HEAT KERNEL OF FRACTIONAL LAPLACIAN WITH HARDY DRIFT VIA DESINGULARIZING WEIGHTS

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ABSTRACT. We establish sharp two-sided bounds on the heat kernel of the fractional Laplacian, perturbed by a drift having critical-order singularity, using the method of desingularizing weights.

1. Introduction

In 1998, Milman and Semënov [MS0] introduced the method of desingularizing weights to establish sharp two-sided weighted bounds on the heat kernel $e^{-tH}(x,y)$ of the Schrödinger operator $H \equiv -\Delta - V$, $V(x) = \delta\big(\frac{d-2}{2}\big)^2|x|^{-2}$, $0<\delta\leq 1$ in $L^2(\mathbb{R}^d,dx)$, $d\geq 3$ [MS1, MS2]. The corresponding C_0 semigroup is not ultra-contractive, but becomes one after transferring it to an appropriate weighted space.

In this paper we use the desingularization method to obtain sharp two-sided weighted bounds on the heat kernel $e^{-t\Lambda}(x,y)$ of the fractional Kolmogorov operator ($1<\alpha<2$)

$$\Lambda \equiv (-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla, \quad b(x) = \kappa|x|^{-\alpha}x,$$

on $\mathbb{R}^d$, $d\geq 3$, where

$$\kappa := \sqrt{\delta}\frac{2}{d-\alpha}c_{\alpha,d}^{-2} > 0, \quad 0<\delta<4,$$

$$c_{\alpha,d} := \frac{\gamma(\frac{d}{2}-\frac{\alpha}{2})}{\gamma(\frac{d}{2})}, \qquad \gamma(\alpha) := \frac{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2}-\frac{\alpha}{2})}.$$

The model vector field b exhibits critical behaviour both at the origin and at infinity. The standard upper bound in terms of the heat kernel of $(-\Delta)^{\frac{\alpha}{2}}$ does not hold. Instead, both bounds depend explicitly on the value of relative bound δ via the presence of a singular weight.

The operator $(-\Delta)^{\frac{\alpha}{2}} + \mathsf{f}\cdot\nabla$, $\mathsf{f}:\mathbb{R}^d\to\mathbb{R}^d$, has been the subject of intense study over the past few decades motivated, in particular, by applications in probability theory. The search for the largest class of admissible f has led to the Kato class corresponding to $(-\Delta)^{\frac{\alpha}{2}}$, which is recognized as the class responsible for existence of two-sided estimates on the heat kernel of $(-\Delta)^{\frac{\alpha}{2}}+\mathsf{f}\cdot\nabla$ in terms of the heat kernel of $(-\Delta)^{\frac{\alpha}{2}}$ [BJ].

The vector field $b(x) = c|x|^{-\alpha}x$, $c\neq 0$, has a stronger singularity than the ones covered by the Kato class, and so the corresponding semigroup $e^{-t\Lambda}$ is not $L^1\to L^\infty$ ultracontractive. It turns out that $e^{-t\Lambda}$ is ultracontractive as a mapping $L^1_{\sqrt{\varphi}}\to L^\infty$, where $L^1_{\sqrt{\varphi}}\equiv L^1(\mathbb{R}^d,\varphi dx)$, for an appropriate singular weight φ. This observation is crucial for the existence of sharp two-sided weighted bounds on $e^{-t\Lambda}(x,y)$, the main result of this paper. The proof of the $L^1_{\sqrt{\varphi}}\to L^\infty$ ultracontractivity depends on a

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"desingularizing" (L^1, L^1) bound on the weighted semigroup $\varphi e^{-t\Lambda}\varphi^{-1}$, and the Sobolev embedding property of Λ. In this regard we note the following:

1. In [MS2], the crucial (L^1, L^1) bound is proved for $-\Delta + V$ by means of the theory of m-sectorial operators and the Stampacchia criterion in L^2. However, attempts to apply that argument to $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$, $\alpha < 2$, are problematic since $(-\Delta)^{\frac{\alpha}{2}}$ lacks the local properties of $-\Delta$ which makes the corresponding approximation and calculational techniques unusable. Below we develop a new approach to the proof of the (L^1, L^1) bound appealing to the Lumer-Phillips Theorem applied to specially constructed C_0 semigroups in L^1 which approximate $\varphi e^{-t\Lambda}\varphi^{-1}$. Thus, in contrast to [MS2], where the (L^1, L^1) bound is proved using the L^2 theory, here we prove it staying within the L^1 theory.

2. In the special case $\delta < 1$, one can construct operator Λ, the generator of a C_0 semigroup in L^2, as the algebraic sum $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$, $D(\Lambda) = D((-\Delta)^{\frac{\alpha}{2}})$. Indeed, $b \cdot \nabla$ is Rellich's perturbation of $(-\Delta)^{\frac{\alpha}{2}}$, as follows from the fractional Hardy-Rellich inequality, and so $e^{-t\Lambda}$ is a holomorphic (contraction) semigroup in L^2 (Section 7). Moreover, for $\delta < 1$, Λ possesses the Sobolev embedding property

$$\mathrm{Re}\langle \Lambda f, f \rangle \geq (1 - \sqrt{\delta}) c_S \|f\|_{2j}^2, \quad j = \frac{d}{d-\alpha},$$

needed to run the variant of the desingularization method in [MS2]. The Sobolev embedding is a consequence of the Hardy-Rellich inequality and the Sobolev inequality. However, these arguments become problematic even for $\delta = 1$. When $1 < \delta < 4$, the operator $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$ ceases to be quasi-accretive in L^2, and the Sobolev embedding property ceases to hold (even for some $1 < j < \frac{d}{d-\alpha}$). Below we show that an operator realization Λ of $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$ can be constructed in L^r, $r \in]\frac{2}{2-\sqrt{\delta}}, \infty[$, for every $0 < \delta < 4$, such that

$$e^{-t\Lambda} := s\text{-}L^r\text{-}\lim_n e^{-t\Lambda(b_n)}, \quad t > 0,$$

where $\{b_n\}$ is appropriate smooth approximation of b. We develop a new non-symmetric variant of the desingularization method that works for all $0 < \delta < 4$. The necessity to work not only in L^2, but within the entire scale of L^r spaces, $1 \leq r \leq \infty$, is characteristic to the non-symmetric situation, and makes the desingularization method of the present paper largely different from the one of [MS2].

Let us now comment on the existing literature. The sharp two-sided weighted bounds on the heat kernel of the Schrödinger operator $-\Delta - \delta(\frac{d-2}{2})^2|x|^{-2}$, $0 < \delta \leq 1$, constructed in L^2, were obtained in [MS0, MS1, MS2]. (It is well known that $\delta = 1$ is the borderline case for the Schrödinger operator, i.e. for $\delta > 1$ solutions to the corresponding parabolic equation blow up instantly at each point in $\mathbb{R}^d$, see [BG, GZ].)

The sharp two-sided weighted bounds on the heat kernel of the fractional Schrödinger operator $H = (-\Delta)^{\frac{\alpha}{2}} - \delta c_{\alpha,d}^{-2}|x|^{-\alpha}$, $0 < \alpha < 2$, $0 < \delta \leq 1$, were obtained in [BGJP].

In [KS2] we give a purely operator-theoretic proof of the result of [BGJP] by specifying the argument of the present paper to H.

Concerning the heat kernel estimates for the operator $(-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}$, $c > 0$, see [CKSV, JW].

The argument in this paper also works for $\alpha = 2$ (cf. Appendix E). The corresponding result, however, is known, see [MSS] and [MSS2].

Let us comment further on the distinction between the local and the non-local cases $\alpha = 2$ and $\alpha < 2$.

The fact that $-\Delta$ is a local operator allowed the authors in [MS0, MS1, MS2] to obtain both upper and lower bounds from the a priori bounds corresponding to smooth approximations of the potential *and* the weights. In the present paper, in the case $\alpha < 2$, one has to work directly with singular coefficients in order to obtain the lower bound.

Concerning the fractional Schrödinger operator H in the borderline case $\delta = 1$, by [FLS, Corollary 2.5] (the fractional variant of the Brezis-Vasquez inequality [BV]) one has $\langle (H+1)f, f\rangle \geq C_d \|f\|_{2j}^2$, $f \in C_c^\infty$, only with $j \in [1, \frac{d}{d-\alpha}[$. The latter yields a sub-optimal weighted Nash initial estimate. Nevertheless, as was observed in [BGJP], this estimate is optimal for $t = 1$, and so the optimal weighted Nash initial estimate follows for all $t > 0$ by the scaling properties of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}$, $0 < \alpha < 2$. This is in contrast to the case $\delta = 1$, $\alpha = 2$: the scaling properties of $e^{t\Delta}$ are different, so one needs an additional argument in order to obtain the optimal upper bound (i.e. to pass to a space of higher dimension where one can appeal to the V. P. Il'in-Sobolev inequality, see [MS2]).

In [MSS], [MSS2], the authors obtain sharp upper and lower bounds on the heat kernel of the operator $-\nabla \cdot (1 + c|x|^{-2}x^t x) \cdot \nabla + \delta_1 \frac{d-2}{2}|x|^{-2}x \cdot \nabla + \delta_2 \frac{(d-2)^2}{4}|x|^{-2}$, $c > -1$, by considering it in the space $L^2(\mathbb{R}^d, |x|^\gamma dx)$ where it becomes symmetric (for appropriate constant γ). This approach, however, does not work for the operator $(-\Delta)^{\frac{\alpha}{2}} + \kappa|x|^{-\alpha}x \cdot \nabla$, $1 < \alpha < 2$.

We note that $\delta = 4$ is the borderline case for the operator $(-\Delta)^{\frac{\alpha}{2}} + \kappa|x|^{-\alpha}x \cdot \nabla$, $1 < \alpha \leq 2$. See e.g. [KS1, sect. 4, remark 3] concerning $\alpha = 2$. The case $1 < \alpha < 2$, $\delta = 4$ requires separate study already with regard to the problem of convergence of approximated semigroups $e^{-t\Lambda(b_n)}$ and constructing the corresponding operator realization of $(-\Delta)^{\frac{\alpha}{2}} + \kappa|x|^{-\alpha}x \cdot \nabla$.

We conclude this introduction by emphasizing the following fact (ensuing from the previous discussion, and well known in the case $\alpha = 2$): for a singular vector field b, the analogy between $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$ and $(-\Delta)^{\frac{\alpha}{2}} - V$ is superficial.

