q\sqrt{\delta}}{4} < \frac{q\sqrt{\delta_f}}{4}, \\
\frac{q^2\sqrt{\delta}}{2} - (q-2)\frac{q^2\delta_f}{4} & \text{if } 1 - \frac{2q\sqrt{\delta}}{4} < 0,
\end{array} \right.
\]

where $Q := \frac{2q\sqrt{\delta}}{2} \left[ q - 2 + \left( \sqrt{\delta_a} + \sqrt{\delta_f} \right) \frac{Q}{2} \right]$, or

\[ -(q-1 - Q) \left( q - 1 \right) \left( q + \sqrt{\delta_f} + \frac{q\sqrt{\delta_f}}{2} \right)^{-1} < c < 0. \]

Then there exist constants $\mu_0 = \mu_0(d, q, c, \delta_2, \delta_a, \delta_f) > 0$ and $K_l = K_l(d, q, c, \delta_2, \delta_a, \delta_f)$, $l = 1, 2$, such that the estimates (11) hold for $u = (\mu + \Lambda_q(a, \nabla a + b))^{-1} f$, $\mu > \mu_0$, $f \in L^q$.

Set $b_n := e^{\epsilon_n \Delta}(1_n b)$, $\epsilon_n \downarrow 0$, $n \geq 1$, where $1_n$ is the indicator of $\{ x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n \}$. Also, set $f_n := (f^i_n)_{i=1}^d$, $f^i_n := e^{\epsilon_n \Delta}(\eta_n f^i)$, $\epsilon_n \downarrow 0$, $n \geq 1$, where

\[ \eta_n(x) := \begin{cases} 
1, & \text{if } |x| < n, \\
n + 1 - |x|, & \text{if } n \leq |x| \leq n + 1, \\
0, & \text{if } |x| > n + 1. 
\end{cases} \quad (x \in \mathbb{R}^d) \]

Theorem 3 ($-a \cdot \nabla + b \cdot \nabla$). (i) In the assumptions of Theorem 2(ii), the formal differential operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $-\Lambda_{C^\infty}(a, b)$ as the generator of a positivity preserving contraction $C_0$ semigroup in $C^\infty$ defined by

\[ e^{-t\Lambda_{C^\infty}(a, b)} := \text{s-C} \lim_n e^{-t\Lambda_{C^\infty}(a_n, b_n)} \quad (\text{loc. uniformly in } t \geq 0), \]

where $a_n := I + c f_n \otimes f_n \in [C^\infty]^{d \times d}$, $\Lambda_{C^\infty}(a_n, b_n) := -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$, $D(\Lambda_{C^\infty}(a_n, b_n)) = (1 - \Delta)^{-1} C^\infty$. 

(ii) [The $L^r$-strong Feller property] \(((\mu + \Lambda_{C_\infty}(a,b))^{-1} \upharpoonright L^r \cap C_\infty)_{L^r \to C_\infty} \in \mathcal{B}(L^r, C^{0,1-\frac{d}{r}})\) for some $r > d - 2$ and all $\mu > \mu_0$.

(iii) The integral kernel of $e^{-t\Lambda_{C_\infty}(a,b)}$ determines the transition probability function of a Feller process.

Corollary 2 \((-a \cdot \nabla^2 + b \cdot \nabla)\). (i) In the assumptions of Corollary 1(ii), the formal differential operator $-a \cdot \nabla^2 + b \cdot \nabla$ has an operator realization $-\Lambda_{C_\infty}(a, \nabla a + b)$ as the generator of a positivity preserving contraction $C_0$ semigroup in $C_\infty$ defined by

$$e^{-t\Lambda_{C_\infty}(a,\nabla a + b)} = s-C_\infty \operatorname*{lim}_{n \to \infty} e^{-t\Lambda_{C_\infty}(a_n, \nabla a_n + b_n)} \quad \text{(loc. uniformly in } t \geq 0),$$

where $a_n = I + cf_n \otimes f_n \subset [C_\infty]^{d \times d}$, $\Lambda_{C_\infty}(a_n, \nabla a_n + b_n) := -a_n \cdot \nabla^2 + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(a_n, \nabla a_n + b_n)) = (1 - \Delta)^{-1} C_\infty$.

(ii) [The $L^r$-strong Feller property] \(((\mu + \Lambda_{C_\infty}(a, \nabla a + b))^{-1} \upharpoonright L^r \cap C_\infty)_{L^r \to C_\infty} \in \mathcal{B}(L^r, C^{0,1-\frac{d}{r}})\) for some $r > d - 2$ and all $\mu > \mu_0$.

(iii) The integral kernel of $e^{-t\Lambda_{C_\infty}(a, \nabla a + b)}$ determines the transition probability function of a Feller process.