2. Desingularizing weights

Let X be a set, and μ a measure on X. Set $L^p = L^p(X, \mu)$, $p \in [1, \infty]$. By $\langle u, v\rangle$ we denote the $(L^p, L^{p'})$ pairing, so that

$$\langle u, v\rangle = \langle u\bar{v}\rangle := \int_X u\bar{v}d\mu \qquad (u \in L^p, v \in L^{p'}).$$

Let $-\Lambda$ be the generator of a C_0 contraction semigroup $e^{-t\Lambda}$, $t > 0$, in L^r for $r \in]r_c, \infty[$, $r_c = \frac{2}{2-\sqrt{\delta}}$. Assume that

$$\|e^{-t\Lambda}\|_{r\to\infty} \leq ct^{-\frac{j'}{r}}, \quad r \in]r_c, \infty[, \tag{S_1}$$

but $e^{-t\Lambda}$ is not ultra-contractive. In this case we will be assuming that there exist a family of real valued weights $\varphi = \{\varphi_s\}_{s>0}$ in X such that, for all $s > 0$,

$$0 \leq \varphi_s, \frac{1}{\varphi_s} \in L^1_{\rm loc}(X, \mu), \tag{S_2}$$

and a constant c_1, independent of s, such that, for all $0 < t \leq s$,

$$\|\varphi_s e^{-t\Lambda}\varphi_s^{-1} f\|_1 \leq c_1\|f\|_1, \quad f \in L^\infty_{\rm com}(X, \mu), \tag{S_3}$$

where $L^\infty_{\rm com}$ is the space of functions in L^∞ having compact support.

The following general theorem is the point of departure for the desingularization method in the non-selfadjoint setting:

Theorem A. *In addition to (S_1)-(S_3) assume that*

$$\inf_{s>0,x\in X}\varphi_s(x)\geq c_0>0. \tag{S_4}$$

Then, for each $t>0$, $e^{-t\Lambda}$ is integral operator, and there is a constant $C=C(j,c_s,c_1,c_0)$ such that the weighted Nash initial estimate

$$|e^{-t\Lambda}(x,y)|\leq Ct^{-j'}\varphi_t(y) \tag{NIE_w}$$

is valid for μ a.e. $x,y\in X$.

The proof of Theorem A uses a weighted variant of the Coulhon-Raynaud Extrapolation Theorem [VSC, Prop. II.2.1, Prop. II.2.2].

Theorem 1. *Let $U^{t,s}$ be a two-parameter evolution family of operators*

$$U^{t,s}f=U^{t,\tau}U^{\tau,s}f,\qquad f\in L^1\cap L^\infty,\quad 0\leq s<\tau<t\leq\infty.$$

Suppose that for some $1\leq p<q<r\leq\infty$, $\nu>0$

$$\|U^{t,s}f\|_p\leq M_1\|f\|_{p,\sqrt{\psi}},\quad 0\leq\psi\in L^1+L^\infty,\quad \|f\|_{p,\sqrt{\psi}}:=\langle|f|^p\psi\rangle^{1/p},$$
$$\|U^{t,s}f\|_r\leq M_2(t-s)^{-\nu}\|f\|_q$$

for all (t,s) and $f\in L^1\cap L^\infty$. Then

$$\|U^{t,s}f\|_r\leq M(t-s)^{-\nu/(1-\beta)}\|f\|_{p,\sqrt{\psi}},$$

where $\beta=\frac{r}{q}\frac{q-p}{r-p}$ and $M=2^{\nu/(1-\beta)^2}M_1M_2^{1/(1-\beta)}$.

Proof of Theorem 1. We have

$$\begin{aligned}\|U^{t,s}f\|_r&\leq M_2(t-t_s)^{-\nu}\|U^{t_s,s}f\|_q\\&\leq M_2(t-t_s)^{-\nu}\|U^{t_s,s}f\|_r^\beta\ \|U^{t_s,s}f\|_p^{1-\beta}\\&\leq M_2M_1^{1-\beta}(t-t_s)^{-\nu}\|U^{t_s,s}f\|_r^\beta\ \|f\|_{p,\sqrt{\psi}}^{1-\beta},\end{aligned}$$

and hence

$$(t-s)^{\nu/(1-\beta)}\|U^{t,s}f\|_r/\|f\|_{p,\sqrt{\psi}}\leq M_2M_1^{1-\beta}2^{\nu/(1-\beta)}\left[(t-s)^{\nu/(1-\beta)}\|U^{t_s,s}f\|_r\ /\|f\|_{p,\sqrt{\psi}}\right]^\beta.$$

Setting $R_{2T}:=\sup_{t-s\in]0,T]}\left[(t-s)^{\nu/(1-\beta)}\|U^{t,s}f\|_r/\|f\|_{p,\sqrt{\psi}}\right]$, we obtain from the last inequality that $R_{2T}\leq M^{1-\beta}(R_T)^\beta$. But $R_T\leq R_{2T}$, and so $R_T\leq M$. The proof of Theorem 1 is completed. □

Proof of Theorem A. By (S_3) and (S_4),

$$\begin{aligned}\|e^{-t\Lambda}h\|_1&\leq c_0^{-1}\|\varphi_se^{-t\Lambda}\varphi_s^{-1}\varphi_sh\|_1\\&\leq c_0^{-1}c_1\|h\|_{1,\sqrt{\varphi_s}},\qquad h\in\varphi_sL^\infty_{\rm com}(X,\mu).\end{aligned}$$

The latter, (S_1) and Theorem 1 with $\psi:=\varphi_s$ yield

$$\|e^{-t\Lambda}f\|_\infty\leq Mt^{-j'}\|\varphi_sf\|_1,\quad 0<t\leq s,\quad f\in\varphi_s^{-1}L^\infty_{\rm com}.$$

Taking $s=t$, we obtain (NIE_w). ☐

The proof of Theorem A, as well as the proofs of Theorems 2 and 3 below, are based on ideas of J.Nash [N].

Remarks. 1. (S_1) can be viewed as a variant of the Sobolev embedding property of Λ.

2. In applications of Theorem A to concrete operators the main difficulty consists in verification of the assumption (S_3).

3. Heat kernel of $(-\Delta)^{\frac{\alpha}{2}}+\kappa|x|^{-\alpha}x\cdot\nabla$, $1<\alpha<2$

We now state in detail our main result concerning $(-\Delta)^{\frac{\alpha}{2}}+b\cdot\nabla$, $1<\alpha<2$,

$$b(x):=\kappa|x|^{-\alpha}x, \qquad \kappa:=\sqrt{\delta}(d-\alpha)^{-1}2c^{-2}\left(\frac{\alpha}{2},2,d\right), \qquad 0<\delta<4,$$

$$c(\alpha,p,d):=\frac{\gamma(\frac{d}{p}-\alpha)}{\gamma(\frac{d}{p})}, \quad \gamma(\alpha):=\frac{2^\alpha\pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2}-\frac{\alpha}{2})}, \quad 1<p<\frac{d}{\alpha}.$$

1. First, we construct an operator realization of $(-\Delta)^{\frac{\alpha}{2}}+b\cdot\nabla$ as the generator of a C_0 semigroup in an appropriate L^r space.

In L^p, $1\le p<\infty$, and $C_u=\{f\in C(\mathbb{R}^d)\mid f \text{ are uniformly continuous and bounded}\}$ (with the sup-norm), define approximating operators

$$P^\varepsilon:=(-\Delta)^{\frac{\alpha}{2}}+b_\varepsilon\cdot\nabla+U_\varepsilon,$$

$$D(P^\varepsilon)=D((-\Delta)_p^{\frac{\alpha}{2}})=\mathcal{W}^{\alpha,p}\equiv\big(1+(-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^p, \qquad D(P^\varepsilon)=\big(1+(-\Delta)^{\frac{\alpha}{2}}\big)^{-1}C_u, \quad \text{respectively,}$$

where $\varepsilon>0$,

$$b_\varepsilon(x)=\kappa|x|_\varepsilon^{-\alpha}x, \quad |x|_\varepsilon:=\sqrt{|x|^2+\varepsilon}, \quad \varepsilon>0, \qquad U_\varepsilon(x):=\alpha\kappa\varepsilon|x|_\varepsilon^{-\alpha-2},$$

and for the weights φ_s defined in Theorem 2.

The potentials U_ε, which become negligible as $\varepsilon\downarrow 0$ (see details below), are needed to carry out the estimates in the proof of Proposition 2.

By the Hille Perturbation Theorem [Ka, Ch. IX, sect. 2.2], $-P^\varepsilon$ is the generator of a holomorphic semigroup in L^p, $1\le p<\infty$, and C_u. Similarly, $-\Lambda^\varepsilon$, where $\Lambda^\varepsilon:=(-\Delta)^{\frac{\alpha}{2}}+b_\varepsilon\cdot\nabla$, generates a holomorphic semigroup in L^p and C_u (for details, if needed, see Remark 1 below). It is well known that $e^{-t\Lambda^\varepsilon}L^p_+\subset L^p_+$, and so $e^{-tP^\varepsilon}L^p_+\subset L^p_+$. Also, $\|e^{-tP^\varepsilon}f\|_\infty\le\|e^{-t\Lambda^\varepsilon}f\|_\infty\le\|f\|_\infty$, $f\in L^p\cap L^\infty$.

For $0<\delta<1$, $b\cdot\nabla$ is Rellich's perturbation of $(-\Delta)^{\frac{\alpha}{2}}$ in L^2, and so the algebraic sum $\Lambda:=(-\Delta)^{\frac{\alpha}{2}}+b\cdot\nabla$, $D(\Lambda)=\mathcal{W}^{\alpha,2}$ $(=\big(1+(-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^2)$, is the (minus) generator of a holomorphic semigroup in L^2, see Proposition 7 below, with the property

$$e^{-t\Lambda}=s\text{-}L^2\text{-}\lim_{\varepsilon\downarrow 0}e^{-tP^\varepsilon}=s\text{-}L^2\text{-}\lim_{\varepsilon\downarrow 0}e^{-t\Lambda^\varepsilon},$$

see Proposition 8 below. Since $e^{-t\Lambda}$ is an L^∞ contraction, $e^{-t\Lambda_r}:=\big[e^{-t\Lambda}\upharpoonright L^2\cap L^r\big]^{\text{clos}}_{L^r\to L^r}$ determines a C_0 semigroup on L^r for every $r\in[2,\infty[$. Then, clearly, $e^{-t\Lambda_r}=s\text{-}L^r\text{-}\lim_{\varepsilon\downarrow 0}e^{-tP^\varepsilon}$.

For $1\le\delta<4$, we prove in Proposition 9 below that for every $r\in]r_c,\infty[$, $r_c:=\frac{2}{2-\sqrt{\delta}}$ the limit

$$e^{-t\Lambda_r}:=s\text{-}L^r\text{-}\lim_i e^{-tP^{\varepsilon_i}} \quad \text{(locally uniformly in } t\ge 0),$$

exists for a sequence $\varepsilon_i \downarrow 0$, and determines a contraction C_0 semigroup in L^r. Λ_r is an operator realization of $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$ in L^r; $v = (\mu + \Lambda_r)^{-1} f$, $\mu > 0$, $f \in L^r$, is a weak solution to the equation $\mu v + (-\Delta)^{\frac{\alpha}{2}} v + b \cdot \nabla v = f$. (We note that in the case $\alpha = 2$ the interval of contractive solvability $]r_c, \infty[$ is known to be sharp, see [KS1, sect. 4].)

By the construction, $e^{-t\Lambda_r}$ is positivity preserving.

2. Define β by $\frac{\gamma(\beta)}{(\beta-\alpha)\gamma(\beta-\alpha)} = \kappa$. (Direct calculations show that such $\beta \in]\alpha, d[$ exists and is unique provided that $0 < \delta \leq 4$. The latter ceases to be true if $\delta > 4$.) This choice of β entails that $|x|^{-d+\beta}$ is a Lyapunov function to the formal operator $\Lambda^* = (-\Delta)^{\frac{\alpha}{2}} - \nabla \cdot b$, i.e. $\Lambda^* |x|^{-d+\beta} = 0$, cf. Appendix B.

Let η be a $C^2(]0,\infty[)$ function such that

$$\eta(r) = \begin{cases} r^{-d+\beta}, & 0 < r < 1, \\ \frac{1}{2}, & r \geq 2. \end{cases}$$

Theorem 2. *Let $0 < \delta < 4$. Then $e^{-t\Lambda_r}$ is an integral operator for each $t > 0$; there exists a constant C such that the weighted Nash initial estimate*

$$e^{-t\Lambda}(x,y) \leq C t^{-j'} \varphi_t(y), \quad j' = \frac{d}{\alpha}, \quad \varphi_t(y) = \eta(t^{-\frac{1}{\alpha}}|y|)$$

is valid for all $x, y \in \mathbb{R}^d$, $y \neq 0$ and $t > 0$.

Having at hand Theorem 2, we obtain below the following.

Theorem 3. *Let $0 < \delta < 4$. Then*

$$e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y), \quad x, y \in \mathbb{R}^d, y \neq 0, \quad t > 0.$$

Here $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}$. ($a(z) \approx b(z)$ means that $c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant $c > 1$ and all admissible z).

In the proof of the lower bound, we use the upper bound and Proposition 10 below: *If $0 < \delta < 4$, then for every $r' \in]1, r_c'[$ there exists an operator realization Λ_r^* of $(-\Delta)^{\frac{\alpha}{2}} - \nabla \cdot b$ in $L^{r'}$ as the (minus) generator of a C_0 semigroup,*

$$e^{-t\Lambda_r^*} = s\text{-}L^{r'}\text{-}\lim_i e^{-t(P^{\varepsilon_i})^*}, \qquad t > 0,$$

$$\langle e^{-t\Lambda_r} f, g \rangle = \langle f, e^{-t\Lambda_r^*} g \rangle, \quad t > 0, \quad f \in L^r, \quad g \in L^{r'}.$$

Remark 1. In the proof that $-P^\varepsilon$ is the generator of a holomorphic semigroup in L^p, $1 \leq p < \infty$, and C_u, we use a well known estimate

$$|\nabla(\zeta + A)^{-1}(x,y)| \leq C(\mathrm{Re}\zeta + A)^{-1+\frac{1}{\alpha}}(x,y), \quad \mathrm{Re}\zeta > 0, \quad A \equiv (-\Delta)^{\frac{\alpha}{2}}.$$

Then (for $Y = L^p$ or C_u)

$$\|b_\varepsilon \cdot \nabla(\zeta + A)^{-1}\|_{Y \to Y} \leq C\|b_\varepsilon\|_\infty \|(\mathrm{Re}\zeta + A)^{-1+\frac{1}{\alpha}})\|_{Y \to Y} \leq C\|b_\varepsilon\|_\infty (\mathrm{Re}\zeta)^{-1},$$

and so $\|b_\varepsilon \cdot \nabla\big(\zeta + A\big)^{-1}\|_{Y\to Y}$, $\mathrm{Re}\zeta \geq c_\varepsilon$, can be made arbitrarily small by selecting c_ε sufficiently large. Similar argument applies to $\|U_\varepsilon(\zeta + A)^{-1}\|_{Y\to Y}$. It follows that the Neumann series for

$$(\zeta + P^\varepsilon)^{-1} = (\zeta + A)^{-1}(1 + T)^{-1}, \quad T := (b_\varepsilon \cdot \nabla + U_\varepsilon)(\zeta + A)^{-1},$$

converges in L^p and C_u and satisfies $\|(\zeta + P^\varepsilon)^{-1}\|_{Y\to Y} \leq C|\zeta|^{-1}$, $\mathrm{Re}\zeta \geq c_\varepsilon$, i.e. $-P^\varepsilon$ is the generator of a holomorphic semigroup.

4. Proof of Theorem 2

First, we are going to verify the assumptions of Theorem A for the operators P^ε, $\varepsilon > 0$.

(S_1): $\|e^{-tP^\varepsilon}\|_{r\to\infty} \leq ct^{-\frac{j'}{r}}$, $j' = \frac{d}{\alpha}$, $r \in]r_c, \infty[$, $r_c := \frac{2}{2-\sqrt{\delta}}$.

Proof. Set $A \equiv (-\Delta)^{\frac{\alpha}{2}}$. We have, for $u = e^{-tP^\varepsilon}f$, $f \in L^1_+ \cap L^\infty$,

$$-\frac{1}{r}\frac{d}{dt}\|u\|_r^r = \langle Au, u^{r-1}\rangle + \frac{2}{r}\kappa\langle |x|_\varepsilon^{-\alpha} x \cdot \nabla u^{\frac{r}{2}}, u^{\frac{r}{2}}\rangle + \langle U_\varepsilon u^{\frac{r}{2}}, u^{\frac{r}{2}}\rangle;$$

$$\langle Au, u^{r-1}\rangle \geq \frac{4}{rr'}\|A^{\frac{1}{2}} u^{\frac{r}{2}}\|_2^2 \qquad \text{by [LS, Theorem 2.1];}$$

$$\kappa\langle |x|_\varepsilon^{-\alpha} x \cdot \nabla u^{\frac{r}{2}}, u^{\frac{r}{2}}\rangle = -\kappa\frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha} u^{\frac{r}{2}}, u^{\frac{r}{2}}\rangle - \frac{1}{2}\langle U_\varepsilon u^{\frac{r}{2}}, u^{\frac{r}{2}}\rangle.$$

Then, by the Hardy-Rellich inequality,

$$-\frac{d}{dt}\|u\|_r^r \geq \left(\frac{4}{r'} - 2\sqrt{\delta}\right)\|A^{\frac{1}{2}} u^{\frac{r}{2}}\|_2^2, \tag{$*$}$$

where $\frac{2}{r'} - \sqrt{\delta} > 0$. In particular, it follows that $\|u(t)\|_r \leq \|f\|_r$, $r \in]r_c, \infty[$, and so

$$\|u(t)\|_r \leq \|f\|_r, \quad r \in [r_c, \infty], \quad t > 0. \tag{$**$}$$

Now, let $r = 2r_c$. Using the Nash inequality $\|A^{\frac{1}{2}}h\|_2^2 \geq C_N\|h\|_2^{2+\frac{2\alpha}{d}}\|h\|_1^{-\frac{2\alpha}{d}}$, we obtain (put $w := \|u\|_r^r$)

$$\frac{d}{dt}w^{-\frac{\alpha}{d}} \geq c_2\|f\|_{\frac{r}{2}}^{-\frac{r\alpha}{d}}, \quad c_2 = C_N\frac{\alpha}{d}\big(\frac{4}{r'} - 2\sqrt{\delta}\big).$$

Integrating this inequality, we obtain

$$\|e^{-tP^\varepsilon}\|_{r_c\to 2r_c} \leq c_2^{-\frac{d}{2\alpha r_c}} t^{-\frac{d}{\alpha}(\frac{1}{r_c} - \frac{1}{2r_c})}, \quad t > 0.$$

Since $\|e^{-tP^\varepsilon}f\|_\infty \leq \|f\|_\infty$, we obtain, by the dual variant of Theorem 1 with $\psi \equiv 1$ (see Appendix A), the required estimate (S_1). □

By the construction of φ,

(S_2), (S_4): $\varphi^{\pm 1} \in L^1_{\mathrm{loc}}$ *and* $\inf_{s>0,\, x\in\mathbb{R}^d}\varphi_s(x) \geq \frac{1}{2}$ are valid.

(S_3): *There exists a constant* $c_1 > 0$ *such that, for all* $0 < t \leq s$

$$\|\varphi_s e^{-tP^\varepsilon}\varphi_s^{-1}h\|_1 \leq c_1\|h\|_1, \quad h \in \varphi_s(L^1 \cap L^2), \quad c_1 \neq c_1(\varepsilon).$$

See the proof of (S_3) below.

Thus, Theorem A applies and yields

$$\|e^{-tP^\varepsilon}f\|_\infty \leq Ct^{-j'}\|\varphi_t f\|_1, \quad C \neq C(\varepsilon), \quad f \in L^1_{\sqrt{\varphi}}. \tag{$\star$}$$

According to Proposition 9, $e^{-tP^{\varepsilon_i}}f \to e^{-t\Lambda}f$ a.e. on $\mathbb{R}^d$. The latter and $(\star)$ clearly yield $\|e^{-t\Lambda}f\|_\infty \leq Ct^{-j'}\|\varphi_t f\|_1$ and hence Theorem 2.

Proof of (S_3). In L^1 define operators

$$P^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon \cdot \nabla + U_\varepsilon, \qquad D(P^\varepsilon) = D((-\Delta)_1^{\frac{\alpha}{2}}) \equiv \mathcal{W}^{\alpha,1} \;(= \big(1 + (-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^1),$$

$$(P^\varepsilon)^* := (-\Delta)^{\frac{\alpha}{2}} - \nabla \cdot b_\varepsilon + U_\varepsilon = (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla - W_\varepsilon, \qquad D((P^\varepsilon)^*) = D((-\Delta)_1^{\frac{\alpha}{2}}),$$

where $W_\varepsilon(x) = (d-\alpha)\kappa|x|_\varepsilon^{-\alpha}$. Recall that, by the Hille Perturbation Theorem, for each $\varepsilon > 0$, both e^{-tP^ε}, $e^{-t(P^\varepsilon)^*}$ can be viewed as C_0 semigroups in L^1 and C_u (with $D(P^\varepsilon) = D((P^\varepsilon)^*) = D((-\Delta)_{C_u}^{\frac{\alpha}{2}})$).

Set

$$\phi_n(x) = \big(e^{-\frac{(P^\varepsilon)^*}{n}}\varphi\big)(x), \qquad \varphi \equiv \varphi_s, \qquad n = 1, 2, \dots$$

Since $\varphi = \varphi_{(1)} + \varphi_{(u)}$, $\varphi_{(1)} \in L^1$, $\varphi_{(u)} \in C_u$, the weights ϕ_n are well defined.

Remark. We emphasize that this choice of ϕ_n, the regularization of φ, is the key observation that allows us to carry out the method in the case $\alpha < 2$.