Remarks. Since our assumptions on $\delta_f$, $\delta_a$ and $\delta$ involve only strict inequalities, we may assume that

\([CC_\delta]\) holds for $f_n$, $\nabla a_n \in F_{\delta_a}$, $b_n \in F_{\delta}$ with $\lambda \neq \lambda(n)$ (5)

for appropriate $\epsilon_n \downarrow 0$. In fact, the proofs work for any approximations $\{f_n\}, \{b_n\} \subset [C_\infty]^d$ such that $\|f_n\|_\infty = 1$, (5) holds, and

$$f_n \to f, \nabla_i f_n \to \nabla_i f \text{ strongly in } [L^2_{\text{loc}}]^d, \quad i = 1, 2, \ldots, d,$$

$$b_n \to b \text{ strongly in } [L^2_{\text{loc}}]^d.$$

1. Proof of Theorem 1

Proof of (i). In what follows, we use notation

$$\langle h \rangle := \int_{\mathbb{R}^d} h(x)dx, \quad \langle h, g \rangle := \langle h \bar{g} \rangle.$$

Define $t[u,v] := \langle \nabla u \cdot a \cdot \nabla \bar{v} \rangle$, $D(t) = W^{1,2}$. There is a unique self-adjoint operator $A \equiv A_2 \geq 0$ on $L^2$ associated with the form $t$: $D(A) \subset D(t)$, $\langle Au, v \rangle = t[u,v]$, $u \in D(A)$, $v \in D(t)$. $-A$ is the generator of a positivity preserving $L^\infty$ contraction $C_0$ semigroup $T^t_0 \equiv e^{-tA}$, $t \geq 0$, on $L^2$. Then $T^t_r := [T^t \upharpoonright L^r \cap L^2]_{L^r \to L^r}$ determines $C_0$ semigroup on $L^r$ for all $r \in [1, \infty]$. The generator $-A_r$ of $T^t_r (\equiv e^{-tA_r})$ is the desired operator realization of $\nabla \cdot a \cdot \nabla$ in $L^r$, $r \in [1, \infty]$. Moreover, $(\mu + A_r)^{-1}$ is well defined on $L^r$ for all $\mu > 0$. This completes the proof of the assertion (i) of the theorem.

Proof of (ii). First, we prove an a priori variant of (CC). Set $u_n := I + cf_n \otimes f_n$, where $f_n$ have been defined in the beginning of the paper. Since our assumption on $\delta_f$ is a strict inequality, we may assume that \([CC_\delta]\) holds for $u_n$ for all $n \geq 1$ with $\lambda \neq \lambda(n)$ for appropriate $\epsilon_n \downarrow 0$. We also note that $\|f_n\|_\infty = 1$.

Set $u \equiv u_n := (\mu + A_n)^{-1} f$, $0 \leq f \in C^2$, where $A_n := -\nabla \cdot a_n \cdot \nabla$, $D(A_n) = W^{2,q}$, $n \geq 1$. Clearly, $0 \leq u_n \in W^{3,q}$.
Denote \( w \equiv w_n := \nabla u_n \). For brevity, below we omit the index \( n \): \( f \equiv f_n, a \equiv a_n, A_q \equiv A^n_q \). Set
\[
I_q := \sum_{r=1}^{d} \langle (\nabla_r w)^2 |w|^{q-2} \rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2} \rangle,
\]
\[
\bar{I}_q := \langle (f \cdot \nabla w)^2 |w|^{q-2} \rangle, \quad \bar{J}_q := \langle (f \cdot \nabla |w|)^2 |w|^{q-2} \rangle.
\]
Set \( [F,G]_- := FG - GF \).