Define operators

$$Q = \phi_n P^\varepsilon \phi_n^{-1}, \quad D(Q) = \phi_n D(P^\varepsilon) = \phi_n D((-\Delta)^{\frac{\alpha}{2}}),$$

where $\phi_n D(P^\varepsilon) := \{\phi_n u \mid u \in D(P^\varepsilon)\}$,

$$F^t_{\varepsilon,n} = \phi_n e^{-tP^\varepsilon}\phi_n^{-1}.$$

Since $\phi_n \geq \frac{1}{2}$ and $\phi_n, \phi_n^{-1} \in L^\infty$, these operators are well defined. In particular, $F^t_{\varepsilon,n}$ is a quasi bounded C_0 semigroup in L^1. Write $F^t_{\varepsilon,n} = e^{-tG}$.

Set

$$\begin{aligned} M &:= \phi_n(1 + (-\Delta)^{\frac{\alpha}{2}})^{-1}[L^1 \cap C_u] \\ &= \phi_n(\lambda_\varepsilon + P^\varepsilon)^{-1}[L^1 \cap C_u], \quad 0 < \lambda_\varepsilon \in \rho(-P^\varepsilon). \end{aligned}$$

Clearly, M is a dense subspace of L^1, $M \subset D(Q)$ and $M \subset D(G)$. Moreover, $Q \upharpoonright M \subset G$. Indeed, for $f = \phi_n u \in M$,

$$Gf = s\text{-}L^1\text{-}\lim_{t\downarrow 0} t^{-1}(1 - e^{-tG})f = \phi_n s\text{-}L^1\text{-}\lim_{t\downarrow 0} t^{-1}(1 - e^{-tP^\varepsilon})u = \phi_n P^\varepsilon u = Qf.$$

Thus $Q \upharpoonright M$ is closable and $\tilde{Q} := (Q \upharpoonright M)^{\mathrm{clos}} \subset G$.

Proposition 1. *The range $R(\lambda_\varepsilon + \tilde{Q})$ is dense in L^1.*

Proof of Proposition 1. If $\langle(\lambda_\varepsilon + \tilde{Q})h, v\rangle = 0$ for all $h \in D(\tilde{Q})$ and some $v \in L^\infty$, $\|v\|_\infty = 1$, then taking $h \in M$ we would have $\langle(\lambda_\varepsilon + Q)\phi_n(\lambda_\varepsilon + P^\varepsilon)^{-1}g, v\rangle = 0$, $g \in L^1 \cap C_u$, or $\langle\phi_n g, v\rangle = 0$. Choosing $g = e^{\frac{\Delta}{k}}(\chi_m v)$, where $\chi_m \in C_c^\infty$ with $\chi_m(x) = 1$ when $x \in B(0,m)$, we would have $\lim_{k\uparrow\infty}\langle\phi_n g, v\rangle = \langle\phi_n\chi_m, |v|^2\rangle = 0$, and so $v \equiv 0$. Thus, $R(\lambda_\varepsilon + \tilde{Q})$ is dense in L^1. □

The following is the main step in the proof.

Proposition 2. *There is a constant $\hat{c} = \hat{c}(d, \alpha, \delta)$ such that for every $\varepsilon > 0$ and all $n \geq n(\varepsilon)$,*

$$\lambda + \tilde{Q} \text{ is accretive whenever } \lambda \geq \hat{c}s^{-1}.$$

Taking Proposition 2 for granted we end the proof of (S_3) as follows.

The fact that $\tilde{Q}$ is closed together with Proposition 1 and Proposition 2 imply $R(\lambda_\varepsilon + \tilde{Q}) = L^1$ (Appendix C). But then, by the Lumer-Phillips Theorem, $\lambda + \tilde{Q}$ is the (minus) generator of a contraction semigroup, and $\tilde{Q} = G$ due to $\tilde{Q} \subset G$. Thus, it follows that

$$\|e^{-tG}\|_{1\to 1} \equiv \|\phi_n e^{-tP^\varepsilon} \phi_n^{-1}\|_{1\to 1} \leq e^{\omega t}, \quad \omega = \hat{c}s^{-1}. \tag{$\star\star$}$$

Clearly, $(\star\star)$ and the Fatou Lemma yields (S_3).

Proof of Proposition 2. Recall that both e^{-tP^ε}, $e^{-t(P^\varepsilon)^*}$ are holomorphic in L^1 and C_u due to Hille's Perturbation Theorem. We have

$$\varphi = \varphi_{(1)} + \varphi_{(u)}, \qquad 0 \leq \varphi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}}), \qquad 0 \leq \varphi_{(u)} \in D((-\Delta)_{C_u}^{\frac{\alpha}{2}})$$

(for instance, fix $\xi \in C_c^\infty$, $0 \leq \xi \leq 1$, $\xi = 1$ on $B(0,r)$, and put

$$\varphi_{(1)} := (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1}\xi v, \quad v(x) := \big(1 + \frac{\gamma(\beta)}{\gamma(\beta-\alpha)}|x|^{-\alpha}\big)\tilde{\varphi}(x), \qquad \tilde{\varphi}(x) := (s^{\frac{1}{\alpha}}|x|^{-1})^{d-\beta}$$

$$\varphi_{(u)} := (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1}\big(1-\xi\big)v + \varphi - \tilde{\varphi} \quad \text{for appropriate } r > 0)$$

(cf. Appendix B). Therefore, $(P^\varepsilon)^*\varphi \ \ (= (P^\varepsilon)^*_{L^1}\varphi_{(1)} + (P^\varepsilon)^*_{C_u}\varphi_{(u)})$ is well defined and belongs to $L^1 + C_u = \{w + v \mid w \in L^1, v \in C_u\}$.

Next, for $f = \phi_n u \in M$, we have

$$\begin{aligned}
\langle Qf, \frac{f}{|f|}\rangle =& \langle \phi_n P^\varepsilon u, \frac{f}{|f|}\rangle = \lim_{t\downarrow 0} t^{-1}\langle \phi_n(1 - e^{-tP^\varepsilon})u, \frac{f}{|f|}\rangle, \\
\mathrm{Re}\langle Qf, \frac{f}{|f|}\rangle \geq& \lim_{t\downarrow 0} t^{-1}\langle (1 - e^{-tP^\varepsilon})|u|, \phi_n\rangle \\
=& \lim_{t\downarrow 0} t^{-1}\langle (1 - e^{-tP^\varepsilon})e^{-\frac{P^\varepsilon}{n}}|u|, \varphi\rangle \\
=& \lim_{t\downarrow 0} t^{-1}\langle e^{-\frac{P^\varepsilon}{n}}|u|, (1 - e^{-t(P^\varepsilon)^*})\varphi\rangle \\
=& \langle e^{-\frac{P^\varepsilon}{n}}|u|, (P^\varepsilon)^*\varphi\rangle.
\end{aligned}$$

Now we are going to estimate $J := \langle e^{-\frac{P^\varepsilon}{n}}|u|, (P^\varepsilon)^*\varphi\rangle$ from below using the representation

$$(-\Delta)^{\frac{\alpha}{2}} = -\Delta I_{2-\alpha} = -I_{2-\alpha}\Delta \tag{$\star\star\star$}$$

on C_c^∞, where $I_\nu \equiv (-\Delta)^{-\frac{\nu}{2}}$.

Define $V(x) := (\beta - \alpha)\kappa|x|^{-\alpha}$ $(= \frac{\gamma(\beta)}{\gamma(\beta-\alpha)}|x|^{-\alpha}$ by the choice of $\beta)$. Then

$$\begin{aligned}
\big\langle(-\Delta)^{\frac{\alpha}{2}}\varphi, h\big\rangle &\big(= \big\langle(-\Delta)_1^{\frac{\alpha}{2}}\varphi_{(1)}, h\rangle\big) + \big\langle(-\Delta)_{C_u}^{\frac{\alpha}{2}}\varphi_{(u)}, h\rangle\big) \qquad 0 \leq h \in C_c^\infty \\
&= \big\langle\tilde{\varphi}, (-\Delta)^{\frac{\alpha}{2}}h\big\rangle + \big\langle\varphi - \tilde{\varphi}, (-\Delta)^{\frac{\alpha}{2}}h\big\rangle \\
&\text{(we are using } (\star\star\star)) \\
&= \big\langle -\Delta I_{2-\alpha}\tilde{\varphi}, h\big\rangle + \big\langle -I_{2-\alpha}\Delta(\varphi - \tilde{\varphi}), h\big\rangle.
\end{aligned}$$

We have, by Appendix B,

$$-\Delta I_{2-\alpha}\tilde{\varphi} = V\tilde{\varphi}.$$

Routine calculation shows that

$$-I_{2-\alpha}\Delta(\varphi-\tilde{\varphi})\ \big(\equiv -I_{2-\alpha}\mathbf{1}_{B^c(0,s^{\frac{1}{\alpha}})}\Delta(\varphi-\tilde{\varphi})\big) \geq -c_0 s^{-1}$$

for a constant c_0.

Also, by straightforward calculation, $-\big(b_\varepsilon\cdot\nabla + W_\varepsilon\big)\varphi \geq -V\tilde{\varphi} - c_1 s^{-1}$ for a constant c_1.

Therefore,

$$(P^\varepsilon)^*\varphi = (-\Delta)^{\frac{\alpha}{2}}\varphi - \big(b_\varepsilon\cdot\nabla + W_\varepsilon\big)\varphi \geq -Cs^{-1}, \qquad C := c_0 + c_1,$$

and so,

$$J = \langle e^{-\frac{P^\varepsilon}{n}}|u|, (P^\varepsilon)^*\varphi\rangle \geq -Cs^{-1}\|e^{-\frac{P^\varepsilon}{n}}|u|\|_1 \geq -Cs^{-1}\|e^{-\frac{P^\varepsilon}{n}}\|_{1\to 1}\|\phi_n^{-1}f\|_1,$$

or due to $\phi_n \geq \frac{1}{2}$,

$$J \geq -2Cs^{-1}\|e^{-\frac{P^\varepsilon}{n}}\|_{1\to 1}\|f\|_1.$$

Noticing that $\|W_\varepsilon\|_\infty \leq c\varepsilon^{-\frac{\alpha}{2}}$, $c := \kappa(d-\alpha)$, we have $\|e^{-\frac{P^\varepsilon}{n}}\|_{1\to 1} \leq e^{c\varepsilon^{-\frac{\alpha}{2}}n^{-1}} = 1 + o(n)$. Taking $\lambda = 3Cs^{-1}$ we see that, for every $\varepsilon > 0$ and for all n larger than some $n(\varepsilon)$,

$$\mathrm{Re}\langle(\lambda + Q)f, \frac{f}{|f|}\rangle \geq 0 \qquad f \in M.$$

The latter holds for all $f \in D(\tilde{Q})$. The proof of Proposition 2 is completed. □

The proof of Theorem 2 is completed. □

The following inequalities, which will be needed in the proof of Theorem 3 below, are simple consequences of (S_3) and $(\star)$:

Corollary 1.

$$e^{-t(P^\varepsilon)^*}\varphi(x) \leq c\varphi(x), \quad \langle e^{-t(P^\varepsilon)^*}(x,\cdot)\rangle \leq 2c\varphi(x) \quad x \neq 0,\ s \geq t > 0.$$

5. Proof of Theorem 3: The upper bound $e^{-t\Lambda_r}(x,y) \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y)$ $(y \neq 0)$.

For brevity, everywhere below $(-\Delta)^{\frac{\alpha}{2}} =: A$.

By the construction of $e^{-t\Lambda_r}$, the result would follow from a priori bound

$$e^{-tP^\varepsilon}(x,y) \leq Ce^{-tA}(x,y)\varphi_t(y), \quad C \neq C(\varepsilon).$$

Since for $u(t,x) := \big(e^{-\lambda t P^\varepsilon}f\big)(t,x)$,

$$u(\lambda t, \lambda^{\frac{1}{\alpha}}x) = e^{-tP^{\lambda^{\frac{2}{\alpha}}\varepsilon}}f_\lambda(x), \quad \lambda > 0,$$

where $f_\lambda(x) := f(\lambda^{\frac{1}{\alpha}}x)$, in view of the scaling properties of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}$ and φ_t, it suffices to prove the a priori bound for $t = 1$. By duality, it suffices to prove

$$e^{-(P^\varepsilon)^*}(x,y) \leq Ce^{-A}(x,y)\varphi(x), \quad C \neq C(\varepsilon), \quad \varphi \equiv \varphi_1.$$

Let $R > 1$ to be chosen later.

The case $|x|, |y| \leq 2R$.

Since $e^{-A}(x,y) \approx 1 \wedge |x-y|^{-d-\alpha}$ $(x \neq y)$, the Nash initial estimate $e^{-t(P^\varepsilon)^*}(x,y) \leq Ct^{-j'}\varphi(x)$ (Theorem 2) yields

$$e^{-(P^\varepsilon)^*}(x,y) \leq C_R e^{-A}(x,y)\varphi(x), \quad C_R \neq C_R(\varepsilon).$$

To consider the other cases we use the Duhamel formula

$$\begin{aligned} e^{-(P^\varepsilon)^*} &= e^{-A} + \int_0^1 e^{-\tau(P^\varepsilon)^*}(B_{\varepsilon,R} + B^c_{\varepsilon,R})e^{-(1-\tau)A}d\tau \\ &=: e^{-A} + K_R + K_R^c, \end{aligned}$$

where $B_{\varepsilon,R} := \mathbf{1}_{B(0,R)}B_\varepsilon$, $B^c_{\varepsilon,R} := \mathbf{1}_{B^c(0,R)}B_\varepsilon$ and $B_\varepsilon := b_\varepsilon \cdot \nabla + W_\varepsilon$ (recall, $W_\varepsilon(x) = \kappa(d-\alpha)|x|_\varepsilon^{-\alpha}$, $b_\varepsilon(x) = \kappa|x|_\varepsilon^{-\alpha}x$).

Below we prove that $K_R(x,y)$, $K_R^c(x,y) \leq C'_R e^{-A}(x,y)\varphi(x)$, which would yield the upper bound. We will need the following.

Lemma 1. *Set* $E^t(x,y) = t\big(|x-y|^{-d-\alpha-1} \wedge t^{-\frac{d+\alpha+1}{\alpha}}\big)$, $E^t f(x) := \langle E^t(x,\cdot)f(\cdot)\rangle$.

Let $0 < t \leq 1$. *Then*

(i) $|\nabla_x e^{-tA}(x,y)| \leq c_0 E^t(x,y)$;

(ii) $\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq c_1 e^{-tA}(x,y)$;

(iii) $\int_0^t \langle E^{t-\tau}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq c_2 E^t(x,y)$.

Proof. For the proof of (i), (ii) see e.g. [BJ]. Essentially the same argument yields (iii). For the sake of completeness, we provide the details:

$$\begin{aligned} E^t(x,z) \wedge E^\tau(z,y) &= (t|x-z|^{-d-\alpha-1} \wedge t^{-\frac{d+1}{\alpha}}) \wedge (\tau|z-y|^{-d-\alpha-1} \wedge \tau^{-\frac{d+1}{\alpha}}) \\ &\leq C_0 \left(\frac{t+\tau}{2}\right)^{-\frac{d+1}{\alpha}} \wedge \left[(t+\tau)\left(\frac{|x-z|+|z-y|}{2}\right)^{-d-\alpha-1}\right] \qquad (C_0 > 1) \\ &\leq C(t+\tau)^{-\frac{d+1}{\alpha}} \wedge \left[(t+\tau)(|x-y|)^{-d-\alpha-1}\right] = CE^{t+\tau}(x,y), \end{aligned}$$

so (iii) follows from the inequality $ac = (a \wedge c)(a \vee c) \leq (a \wedge c)(a+c)$ $(a, c \geq 0)$:

$$\int_0^t \langle E^{t-\tau}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq E^{t+\tau}(x,y)\int_0^t \langle E^{t-\tau}(x,\cdot) + E^\tau(\cdot,y)\rangle d\tau,$$

where, routine calculation shows, $\int_0^t \langle E^{t-\tau}(x,\cdot) + E^\tau(\cdot,y)\rangle d\tau \leq c_2 < \infty$ (we use that $t \leq 1$). ☐

The case $|y| > 2R$, $0 < |x| \leq |y|$.

Claim 1. *If* $|y| > 2R$, $0 < |x| \leq |y|$, *then*

$$K_R(x,y) \equiv \int_0^1 \big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)B_{\varepsilon,R}(\cdot)e^{-(1-\tau)A}(\cdot,y)\big\rangle d\tau \leq \hat{C}e^{-A}(x,y)\varphi(x), \quad \hat{C} \neq \hat{C}(\varepsilon).$$

Proof. Claim 1 clearly follows from

(j) $\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle d\tau \leq c_4 e^{-tA}(x,y)\varphi(x)$,

and, in view of Lemma 1(i), from

(jj) $\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau \leq c_3 e^{-tA}(x,y)\varphi(x)$, where $Z_\varepsilon(x) := |x|_\varepsilon^{-\alpha}|x|$.

Let us prove (jj):

$$\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau$$

(we are using $E^{t-\tau}(\cdot,y)\leq Ce^{-(t-\tau)A}(\cdot,y)|\cdot-y|^{-1}$)

$$\leq C\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)|\cdot-y|^{-1}\rangle d\tau$$

(we are using $\mathbf{1}_{B(0,R)}(\cdot)|\cdot-y|^{-1}\leq|\cdot|^{-1}$)

$$\leq C'\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle d\tau$$

(we are using $\mathbf{1}_{B(0,R)}(\cdot)e^{-(t-\tau)A}(\cdot,y)\leq e^{-tA}(x,y)$)

$$\leq C''e^{-tA}(x,y)\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)W_\varepsilon(\cdot)\rangle d\tau.$$

According to the Duhamel formula $e^{-t(P^\varepsilon)^*}=e^{-tA}+\int_0^t e^{-\tau(P^\varepsilon)^*}(b_\varepsilon\cdot\nabla+W_\varepsilon)e^{-(t-\tau)A}d\tau$,

$$1+\int_0^t\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)W_\varepsilon(\cdot)\rangle d\tau=\langle e^{-t(P^\varepsilon)^*}(x,\cdot)\rangle.$$

Using the inequality $\langle e^{-t(P^\varepsilon)^*}(x,\cdot)\rangle\leq 2c\varphi(x)$ from Corollary 1, it is seen that

$$\int_0^t\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)W_\varepsilon(\cdot)\rangle d\tau\leq 2c\varphi(x).$$

The latter and the previous estimate yield (jj). Incidentally, we have also proved (j). □

Claim 2. *If* $|y|>2R$, $|x|\leq|y|$, *then*

$$K_R^c(x,y)\equiv\int_0^1\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)B^c_{\varepsilon,R}(\cdot)e^{-(1-\tau)A}(\cdot,y)\rangle d\tau\leq Ce^{-A}(x,y)\varphi(x).$$

Proof. Lemma 1(i) yields

$$|B^c_{\varepsilon,R}(\cdot)e^{-(\tau-\tau')A}(\cdot,y)|\leq C_0\big(R^{-\alpha}e^{-(\tau-\tau')A}(\cdot,y)+R^{-\alpha+1}E^{\tau-\tau'}(\cdot,y)\big), \tag{$*$}$$

$$\begin{aligned}K_R^c(x,y)&\equiv\int_0^1\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)B^c_{\varepsilon,R}(\cdot)e^{-(1-\tau)A}(\cdot,y)\rangle d\tau\\&\leq C_0R^{-\alpha}\int_0^1\big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)e^{-(1-\tau)A}(\cdot,y)\big\rangle d\tau+C_0R^{-\alpha+1}\int_0^1\big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\big\rangle d\tau.\end{aligned} \tag{$**$}$$

1. Let us estimate the first term in the RHS of $(**)$. By the Duhamel formula,

$$\begin{aligned}&\int_0^1 e^{-\tau(P^\varepsilon)^*}e^{-(1-\tau)A}d\tau\\&=\int_0^1 e^{-\tau A}e^{-(1-\tau)A}d\tau+\int_0^1\int_0^\tau e^{-\tau'(P^\varepsilon)^*}(B_{\varepsilon,R}+B^c_{\varepsilon,R})e^{-(\tau-\tau')A}d\tau' e^{-(1-\tau)A}d\tau\\&\equiv e^{-A}+I_R+I_R^c.\end{aligned}$$

We have $I_R = \int_0^1 I_R^\tau e^{-(1-\tau)A} d\tau$, where $I_R^\tau := \int_0^\tau e^{-\tau'(P^\varepsilon)^*} B_{\varepsilon,R} e^{-(\tau-\tau')A} d\tau'$. By Claim 1,

$$|I_R^\tau(x,y)| \leq \hat{C} e^{-\tau A}(x,y)\varphi(x) \qquad \text{and so } |I_R(x,y)| \leq \hat{C} e^{-A}(x,y)\varphi(x).$$

In turn, $I_R^c = \int_0^1 (I_R^c)^\tau e^{-(1-\tau)A} d\tau$, where $(I_R^c)^\tau := \int_0^\tau e^{-\tau'(P^\varepsilon)^*} B_{\varepsilon,R}^c e^{-(\tau-\tau')A} d\tau'$, so

$$\begin{aligned}|(I_R^c)^\tau(x,y)| &\leq C_0 R^{-\alpha} \int_0^\tau \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) e^{-(\tau-\tau')A}(\cdot,y)\big\rangle d\tau' \\ &\quad + C_0 R^{-\alpha+1} \int_0^\tau \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) E^{\tau-\tau'}(\cdot,y)\big\rangle d\tau'.\end{aligned}$$

Then

$$\begin{aligned}|I_R^c(x,y)| &\leq C_0 R^{-\alpha} \int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} e^{-(\tau-\tau')A} e^{-(1-\tau)A}\big)(x,y) d\tau' d\tau \\ &\quad + C_0 R^{-\alpha+1} \int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} E^{\tau-\tau'} e^{-(1-\tau)A}\big)(x,y) d\tau' d\tau,\end{aligned}$$

where we estimate the first and second integrals as follows.