1. We multiply \( \mu u + A_q u = f \) by \( \phi := -\nabla \cdot (w|w|^{q-2}) \) and integrate:
\[
\mu \langle |w|^q \rangle + \langle A_q w, w|w|^{q-2} \rangle + \langle (\nabla, A_q)_w u, w|w|^{q-2} \rangle = \langle f, \phi \rangle,
\]
\[
\mu \langle |w|^q \rangle + I_q + cI_q + (q-2)(J_q + c\bar{J}_q) + \langle (\nabla, A_q)_w u, w|w|^{q-2} \rangle = \langle f, \phi \rangle.
\]
The term to evaluate is this:
\[
\langle [\nabla, A_q]_w u, w|w|^{q-2} \rangle := \sum_{r=1}^{d} \langle [\nabla_r, A_q]_w u, w_r|w|^{q-2} \rangle.
\]
From now on, we omit the summation sign in repeated indices. Note that
\[
[\nabla_r, A_q]_w = -\nabla \cdot (\nabla_r a) \cdot \nabla, \quad (\nabla_r a)_{il} = c(\nabla_r f^i)^l + cf^i \nabla_r f^l.
\]
Thus,
\[
\langle [\nabla_r, A_q]_w u, w_r|w|^{q-2} \rangle = c \left\langle \left( [\nabla_r f^l]^l + f^l \nabla_r f^l \right) w_i, \nabla_i (w_r|w|^{q-2}) \right\rangle =: S_1 + S_2,
\]
\[
S_1 = c \langle (\nabla_r f) \cdot (\nabla_r w)(f \cdot w)|w|^{q-2} \rangle + c(q-2)\langle (\nabla_r f) \cdot (\nabla |w|)(f \cdot w)w_r|w|^{q-3} \rangle,
\]
\[
S_2 = c \langle (\nabla_r f) \cdot w, (f \cdot \nabla w_r)|w|^{q-2} \rangle + c(q-2)\langle (\nabla_r f) \cdot w, w_r|w|^{q-3} f \cdot \nabla |w| \rangle.
\]
By the quadratic estimates and the condition \([C_\delta_q]\),
\[
S_1 \leq |c| \left[ \alpha \left( \delta \frac{q^2}{4} J_q + \lambda \delta |w|^q \right) + \frac{1}{4\alpha} I_q \right] + |c|(q-2) \left[ \alpha_1 \left( \delta \frac{q^2}{4} J_q + \lambda \delta |w|^q \right) + \frac{1}{4\alpha_1} J_q \right], \quad \alpha, \alpha_1 > 0
\]
\[
S_2 \leq |c| \left[ \gamma \left( \delta \frac{q^2}{4} J_q + \lambda \delta |w|^q \right) + \frac{1}{4\gamma} I_q \right] + |c|(q-2) \left[ \gamma_1 \left( \delta \frac{q^2}{4} J_q + \lambda \delta |w|^q \right) + \frac{1}{4\gamma_1} J_q \right], \quad \gamma, \gamma_1 > 0.
\]
Thus, selecting \( \alpha = \alpha_1 = \frac{1}{\sqrt{q^2}} \), we obtain the inequality
\[
\mu |w|^q + I_q + cI_q + (q-2)(J_q + c\bar{J}_q)
\]
\[
\leq |c| \left[ \frac{\sqrt{\delta}}{4} J_q + \frac{q\sqrt{\delta}}{4} I_q \right] + |c|(q-2) \frac{q\sqrt{\delta}}{2} J_q
\]
\[
+ |c| \left[ \gamma \delta \frac{q^2}{4} J_q + \frac{1}{4\gamma} I_q \right] + |c|(q-2) \left[ \gamma_1 \delta \frac{q^2}{4} J_q + \frac{1}{4\gamma_1} J_q \right]
\]
\[
+ \mu_0 |w|^q + \langle f, \phi \rangle
\]
\[
\text{where } \mu_0 := |c| \lambda \sqrt{\delta} (q^{-1} + \gamma \sqrt{\delta}) + |c|(q-2) \lambda \sqrt{\delta} (q^{-1} + \gamma_1 \sqrt{\delta}).
\]
2. Let us prove that there exists constant \( \eta > 0 \) such that
\[
(\mu - \mu_0) |w|^q + \eta J_q \leq \langle f, \phi \rangle. \tag{6}
\]
Lemma 1. For each $\varepsilon_0 > 0$ there exists a constant $C = C(\varepsilon_0) < \infty$ such that

$$\langle f, \phi \rangle \leq \varepsilon_0 I_q + C \|w\|_q^{g-2} \|f\|_q^2.$$ 

Proof of Lemma 1. We have:

$$\langle f, \phi \rangle = \langle -\Delta u, |w|^{g-2} f \rangle + (q-2) \langle |w|^{g-3} w \cdot \nabla |w|, f \rangle =: F_1 + F_2.$$
Due to $|\Delta u|^2 \leq d(\nabla, w)^2$ and $(|w|^{q-2}f) \leq \|w\|^{q-2}_q \|f\|_q^2$,

$$F_1 \leq \sqrt{dI_q^b} \|w\|^{\frac{q-2}{q^2}}_q \|f\|_q, \quad F_2 \leq (q-2)J_q^b \|w\|^{\frac{q-2}{q^2}}_q \|f\|_q,$$

Now the standard quadratic estimates yield the lemma. \hfill \Box

We choose $\varepsilon_0 > 0$ in Lemma \textnormal{4} so small that in the estimates below we can ignore $\varepsilon_0 I_q$.

4. Clearly, \textnormal{5} yields the inequalities

$$\|\nabla u_n\|_{q} \leq K_1 (\mu - \mu_0)^{-\frac{1}{2}} \|f\|_q, \quad K_1 := C^\frac{1}{2},$$

$$\|\nabla u_n\|_{aq} \leq K_2 (\mu - \mu_0)^{-\frac{1}{2+\frac{1}{q}}} \|f\|_q, \quad K_2 := C_S q^{-\frac{1}{q}} (q^2 / 4)^{\frac{1}{q}} C^\frac{1}{2} \frac{1}{2},$$

where $C_S$ is the constant in the Sobolev Embedding Theorem. So, \textnormal{[KIS, Theorem 3.5]} $((\mu + A_q)^{-1} = s-L^q$-$\lim_n (\mu + A_n^q)^{-1})$ yields \textnormal{5}. The proof of Theorem \textnormal{1} is completed.

2. PROOF OF THEOREM \textnormal{2}

Proof of (i). Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2 := [-\nabla \cdot a, \nabla | C^\infty_{\text{loc}} \cap L^1]$, if $b^2_a := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that

$$\|b_a(\lambda + A)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta_1}.$$ 

It is easily seen that if $b \in \mathbf{F}_{\delta_1}$, then $b \in \mathbf{F}_{\delta_1}(A)$, with $\delta_1 := [1 \vee (1 + c)^{-2}] \delta$. By the assumptions of the theorem, $\delta_1 < 4$. Therefore, by \textnormal{[KIS, Theorem 3.2]}, $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $\Lambda_q(a, b)$ in $L^q$, $q \in \left[\frac{2}{2-\sqrt{\delta_1}}, \infty \right]$, as the (minus) generator of a positivity preserving $L^\infty$ contraction quasi contraction $C_0$ semigroup. Moreover, $(\mu + \Lambda_q(a, b))^{-1}$ is well defined on $L^q$ for all $\mu > \frac{\lambda \delta}{q(q-1)^2}$. This completes the proof of (i).