$$\begin{aligned}&\int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} e^{-(1-\tau')A}\big)(x,y) d\tau' d\tau \\ &\leq \int_0^1 \int_0^1 \big(e^{-\tau'(P^\varepsilon)^*} e^{-(1-\tau')A}\big)(x,y) d\tau' d\tau = \int_0^1 \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau')A}(\cdot,y)\big\rangle d\tau',\end{aligned}$$

$$\begin{aligned}&\int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} E^{\tau-\tau'} e^{-(1-\tau)A}\big)(x,y) d\tau' d\tau \\ &\text{(we are changing the order of integration in } \tau \text{ and } \tau') \\ &= \int_0^1 \int_{\tau'}^1 \big(e^{-\tau'(P^\varepsilon)^*} E^{\tau-\tau'} e^{-(1-\tau)A}\big)(x,y) d\tau d\tau' \\ &\text{(by Lemma 1}(ii)\text{, } \int_{\tau'}^1 (E^{\tau-\tau'} e^{-(1-\tau)A})(\cdot,y) d\tau \leq c_1 e^{-(1-\tau')A}(\cdot,y)) \\ &\leq c_1 \int_0^1 \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau')A}(\cdot,y)\big\rangle d\tau'.\end{aligned}$$

Thus,

$$|I_R^c(x,y)| \leq C_0(R^{-\alpha} + c_1 R^{-\alpha+1}) \int_0^1 \big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\big\rangle d\tau.$$

Therefore, for $R > 1$ such that $C_0(R^{-\alpha} + c_1 R^{-\alpha+1}) \leq \frac{1}{2}$,

$$\begin{aligned}&\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \\ &\leq e^{-A}(x,y) + \frac{1}{2} \int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\rangle d\tau + \hat{C} e^{-A}(x,y)\varphi(x),\end{aligned}$$

i.e. $\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \leq 2(2+\hat{C}) e^{-A}(x,y)\varphi(x)$.

2. Let us estimate the second term in the RHS of $(**)$. By the Duhamel formula

$$\begin{aligned}&\int_0^1 e^{-\tau (P^\varepsilon)^*}E^{1-\tau}d\tau \\ &= \int_0^1 e^{-\tau A}E^{1-\tau}d\tau + \int_0^1\int_0^\tau e^{-\tau'(P^\varepsilon)^*}(B_{\varepsilon,R}+B^c_{\varepsilon,R})e^{-(\tau-\tau')A}d\tau' E^{1-\tau}d\tau \\ &\equiv \int_0^1 e^{-\tau A}E^{1-\tau}d\tau + J_R + J_R^c,\end{aligned}$$

where, by Lemma 1(ii), $\int_0^1 \langle e^{-\tau A}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle ds \leq c_1 e^{-A}(x,y)$. Let us estimate J_R and J_R^c.

We have $J_R = \int_0^1 J_R^\tau E^{1-\tau}d\tau$, where $J_R^\tau := \int_0^\tau e^{-\tau'(P^\varepsilon)^*}B_{\varepsilon,R}e^{-(\tau-\tau')A}d\tau'$. By Claim 1,

$$|J_R^\tau(x,y)| \leq \hat{C}e^{-\tau A}(x,y)\varphi(x), \qquad \text{and so by Lemma 1}(ii),$$

$$|J_R(x,y)| \leq C_1 e^{-A}(x,y)\varphi(x).$$

In turn, $J_R^c = \int_0^1 (J_R^c)^\tau E^{1-\tau}d\tau$, where $(J_R^c)^\tau := \int_0^\tau e^{-\tau'(P^\varepsilon)^*}B^c_{\varepsilon,R}e^{-(\tau-\tau')A}d\tau'$. By $(*)$ and Lemma 1(ii), $|(J_R^c)^\tau(x,y)| \leq C_0R^{-\alpha}\int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*}e^{-(\tau-\tau')A}\big)(x,y)d\tau' + C_0R^{-\alpha+1}\int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*}E^{\tau-\tau'}\big)(x,y)d\tau'$. Due to Lemma 1$(ii)$,$(iii)$,

$$\begin{aligned}|J_R^c(x,y)| \leq\ & C_0c_1R^{-\alpha}\int_0^1 \langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot)e^{-(1-\tau')A}(\cdot,y)\rangle d\tau' \\ &+ C_0c_2R^{-\alpha+1}\int_0^1 \langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot)E^{1-\tau'}(\cdot,y)\rangle d\tau'.\end{aligned}$$

Thus, for $R>1$ such that $C_0c_1R^{-\alpha}, C_0c_2R^{-\alpha+1} \leq \frac{1}{2}$,

$$\begin{aligned}&\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle d\tau \leq c_1e^{-A}(x,y) + \frac{1}{2}\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \\ &+ \frac{1}{2}\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle d\tau + C_1e^{-A}(x,y)\varphi(x).\end{aligned}$$

Using **1** we arrive at $\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle d\tau \leq 2(2c_1+2+\hat{C}+C_1)e^{-A}(x,y)\varphi(x)$.

Now **1** and **2** applied in $(**)$ yield Claim 2. □

The case $|x|>2R$, $|y|\leq|x|$ is treated similarly, so we omit the details.

The proof of the upper bound is completed.

6. Proof of Theorem 3: The lower bound $e^{-t\Lambda_r}(x,y) \geq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y)$ $(C>0, x,y\neq 0)$.

For brevity, put $\Lambda \equiv \Lambda_r$ and $A := (-\Delta)^{\frac{\alpha}{2}}$.

Proposition 3. *Define* $g=\varphi h$, $\varphi\equiv\varphi_s$, $0\leq h\in\mathcal{S}$*-the L. Schwartz space of test functions. There is a constant* $0<\hat{\mu}$ *such that, for all* $0<t\leq s$,

$$e^{-\frac{\hat{\mu}}{s}t}\langle g\rangle \leq \langle \varphi e^{-t\Lambda}\varphi^{-1}g\rangle.$$

Proof. Set $g_n = \phi_n h$, $\phi_n(x) = (e^{-\frac{(P^\varepsilon)^*}{n}}\varphi)(x)$. Then

$$\langle g_n\rangle - \langle \phi_n e^{-t(P^\varepsilon-\mu)}h\rangle = -\mu\int_0^t \langle \varphi, e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau + \int_0^t \langle \varphi, P^\varepsilon e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau,$$

where $\mu = \frac{\hat{\mu}}{s} > 0$ is to be chosen. Let $\tilde{\varphi}(x) = (s^{-\frac{1}{\alpha}}|x|)^{-d+\beta}$. Write $(P^\varepsilon)^*\varphi = (P^\varepsilon)^*\tilde{\varphi} + (P^\varepsilon)^*(\varphi-\tilde{\varphi}) = \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}(V-V_\varepsilon)\varphi + v_\varepsilon$, $V(x) \equiv V(|x|) = \kappa(\beta-\alpha)|x|^{-\alpha}$, $V_\varepsilon(x) \equiv V_\varepsilon(|x|) := V(|x|_\varepsilon)$. Routine calculation shows that $\|v_\varepsilon\|_\infty \leq \frac{\mu_1}{s}$ for a $\mu_1 \neq \mu_1(\varepsilon)$ (cf. the proof of Proposition 2). Thus

$$\int_0^t \langle v_\varepsilon, e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau \leq \frac{\mu_1}{s}\int_0^t \langle e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau \leq \frac{2\mu_1}{s}\int_0^t \langle \varphi, e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau.$$

Taking $\hat{\mu} = 2\mu_1$, we have

$$\langle g_n\rangle - \langle \phi_n e^{-t(P^\varepsilon-\mu)}h\rangle \leq \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}(V-V_\varepsilon)\varphi, e^{-(\tau+\frac{1}{n})P^\varepsilon}h\rangle e^{\mu\tau}d\tau,$$

or, sending $n \to \infty$,

$$\langle g\rangle - e^{\frac{\hat{\mu}}{s}t}\langle \varphi e^{-tP^\varepsilon}h\rangle \leq e^{\hat{\mu}}\int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}(V-V_\varepsilon)\varphi, e^{-\tau P^\varepsilon}h\rangle d\tau. \qquad (\diamond)$$

It remains to take $\varepsilon \downarrow 0$ in $(\diamond)$. Since $\|e^{-\tau P^\varepsilon}h\|_\infty \leq \|h\|_\infty$ and

$$\mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}|V-V_\varepsilon|\varphi \leq 2\varphi\mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}V \leq C\mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}(s^{-\frac{1}{\alpha}}|x|)^{-d+\beta}|x|^{-\alpha}, \qquad d-\beta+\alpha < d,$$

the RHS of $(\diamond)$ tends to 0 as $\varepsilon \downarrow 0$ due to the Dominated Convergence Theorem. The latter, $e^{-tP^{\varepsilon_i}}h \to e^{-t\Lambda}h$ a.e. on $\mathbb{R}^d$ and (S_3) yield Proposition 3. ☐

We also need the following consequence of the upper bound and Proposition 3.

Proposition 4. *There exist constants* $0 < r < R_0 < R$ *such that for all* $g := \varphi_t h$, $0 \leq h \in \mathcal{S}$ *with* $\operatorname{sprt} h \subset B(0, R_{0,t})$, $R_{0,t} := R_0\frac{1}{2}(1+t^{\frac{1}{\alpha}})$ *we have*

$$e^{-\hat{\mu}-1}\langle g\rangle \leq \langle \mathbf{1}_{R_t,r_t}\varphi_t e^{-t\Lambda}\varphi_t^{-1}g\rangle,$$

where $r_t := rt^{\frac{1}{\alpha}}$, $R_t := R\frac{1}{2}(1+t^{\frac{1}{\alpha}})$, $\mathbf{1}_{R_t,r_t} := \mathbf{1}_{B(0,R_t)} - \mathbf{1}_{B(0,r_t)}$.

In particular,

$$e^{-\hat{\mu}-1}\varphi_t(x) \leq e^{-t\Lambda^*}\varphi_t\mathbf{1}_{R_t,r_t}(x) \quad \text{for all } x \in B(0,R_{0,t}).$$

Proof. By the upper bound,

$$\begin{aligned}
\langle \mathbf{1}_{B(0,r_t)}\varphi_t e^{-t\Lambda}\varphi_t^{-1}g\rangle &\leq C\langle \mathbf{1}_{B(0,r_t)}\varphi_t, e^{-tA}g\rangle \\
&\leq CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r_t)}\varphi_t\|_1\|g\|_1 \\
&= CC_1\|\mathbf{1}_{B(0,r)}\varphi\|_1\|g\|_1, \quad \|\mathbf{1}_{B(0,r)}\varphi\|_1 \to 0 \text{ as } r \downarrow 0.
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{1}_{B^c(0,R_t)}\varphi_t e^{-t\Lambda}\varphi_t^{-1}g\rangle &\leq C\langle \mathbf{1}_{B^c(0,R_t)}\varphi_t, e^{-tA}g\rangle \\
&\leq C\langle e^{-tA}\mathbf{1}_{B^c(0,R_t)}, g\mathbf{1}_{B(0,R_{0,t})}\rangle, \\
&\leq C\sup_{x\in B(0,R_{0,t})} e^{-tA}\mathbf{1}_{B^c(0,R_t)}(x)\|g\|_1 \\
&\leq C(R_0,R)\|g\|_1, \quad C(R_0,R) \to 0 \text{ as } R-R_0 \uparrow \infty
\end{aligned}$$

due to $e^{-tA}(x,y) \le \tilde{C}(t|x-y|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}}) \le C(t|R-R_0|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}})$ if $y \in B(0,R_{0,t})$, $x \in B(0,R_t)$.

It remains to apply Proposition 3. □

Proposition 5. *$\langle h\rangle = \langle e^{-t\Lambda^*}h\rangle$ for every $h \in L^1$, $t>0$.*

Proof. We have, for $h \in \mathcal{S}$,

$$\begin{aligned}\langle h\rangle - \langle e^{-t(P^\varepsilon)^*}h\rangle &= \int_0^t \langle 1, (P^\varepsilon)^* e^{-\tau(P^\varepsilon)^*}h\rangle d\tau = \int_0^t \langle U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau \\ &= \int_0^t \langle \mathbf{1}_{B^c(0,1)}U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau + \int_0^t \langle \mathbf{1}_{B(0,1)}U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau.\end{aligned}$$

It is clear that $\langle \mathbf{1}_{B^c(0,1)}U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle \le \|\mathbf{1}_{B^c(0,1)}U_\varepsilon\|_\infty \|h\|_1 \to 0$ as $\varepsilon \downarrow 0$, and so the first integral converges to 0. Let us estimate the second integral:

$$\int_0^t \langle \mathbf{1}_{B(0,1)}U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau = \int_0^t \langle e^{-\tau P^\varepsilon}\mathbf{1}_{B(0,1)}U_\varepsilon, h\rangle d\tau$$

(we are using the upper bound $e^{-tP^\varepsilon}(x,y) \le Ce^{-tA}(x,y)\varphi_t(y)$)

$$\le C\int_0^t \langle e^{-\tau A}\varphi \mathbf{1}_{B(0,1)}U_\varepsilon, |h|\rangle d\tau$$

$$\le Ct\|h\|_\infty \|\varphi \mathbf{1}_{B(0,1)}U_\varepsilon\|_1 \to 0 \text{ as } \varepsilon \downarrow 0 \text{ due to } d-\beta+\alpha<d.$$

Thus, $\langle h\rangle = \lim_\varepsilon \langle e^{-t(P^\varepsilon)^*}h\rangle$. By Proposition 10, we have $e^{-t(P^{\varepsilon_i})^*}h \to e^{-t\Lambda^*}h$ a.e. on $\mathbb{R}^d$. The upper bound $e^{-t(P^\varepsilon)^*}(x,y) \le Ce^{-tA}(x,y)\varphi_t(x)$ yields $|e^{-t(P^\varepsilon)^*}h| \le C\varphi_t e^{-tA}|h| \in L^1$, and so $\lim_i \langle e^{-t(P^{\varepsilon_i})^*}h\rangle = \langle e^{-t\Lambda^*}h\rangle$ by the Dominated Convergence Theorem. Thus, equality $\langle h\rangle = \langle e^{-t\Lambda^*}h\rangle$ holds for every $h \in \mathcal{S}$ and hence for every $h \in L^1$. □

Proposition 6. *There exist constants $0<r<R_0<R$ such that for all $0 \le h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0,R_{0,t})$, $R_{0,t} := R_0\frac{1}{2}(1+t^{\frac{1}{\alpha}})$ we have*

$$\frac{1}{2}\langle h\rangle \le \langle \mathbf{1}_{R_t,r_t}e^{-t\Lambda^*}h\rangle.$$

where $r_t := rt^{\frac{1}{\alpha}}$, $R_t := R\frac{1}{2}(1+t^{\frac{1}{\alpha}})$, $\mathbf{1}_{R_t,r_t} := \mathbf{1}_{B(0,R_t)} - \mathbf{1}_{B(0,r_t)}$.

In particular,

$$\frac{1}{2} \le e^{-t\Lambda}\mathbf{1}_{R_t,r_t}(x) \quad \text{for all } x \in B(0,R_{0,t}).$$

Proof. We follow the argument in the proof of Proposition 4. By the upper bound,

$$\begin{aligned}\langle \mathbf{1}_{B(0,r_t)}e^{-t\Lambda^*}h\rangle &\le C\langle \mathbf{1}_{B(0,r_t)}\varphi_t, e^{-tA}h\rangle \\ &\le CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r_t)}\varphi_t\|_1\|h\|_1 \\ &= o(r)\|h\|_1, \quad o(r) \to 0 \text{ as } r \downarrow 0;\end{aligned}$$

$$\begin{aligned}\langle \mathbf{1}_{B^c(0,R_t)}e^{-t\Lambda^*}h\rangle &\le C\langle \mathbf{1}_{B^c(0,R_t)}\varphi_t, e^{-tA}h\rangle \\ &\le C\langle e^{-tA}\mathbf{1}_{B^c(0,R_t)}, h\mathbf{1}_{B(0,R_{0,t})}\rangle \\ &\le C\sup_{x\in B(0,R_{0,t})} e^{-tA}\mathbf{1}_{B^c(0,R_t)}(x)\|h\|_1 \\ &= C(R_0,R)\|h\|_1, \quad C(R_0,R) \to 0 \text{ as } R-R_0 \uparrow \infty\end{aligned}$$

due to $e^{-tA}(x,y) \leq \tilde{C}(t|x-y|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}}) \leq C(t|R-R_0|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}})$ if $y \in B(0,R_{0,t})$, $x \in B(0,R_t)$.

The last two estimates and Proposition 5 yield $\frac{1}{2}\langle h\rangle \leq \langle \mathbf{1}_{R_t,r_t} e^{-t\Lambda^*} h\rangle$. □

Claim 3. *For every $r>0$ there exist a constant $t(r)>0$ such that*

$$e^{-t\Lambda^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y) \quad \text{for all } |x|\geq r,\ |y|\geq r, \quad 0<t\leq t(r).$$

Proof. By the Duhamel formula,

$$e^{-t(P^\varepsilon)^*}(x,y) \geq e^{-tA}(x,y) + M_t(x,y), \qquad M_t(x,y) \equiv \int_0^t \langle e^{-(t-\tau)(P^\varepsilon)^*}(x,\cdot) b_\varepsilon(\cdot)\cdot\nabla . e^{-\tau A}(\cdot,y)\rangle d\tau.$$

By Lemma 1(i),

$$\begin{aligned}
|M_t(x,y)| &\leq c_1 \int_0^t \big\langle e^{-(t-\tau)(P^\varepsilon)^*}(x,\cdot)|\cdot|^{1-\alpha}E^\tau(\cdot,y)\big\rangle d\tau \\
&\quad \text{(we apply the upper bound)} \\
&\leq c_1 C \int_0^t \varphi_{t-\tau}(x)\big\langle e^{-(t-\tau)A}(x,\cdot)|\cdot|^{1-\alpha}E^\tau(\cdot,y)\big\rangle d\tau \\
&\quad \text{(since } |x|\geq r \text{, we may select } t=t(r)>0 \text{ sufficiently small so that } \varphi_{t-\tau}(x)=\tfrac{1}{2}) \\
&\leq \frac{c_1 C}{2}\int_0^t \big\langle e^{-(t-\tau)A}(x,\cdot)|\cdot|^{1-\alpha}E^\tau(\cdot,y)\big\rangle d\tau =: J(|\cdot|^{1-\alpha}).
\end{aligned}$$

Next, select $\gamma>0$ sufficiently small ($\gamma \ll r$) so that, for all $0<\tau<t$, $|x|$, $|y|\geq r$,

$$\begin{aligned}
\mathbf{1}_{B(0,\gamma)}(\cdot)e^{-(t-\tau)A}(x,\cdot) &\leq C_5 e^{-tA}(x,0),\\
\mathbf{1}_{B(0,\gamma)}(\cdot)e^{-\tau A}(\cdot,y) &\leq C_6 e^{-tA}(0,y),\\
\mathbf{1}_{B(0,\gamma)}(\cdot)E^\tau(\cdot,y) &\leq C_7 e^{-tA}(0,y).
\end{aligned}$$

Using the inequality

$$e^{-tA}(x,z)e^{-\tau A}(z,y) \leq K e^{-(t+\tau)A}(x,y)\big(e^{-tA}(x,z)+e^{-\tau A}(z,y)\big), \tag{$*$}$$

which holds for a constant $K=K(d,\alpha)$, all $x,z,y\in\mathbb{R}^d$ and $t,\tau>0$ (see e.g. [BJ]), we have

$$\begin{aligned}
J(\mathbf{1}_{B(0,\gamma)}|\cdot|^{1-\alpha}) &\leq c\int_0^t \langle \mathbf{1}_{B(0,\gamma)}(\cdot)|\cdot|^{1-\alpha}\rangle d\tau (e^{-tA}(x,0)+e^{-tA}(0,y))e^{-2tA}(x,y) \\
&\leq cC(r)\gamma^{d-\alpha+1}te^{-tA}(x,y). \qquad (**)
\end{aligned}$$

In turn,

$$J(\mathbf{1}_{B^c(0,\gamma)}|\cdot|^{1-\alpha}) \leq \frac{c_1 C}{2}C_0\gamma^{1-\alpha}t^{1-\frac{1}{\alpha}}e^{-tA}(x,y), \tag{$***$}$$

follows immediately from

$$\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq C_0 t^{1-\frac{1}{\alpha}}e^{-tA}(x,y)$$

proved in Appendix D.