Proof of (ii). First, we prove an a priori variant of \textnormal{5}. Set $a_n := I + c \varepsilon_n \otimes f_n$, where $f_n$ have been defined in the beginning of the paper. Since our assumptions on $\varepsilon_1, \varepsilon_2$ and $\delta$ involve only strict inequalities, we may assume that $(C_{b_n})$ holds for $f_n$, $\nabla a_n \in \mathbf{F}_{\delta_n}$, $b_n \in \mathbf{F}_\delta$ with $\lambda \neq \lambda(n)$ for appropriate $\varepsilon_n \downarrow 0$. We also note that $\|f_n\|_\infty = 1$.

Denote $A_n^\alpha := -\nabla \cdot a_n \cdot \nabla, D(A_n^\alpha) = W^2,q$. Set $u \equiv u_n := (\mu + \Lambda_q(a_n, b_n))^{-1} f, 0 \leq f \in C^1_c, n \geq 1$, where $\Lambda_q(a_n, b_n) = A_n^\alpha + b_n \cdot \nabla, D(\Lambda_q(a_n, b_n)) = D(A_n^\alpha)$. Clearly, $0 \leq u_n \in W^{3,q}$. It is easily seen that $b_n \in \mathbf{F}_{\delta_1}(A_n^\alpha)$ with $\lambda \neq \lambda(n)$, so $(\mu + \Lambda_q(a_n, b_n))^{-1}$ are well defined on $L^q$ for all $n \geq 1, \mu > \frac{\lambda \delta}{q(q-1)^2}$.

1. Denote $w \equiv w_n := \nabla u_n$. Below we omit the index $n$: $f \equiv f_n$, $a \equiv a_n$, $b \equiv b_n$, $A_q \equiv A_n^\alpha$. Set

$$I_q := \langle |(\nabla \cdot w)^2| w|^{q-2} \rangle, \quad J_q := \langle (|\nabla w|^2)^q_2 \rangle,$$

Arguing as in the proof of Theorem \textnormal{1} we arrive at
\[
\mu \langle |w|^q \rangle + I_q + cI_q + (q - 2)(J_q + cJ_q)
\leq |c|\left[ \alpha \delta r \frac{q^2}{4} J_q + \frac{1}{4\alpha} I_q \right] + |c|(q - 2)\left[ \alpha \delta r \frac{q^2}{4} J_q + \frac{1}{4\alpha} I_q \right]
\]
\[
+ |c|\left[ \gamma \delta r \frac{q^2}{4} J_q + \frac{1}{4\gamma} I_q \right] + |c|(q - 2)\left[ \gamma \delta r \frac{q^2}{4} J_q + \frac{1}{4\gamma} I_q \right]
\]
\[
+ \mu_0 \|w\|_q^2 + \langle -b \cdot w, \phi \rangle + \langle f, \varphi \rangle, \quad \text{with } \alpha = \alpha_1 := \frac{1}{q \sqrt{\delta r}}.
\]
where \(\mu_0 := |c|\lambda \sqrt{\delta r}(q^{-1} + \gamma \sqrt{\delta r}) + |c|(q - 2)\lambda \sqrt{\delta r}(q^{-1} + \gamma_1 \sqrt{\delta r})\), and \(\gamma, \gamma_1 > 0\) are to be chosen.

**2.** We estimate the term \(\langle -b \cdot w, \phi \rangle\) as follows.

**Lemma 2.** There exist constants \(C_i (i = 0, 1)\) such that
\[
\langle -b \cdot w, \phi \rangle \leq \left( \sqrt{\delta \sqrt{\delta a} + \delta} \right) \frac{q^2}{4} + (q - 2)\frac{q \sqrt{\delta}}{2} J_q + |c|\left[ \frac{q \sqrt{\delta}}{2} J_q \right] + C_0 \|w\|_q^2 + C_1 \|w\|_q^{q - 2} \|f\|_q^2.
\]

**Proof.** We have:
\[
\langle -b \cdot w, \phi \rangle = \langle -\Delta u, |w|^{q - 2}(b \cdot w) \rangle + (q - 2)\langle |w|^{q - 2}w \cdot \nabla |w|, -b \cdot w \rangle
\]
\[
= F_1 + F_2.
\]

Set \(B_q := (|b \cdot w|^2 |w|^{q - 2})\). We have
\[
F_2 \leq (q - 2)B_q^{1/2} J_q^{1/2}.
\]

Next, we bound \(F_1\). Recall that \(\nabla a = c[(\text{div} f)f \cdot f - \text{div} f]\). We represent \(-\Delta u = \nabla \cdot (a - 1) \cdot w - \mu u - b \cdot w + f\), and evaluate:
\[
\nabla \cdot (a - 1) \cdot w = \nabla a \cdot w + c f \cdot (f \cdot \nabla w),
\]
\[
F_1 = \langle \nabla \cdot (a - 1) \cdot w, |w|^{q - 2}(b \cdot w) \rangle + \langle (-\mu u - b \cdot w + f), |w|^{q - 2}(b \cdot w) \rangle
\]
\[
= \langle \nabla a \cdot w, |w|^{q - 2}(b \cdot w) \rangle
\]
\[
+ c \langle f \cdot (f \cdot \nabla w), |w|^{q - 2}(b \cdot w) \rangle
\]
\[
+ \langle (-\mu u - b \cdot w + f), |w|^{q - 2}(b \cdot w) \rangle.
\]

Set \(P_q := (|\nabla a \cdot w|^2 |w|^{q - 2})\). We bound \(F_1\) from above by applying consecutively the following estimates:

\(1^o\) \(\langle \nabla a \cdot w, |w|^{q - 2}(b \cdot w) \rangle \leq P_q^{1/2} B_q^{1/2} B_q^{1/2}\).

\(2^o\) \(\langle f \cdot (f \cdot \nabla w), |w|^{q - 2}(b \cdot w) \rangle \leq \tilde{I}_q^{1/2} B_q^{1/2} B_q^{1/2}\).

\(3^o\) \(\langle \mu u, |w|^{q - 2}(b \cdot w) \rangle \leq \frac{\mu}{\mu - \omega_q} B_q^{1/2} \|w\|_q^{2} \|f\|_q^2 \) (here \(\frac{2}{q - 3} < q \Rightarrow \|u\|_q \leq (\mu - \omega_q)^{-1} \|f\|_q\)).

\(4^o\) \(\langle b \cdot w, |w|^{q - 2}b \cdot w \rangle = B_q\).

\(5^o\) \(\langle f, |w|^{q - 2}(b \cdot w) \rangle \leq B_q^{1/2} \|w\|_q^{2} \|f\|_q^2 \).

In \(3^o\) and \(5^o\) we estimate \(B_q^{1/2} \|w\|_q^{2} \|f\|_q^2 \leq \varepsilon_0 B_q + \frac{1}{4\varepsilon_0} \|w\|_q^{q - 2} \|f\|_q^2 (\varepsilon_0 > 0)\).

Therefore,
\[
\langle -b \cdot w, \phi \rangle \leq P_q^{1/2} B_q^{1/2} + |c|I_q^{1/2} B_q^{1/2} + B_q + (q - 2)B_q^{1/2} J_q^{1/2} + \varepsilon_0 B_q + C_1(\varepsilon_0) \|w\|_q^{q - 2} \|f\|_q^2.
\]
It is easily seen that $b \in F_{\delta}$ is equivalent to the inequality
\[
\langle b^2 | \varphi |^2 \rangle \leq \delta \langle |\nabla \varphi |^2 \rangle + \lambda \delta \langle |\varphi |^2 \rangle, \quad \varphi \in W^{1,2}.
\]
Thus,
\[
B_q \leq \| b |w|_2^2 \|^2_2 \leq \delta \| \nabla |w|_2^2 \|_2 + \lambda \delta \|w\|_q^q = \frac{q^2 \delta}{4} J_q + \lambda \delta \|w\|_q^q.
\]
Similarly, using that $\nabla a \in F_{\delta_a}$, we obtain
\[
P_q \leq \| (\nabla a) |w|_2^2 \|^2_2 \leq \delta_a \| \nabla |w|_2^2 \|_2 + \lambda \delta_a \|w\|_q^q = \frac{q^2 \delta_a}{4} J_q + \lambda \delta_a \|w\|_q^q.
\]
Then selecting $\varepsilon_0 > 0$ sufficiently small, and noticing that the assumption on $\delta, \delta_a$ in the theorem are strict inequalities, we can and will ignore below the terms multiplied by $\varepsilon_0$. The proof of Lemma \((\ref{eq:1})\) is completed.

In \((\ref{eq:2})\), we apply Lemma \((\ref{eq:1})\) where the inequality $\frac{q^2 \delta}{4} J_q + \frac{1}{4 \gamma_2} I_q, \gamma_2 > 0$, is used. Thus, we have
\[
\mu \|w\|_q^q + I_q + c I_q + (q - 2) (J_q + c J_q)
\]
\[
\leq |c| \left[ \frac{q \sqrt{\delta_f}}{4} J_q + \frac{q \sqrt{\delta_f}}{4} I_q \right] + |c| (q - 2) \frac{q \sqrt{\delta_f}}{2} J_q
\]
\[
+ |c| \left[ (\gamma \delta_f + \gamma_2 \delta) \frac{q^2}{4} J_q + \left( \frac{1}{4 \gamma^2} + \frac{1}{4 \gamma_2} \right) I_q \right] + |c| (q - 2) \left[ \gamma_1 \delta_f \frac{q^2}{4} J_q + \frac{1}{4 \gamma^2} I_q \right]
\]
\[
+ \left[ (\sqrt{\delta} \sqrt{\delta_a} + \delta) \frac{q^2}{4} + (q - 2) \frac{2 \sqrt{\delta}}{2} \right] J_q + \mu_0 \|w\|_q^q + C_1 \|w\|_q^{q - 2} \|f\|_q^2 + \langle f, \phi \rangle,
\]
where $\mu_0 := |c| \lambda \sqrt{\delta_f} (q^{-1} + \gamma \sqrt{\delta_f}) + |c| (q - 2) \lambda \sqrt{\delta_f} (q^{-1} + \gamma_1 \sqrt{\delta_f}) + C_0$.

3. Let us prove that there exists constant $\eta > 0$ such that
\[
(\mu - \mu_0) \|w\|_q^q + \eta J_q \leq C_1 \|w\|_q^{q - 2} \|f\|_q^2 + \langle f, \phi \rangle.
\]

Set $Q := (\sqrt{\delta} \sqrt{\delta_a} + \delta) \frac{q^2}{4} + (q - 2) \frac{2 \sqrt{\delta}}{2}$.

**Case** $c > 0$. First, suppose that $1 - \frac{q \sqrt{\delta}}{4} - \frac{q \sqrt{\delta}}{4} \geq 0$. We select $\gamma, \gamma_2 > 0$ such that $\frac{1}{4 \gamma} + \frac{1}{4 \gamma_2} = 1$ while $\gamma \delta_f + \gamma_2 \delta$ attains its minimal value. It is easily seen that $\gamma = \frac{1}{4} (1 + \sqrt{\frac{2}{q}}), \gamma_2 = \frac{1}{4} (1 + \sqrt{\frac{2}{q}})$. We have $1 - \frac{q \sqrt{\delta}}{4} \geq 0$, and select $\gamma_1 = \frac{1}{4}$. Thus, the terms $I_q, J_q$ are no longer present in \((\ref{eq:2})\):
\[
\mu \|w\|_q^q + \left( 1 - c \frac{q \sqrt{\delta_f}}{4} \right) I_q
\]
\[
+ \left[ q - 2 - c \frac{q \sqrt{\delta_f}}{4} - c(q - 2) \frac{q \sqrt{\delta_f}}{2} - c(\delta_f + 2 \sqrt{\delta_f} + \delta) \frac{q^2}{16} - c(q - 2) \frac{q^2 \delta_f}{16} - Q \right] J_q
\]
\[
\leq \mu_0 \|w\|_q^q + C_1 \|w\|_q^{q - 2} \|f\|_q^2 + \langle f, \phi \rangle.
\]
By the assumptions of the theorem, $1 - c q \sqrt{\delta t} \geq 0$, so by $J_q \leq I_q$ we obtain

$$
\mu \|w\|_q^q + \left[ q - 1 - c(q - 1) - c(q - 2) \frac{q \sqrt{\delta t}}{2} + c(\delta t + 2 \sqrt{\delta t} + \delta) \right] J_q \\
\leq \mu_0 \|w\|_q^q + C I_q \|w\|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
$$

Next, suppose that $1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} < 0$, but $1 - \frac{q \sqrt{\delta t}}{4} \geq 0$. We select $\gamma = \frac{1}{q \sqrt{\delta t}}$, $\gamma_2 = \frac{1}{q \sqrt{\delta}}$, and $\gamma_1 = \frac{1}{q}$. Then the term $\bar{J}_q$ is no longer present, so using $\bar{I}_q \leq I_q$ we obtain

$$
\mu \|w\|_q^q + \left[ 1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \right] J_q \\
+ \left[ q - 2 - c \frac{q \sqrt{\delta t}}{2} - c(q - 2) \frac{q \sqrt{\delta t}}{2} - c \frac{q \sqrt{\delta t} + q \sqrt{\delta}}{4} - c(q - 2) \frac{q^2 \delta t}{16} - Q \right] J_q \\
\leq \mu_0 \|w\|_q^q + C I_q \|w\|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
$$

Thus, since $1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \geq 0$ by the assumptions of the theorem, we have using $J_q \leq I_q$

$$
\mu \|w\|_q^q + \left[ q - 1 + c - c \frac{q \sqrt{\delta t}}{2} - c \frac{q^2 \sqrt{\delta t}}{2} - c(q - 2) \frac{q^2 \delta t}{16} - Q \right] J_q \\
\leq \mu_0 \|w\|_q^q + C I_q \|w\|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
$$

Finally, suppose that $1 - \frac{q \sqrt{\delta t}}{4} < 0$. We select $\gamma = \gamma_1 = \frac{1}{q \sqrt{\delta t}}$, $\gamma_2 = \frac{1}{q \sqrt{\delta}}$. Then using $\bar{I}_q \leq I_q$, $\bar{J}_q \leq J_q$ we obtain

$$
\mu \|w\|_q^q + \left[ 1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \right] I_q + \left[ q - 2 + c(q - 2) \left( 1 - \frac{q \sqrt{\delta t}}{4} \right) \right] J_q \\
- \frac{c q \sqrt{\delta t}}{2} - c(q - 2) \frac{q \sqrt{\delta t}}{2} - c \frac{q \sqrt{\delta t} + q \sqrt{\delta}}{4} - c(q - 2) \frac{q \sqrt{\delta t}}{4} - Q \right] J_q \\
\leq \mu_0 \|w\|_q^q + C I_q \|w\|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
$$

Since $1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \geq 0$ by the assumptions of the theorem, we have using $J_q \leq I_q$

$$
\mu \|w\|_q^q + \left[ q - 1 + c(q - 1) - c \frac{q \sqrt{\delta t}}{2} - c(q - 1) q \sqrt{\delta t} - Q \right] J_q \\
\leq \mu_0 \|w\|_q^q + C I_q \|w\|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
$$

In all three cases, the coefficient of $J_q$ is positive. We have proved (2).

**Case** $c < 0$. In (3), select $\gamma = \gamma_1 = \frac{1}{q \sqrt{\delta t}}$, $\gamma_2 = \frac{1}{q \sqrt{\delta}}$:

$$
\mu \|w\|_q^q + \left( 1 - |c| \frac{q \sqrt{\delta t}}{4} \right) I_q \\
+ \left[ q - 2 - |c|(q - 1) \frac{q \sqrt{\delta t}}{2} - |c|(q - 2) \frac{q \sqrt{\delta t}}{4} - |c| \frac{q \sqrt{\delta}}{4} - Q \right] J_q \\
- |c| \left( 1 + \frac{q \sqrt{\delta t}}{4} + \frac{q \sqrt{\delta}}{4} \right) I_q - |c|(q - 2) \left( 1 + \frac{q \sqrt{\delta t}}{4} \right) J_q \leq \mu_0 \|w\|_q^q + C I_q \|w\|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
$$
Using $I_q \geq \tilde{I}_q$, $J_q \geq \tilde{J}_q$, we obtain
\[
\mu\|w\|_q^2 + \left(1 - |c|\left(1 + \frac{q\sqrt{\delta_t}}{2} + \frac{q\sqrt{\delta}}{4}\right)\right)I_q
\]
\[+ \left[q - 2 - |c|(q - 1)\frac{q\sqrt{\delta_t}}{2} - |c|(q - 2)\frac{q\sqrt{\delta}}{4} - |c|\frac{q\sqrt{\delta}}{4} - |c|(q - 2)\left(1 + \frac{q\sqrt{\delta_t}}{4}\right) - \bar{Q}\right]J_q
\]
\[\leq \mu_0\|w\|_q^2 + C_1\|w\|_q^{q-2}\|f\|_q^2 + \langle f, \phi \rangle.
\]
By the assumptions of the theorem, $1 - |c|(1 + \frac{q\sqrt{\delta_t}}{2} + \frac{q\sqrt{\delta}}{4}) \geq 0$. Therefore, by $I_q \geq J_q$,
\[
\mu\|w\|_q^2 + \left[q - 2 - |c|(q - 1)\frac{q\sqrt{\delta_t}}{2} - |c|(q - 2)\frac{q\sqrt{\delta}}{4} - |c|\frac{q\sqrt{\delta}}{4} - |c|(q - 2)\left(1 + \frac{q\sqrt{\delta_t}}{4}\right) - \bar{Q}\right]J_q
\]
\[\leq \mu_0\|w\|_q^2 + C_1\|w\|_q^{q-2}\|f\|_q^2 + \langle f, \phi \rangle,
\]
where the coefficient of $J_q$ is strictly positive by the assumptions of the theorem. We have proved \(\Box\).

4. We estimate the term $\langle f, \phi \rangle$ by Lemma \[\text{I}\]. For each $\varepsilon_0 > 0$ there exists a constant $C = C(\varepsilon_0) < \infty$ such that
\[
\langle f, \phi \rangle \leq \varepsilon_0 I_q + C\|w\|_q^{q-2}\|f\|_q^2.
\]
We choose $\varepsilon_0 > 0$ so small that in the estimates below we can ignore $\varepsilon_0 I_q$.

Then \(\Box\) yields the inequalities
\[
\|\nabla u_n\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{q}}\|f\|_q, \quad K_1 := (C + C_1)^{\frac{1}{q}},
\]
\[
\|\nabla u_n\|_{qj} \leq K_2(\mu - \mu_0)^{\frac{1}{q} - \frac{1}{q_j}}\|f\|_q, \quad K_2 := C_2\eta^{-\frac{1}{q}}(q^2/4)^{\frac{1}{q}}(C + C_1)^{\frac{1}{2} - \frac{1}{q}},
\]
where $C_S$ is the constant in the Sobolev Embedding Theorem.

If $c > 0$ then $\delta_1 = \delta < 1$. If $c < 0$ then elementary arguments show that, by the assumptions of the theorem, $\delta_1 = (1 - |c|)^{-2}\delta < 1$. Therefore, [KiS, Theorem 3.5] $((\mu + \Lambda_q(a, b))^{-1} = sL^q$-lim$_n(\mu + \Lambda_q(a_n, b_n))^{-1})$ yields \(\Box\). The proof of Theorem \[\text{II}\] is completed.

3. The iteration procedure

The following is a direct extension of the iteration procedure in [KS]. Let $a \in (H_u)$.

Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $F_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2 := [-\nabla \cdot a, \nabla | C_c^{\text{loc}}]_{2 \to 2}$), if $b_a := b \cdot a^{-1} \cdot b \in L_{4c}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that $\|b_a(\lambda + A)^{-\frac{1}{q}}\|_{2 \to 2} \leq \sqrt{\delta_1}$.

Consider
\[
\{a_n\}_{n=1}^\infty \subset [C^1]^d \cap (H_{u, \sigma, \xi})
\]
and
\[
\{b_n\}_{n=1}^\infty \subset [C^1]^d \cap \bigcap_{m \geq 1} F_{\delta_1}(A^m), \quad \delta_1 < 4, \quad \lambda \neq \lambda(n, m).
\]
Here $A^n \equiv A(a_n)$. 

By [KiS, Theorem 3.2], \(-\Lambda_r(a_n, b_n) := \nabla \cdot a_n \cdot \nabla - b_n \cdot \nabla, D(\Lambda_r(a_n, b_n)) = W^{2r}\), is the generator of a positivity preserving \(L^\infty\) contraction quasi contraction \(C_0\) semigroup on \(L^r, r \in \left[\frac{2}{2-\sqrt{\delta_1}}, \infty\right]\), with the resolvent set of \(-\Lambda_r(a_n, b_n)\) containing \(\mu > \omega_r := \frac{\lambda_1}{2(r-1)}\) for all \(n \geq 1\).

Set \(u_n := (\mu + \Lambda_r(a_n, b_n))^{-1}f, f \in L^1 \cap L^\infty\) and \(g := u_m - u_n\).

**Lemma 3.** There are positive constants \(C = C(d), k = k(\delta_1)\) such that

\[
\|g\|_{r_j} \leq (C\sigma^{-1}(\delta_1 + 2\xi\sigma^{-1})(1 + 2\xi)\|\nabla u_m\|_{q_j}^2)\left(2k\right)^{\frac{k-2}{k}}\|g\|_{r'(r-2)},
\]

where \(q \in \left[\frac{2}{2-\sqrt{\delta_1}} \vee (d-2), \frac{2}{\sqrt{\delta_1}}\right], \quad 2x = qj, j = \frac{d-2}{x-2}, \quad x' := \frac{x}{x-2} \text{ and } x'(r-2) > \frac{2}{2-\sqrt{\delta_1}} \mu > \lambda_1\).

The proof follows closely [KiS, proof of Lemma 3.12] or [KS, proof of Lemma 6].

Iterating the inequality of Lemma 3, we arrive at

**Lemma 4.** In the notation of Lemma 3, assume that \(\sup_m \|\nabla u_m\|_{q_j}^2 < \infty, \mu > \mu_0\). Then for any \(r_0 > \frac{2}{2-\sqrt{\delta_1}}\)

\[
\|g\|_{\infty} \leq B\|g\|_{r_0}^\gamma, \quad \mu \geq 1 + \mu_0 \vee \lambda_1,
\]

where \(\gamma = (1 - \frac{x'}{2})\left(1 - \frac{x'}{2} + \frac{2x'}{r_0}\right)^{-1} > 0\), and \(B = B(d, \delta_1) < \infty\).

The proof repeats [KiS, proof of Lemma 3.13] or [KS, proof of Lemma 7].

**Remark.** The assumption \(\sup_m \|\nabla u_m\|_{q_j}^2 < \infty\) in Lemma 4 is crucial and holds e.g. in the assumptions of Theorem 2(ii).

4. PROOF OF THEOREM 3

By Lemma 4 and the second inequality in (xx), we have for all \(r_0 > \frac{2}{2-\sqrt{\delta_1}}\)

\[
\|u_n - u_m\|_{\infty} \leq B\|u_n - u_m\|_{r_0}^\gamma, \quad \mu \geq 1 + \mu_0 \vee \lambda_1,
\]

where \(\gamma > 0, B < \infty, \text{ and } u_n := (\mu + \Lambda_{r_0}(a_n, b_n))^{-1}f, f \in L^1 \cap L^\infty\). By [KiS, Theorem 3.5],

\[
(\mu + \Lambda_{r_0}(a, b))^{-1} = s_{L_{r_0}} \lim_n (\mu + \Lambda_{r_0}(a_n, b_n))^{-1},
\]

so \(\{u_n\}\) is fundamental in \(C_\infty\).

**Lemma 5.** \(s_{C_{\infty}} \lim_{\mu \uparrow \infty} (\mu + \Lambda_{C_{\infty}}(a_n, b_n))^{-1} = 1\) uniformly in \(n\).

The proof follows closely [KiS, proof of Lemma 3.16].

We are in position to complete the proof of Theorem 3. The assertion (i) follows from the fact that \(\{u_n\}\) is fundamental in \(C_\infty\) and Lemma 5 by applying the Trotter Approximation Theorem. (ii) is Theorem 2(ii). The proof of (iii) is standard. The proof of Theorem 3 is completed.

**Remark.** The arguments of the present paper extend more or less directly to the time-dependent case \(\partial_t - \nabla \cdot a(t, x) \cdot \nabla + b(t, x) \cdot \nabla\), cf. [Ki].
References


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