Thus, putting $t=\gamma^{2\alpha}$ and selecting $\gamma>0$ sufficiently small in $(**)$ and $(***)$, we have

$$|M_t(x,y)| \leq \frac{1}{2}e^{-tA}(x,y).$$

Thus,

$$e^{-t(P^\varepsilon)^*}(x,y) \geq \frac{1}{2} e^{-tA}(x,y), \quad |x| \geq r,\ |y| \geq r,\ 0 < t \leq t(r).$$

Finally, using the a.e. convergence $e^{-t(P^{\varepsilon_i})^*}h \to e^{-t\Lambda^*}h$ (Proposition 10), we complete the proof of the Claim. □

Claim 4. For every $r > 0$ there exists a constant $c(r) > 0$ such that

$$e^{-\Lambda^*}(x,y) \geq c(r) e^{-A}(x,y) \quad \text{for all } |x| \geq r,\ |y| \geq r, \quad x \neq y.$$

Proof. By the reproduction property,

$$\begin{aligned} e^{-2t_0\Lambda^*}(x,y) &\geq \langle e^{-t_0\Lambda^*}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot) e^{-t_0\Lambda^*}(\cdot,y)\rangle \\ &\quad \text{(we are applying Claim 3)} \\ &\geq c_1^2 \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot) e^{-t_0 A}(\cdot,y)\rangle, \qquad c_1 := \frac{1}{2},\ t_0 = t(r). \end{aligned}$$

Consider the following cases:

1) If $(r \leq)$ $|x|, |y| \leq r_m$, where r_m $(> r)$ is to be chosen, then the above inequality yields $e^{-2t_0\Lambda^*}(x,y) \geq C_{r_m} > 0$, and so

$$e^{-2t_0\Lambda^*}(x,y) \geq C_{1,r_m} e^{-2t_0 A}(x,y), \quad C_{1,r_m} > 0.$$

2) If $|x|, |y| > r_m$, then

$$\begin{aligned} e^{-2t_0\Lambda^*}(x,y) &\geq c_1^2 \big(e^{-2t_0 A}(x,y) - \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B(0,r)}(\cdot) e^{-t_0 A}(\cdot,y)\rangle \big) \\ &\quad \text{(we are applying } (*)) \\ &\geq c_1^2 e^{-2t_0 A}(x,y) \big(1 - K \langle \mathbf{1}_{B(0,r)}(\cdot)(e^{-t_0 A}(x,\cdot) + e^{-t_0 A}(\cdot,y))\rangle \big) \\ &\geq c_1^2 e^{-2t_0 A}(x,y) \big(1 - K_1 \langle \mathbf{1}_{B(0,r)}\rangle (r_m - r)^{-d-\alpha} \big) \\ &\quad \text{(we select } r_m \text{ sufficiently large)} \\ &\geq C_{2,r_m} e^{-2t_0 A}(x,y) \quad C_{2,r_m} > 0. \end{aligned}$$

3) If $r \leq |x| \leq r_m$, $|y| > r_m$, then

$$\begin{aligned} e^{-2t_0\Lambda^*}(x,y) &\geq c_1^2 \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot) e^{-t_0 A}(\cdot,y)\rangle \\ &\geq C_{3,r_m} \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot)\rangle (r + |y|)^{-d-\alpha} \\ &\geq C_{4,r_m} e^{-2t_0 A}(0,y) \geq C_{5,r_m} e^{-2t_0 A}(x,y), \quad C_{i,r_m} > 0 \ (i = 3,4,5). \end{aligned}$$

4) If $r \leq |y| \leq r_m$, $|x| > r_m$, then, by the symmetry of $e^{-t_0 A}$, $e^{-2t_0\Lambda^*}(x,y) \geq C_{5,r_m} e^{-2t_0 A}(x,y)$.

Thus, we have proved that $e^{-2t_0\Lambda^*}(x,y) \geq c_2 e^{-2t_0 A}(x,y)$, $c_2 > 0$, for all $|x|, |y| \geq r$. Continuing this process, we obtain the assertion of the claim. □

We are in position to complete the proof of the lower bound using the so-called $3q$ argument.

Set $q_t(x,y) := \varphi^{-1}(x) e^{-t\Lambda^*}(x,y)$ $(\varphi \equiv \varphi_1)$.

(a) Let $x, y \in B^c(0,1)$, $x \neq y$. Then, using that $\varphi^{-1} \geq c_0 > 0$ on $B^c(0,1)$, we have by Claim 4

$$q_3(x,y) \geq c_0 e^{-3\Lambda^*}(x,y) \geq c e^{-3A}(x,y).$$

Now, fix $R_0 = 1$.
(b) Let $x \in B(0,1)$, $|y| \geq r$, $x \neq y$. By the reproduction property,

$$\begin{aligned} q_2(x,y) &\geq \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)\varphi^{-1}(\cdot)\varphi(\cdot)e^{-\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\geq \varphi^{-1}(x)\varphi^{-1}(y)\langle e^{-\Lambda^*}(x,\cdot)\varphi(\cdot)e^{-\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are applying Proposition 4)} \\ &\geq e^{-\hat{\mu}-1}\varphi^{-1}(y)\inf_{r\leq|z|\leq R} e^{-\Lambda^*}(z,y) \\ &\text{(we are applying Claim 4)} \\ &\geq e^{-\hat{\mu}-1}\varphi^{-1}(y)c(r)e^{-A}(x,y) \\ &\geq C_1(r)e^{-A}(x,y). \end{aligned}$$

(b') Let $x \in B(0,1), |y| \geq 1$ $(> r)$, $x \neq y$. Arguing as in (b), we obtain

$$q_3(x,y) \geq C_2 e^{-3A}(x,y).$$

(c) Let $|x| \geq r$, $y \in B(0,1)$, $x \neq y$. We have

$$\begin{aligned} q_2(x,y) &\geq \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)e^{-\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &= \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)e^{-\Lambda}(y,\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are applying Claim 4)} \\ &\geq \varphi^{-1}(x)c(r)\langle e^{-A}(x,\cdot)e^{-\Lambda}(y,\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\geq C_3(r)(R+|x|)^{-d-\alpha}\langle e^{-\Lambda}(y,\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are applying Proposition 6)} \\ &\geq C_3(r)2^{-1}(R+|x|)^{-d-\alpha} \geq C_4(r)e^{-2A}(x,y). \end{aligned}$$

(c') Let $|x| \geq 1(> r)$, $y \in B(0,1)$, $x \neq y$. Arguing as in (c), we obtain

$$q_3(x,y) \geq C_5(r)e^{-3A}(x,y).$$

(d) Let $x, y \in B(0,1)$, $x \neq y$. By the reproduction property,

$$\begin{aligned} q_3(x,y) &\geq \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)e^{-2\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are using (c))} \\ &\geq \varphi^{-1}(x)C_4(r)\langle e^{-\Lambda^*}(x,\cdot)\varphi(\cdot)e^{-2A}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are using } e^{-2A}(z,y) \geq c_{r,R} > 0 \text{ for } r \leq |z| \leq R,\ |y| \leq 1) \\ &\geq C_4 c_{r,R}\varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)\mathbf{1}_{R,r}(\cdot)\varphi(\cdot)\rangle \\ &\text{(we are applying Proposition 4)} \\ &\geq C_4 c_{r,R} e^{-\hat{\mu}-1} \geq C_5(r,R)e^{-3A}(x,y). \end{aligned}$$

By (a), (b'), (c'), (d), $q^3(x,y) \geq Ce^{-3A}(x,y)$ for all $x, y \in \mathbb{R}^d$, $x \neq y$, and so $e^{-3\Lambda^*}(x,y) \geq Ce^{-3A}(x,y)\varphi(x)$. Now the scaling argument yields the lower bound.

7. Operator realization of $(-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$ in L^2

Proposition 7. *Let $0<\delta<1$. Then the algebraic sum $\Lambda := (-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$, $D(\Lambda)=\mathcal{W}^{\alpha,2}$ $\big(= \big(1+(-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^2\big)$, is the (minus) generator of a holomorphic semigroup in L^2.*

Proof of Proposition 7. We show that $b\cdot\nabla$ is Rellich's perturbation of $(-\Delta)^{\frac{\alpha}{2}}$.

For brevity, write $\|\cdot\|\equiv\|\cdot\|_{2\to 2}$ and $A\equiv(-\Delta)^{\frac{\alpha}{2}}$ in L^2.

Define $T=b\cdot\nabla(\zeta+A)^{-1}$, $\mathrm{Re}\zeta>0$, and note that

$$
\begin{aligned}
\|T\| &\leq \||b|(\zeta+A)^{-1+\frac{1}{\alpha}}\|\|\nabla(\zeta+A)^{-\frac{1}{\alpha}}\| \\
&\quad(\text{we are using } \|\nabla g\|_2=\|(-\Delta)^{\frac12}g\|_2) \\
&\leq \||b|(\mathrm{Re}\zeta+A)^{-1+\frac{1}{\alpha}}\|\|A^{\frac{1}{\alpha}}(\zeta+A)^{-\frac{1}{\alpha}}\| \\
&\quad(\text{by the Spectral Theorem, } \|A^{\frac{1}{\alpha}}(\zeta+A)^{-\frac{1}{\alpha}}\|\leq 1) \\
&\leq \||b|(-\Delta)^{-\frac{\alpha-1}{2}}\| \\
&\quad(\text{we are using [KPS, Lemma 2.7]}) \\
&= \kappa c(\alpha-1,2,d) < \sqrt{\delta}
\end{aligned}
$$

because $c(\alpha-1,2,d)<(d-\alpha)2^{-1}c^2(\frac{\alpha}{2},2,d)$ or, equivalently,

$$
F(\alpha)\equiv(d-\alpha)\Gamma\big(\frac{d-2+2\alpha}{4}\big)\big[\Gamma\big(\frac{d-\alpha}{4}\big)\big]^2-4\Gamma\big(\frac{d+2-2\alpha}{4}\big)\big[\Gamma\big(\frac{d+\alpha}{4}\big)\big]^2>0
$$

(the latter is due to $\frac{d^2}{dt^2}\log\Gamma(t)\geq 0$ and $F(2)=0$ $\big((d-2)\Gamma(\frac{d-2}{4})=4\Gamma(\frac{d+2}{4})\big)$).

Thus, the Neumann series for $(\zeta+\Lambda)^{-1}=(\zeta+A)^{-1}(1+T)^{-1}$ converges, and

$$
\|(\zeta+\Lambda)^{-1}\|\leq(1-\sqrt{\delta})^{-1}|\zeta|^{-1},\quad \mathrm{Re}\zeta>0,
$$

i.e. $-\Lambda$ is the generator of a holomorphic semigroup. □

Proposition 8. *In the assumptions of Proposition 7, $e^{-tP^\varepsilon}\overset{s}{\to}e^{-t\Lambda}$.*

Proof. It suffices to show that $(\mu+P^\varepsilon)^{-1}\overset{s}{\to}(\mu+\Lambda)^{-1}$ for a $\mu>0$.

First, we show that $(\mu+\Lambda^\varepsilon)^{-1}\overset{s}{\to}(\mu+\Lambda)^{-1}$. We will use notation introduced in the proof of Proposition 7 above. Recall: $(\mu+\Lambda)^{-1}=(\mu+A)^{-1}(1+T)^{-1}$, $\|(\mu+\Lambda)^{-1}\|\leq(1-\sqrt{\delta})^{-1}\mu^{-1}$. Since $\|(T-T_\varepsilon)f\|_2\leq\||b-b_\varepsilon|(\mu+A)^{-1}|\nabla f|\|_2\to 0$ for every $f\in C_c^\infty$ by the Dominated Convergence Theorem, we have $T_\varepsilon\overset{s}{\to}T$. Therefore, $(\mu+\Lambda^\varepsilon)^{-1}\overset{s}{\to}(\mu+\Lambda)^{-1}$.

We show that $(\mu+P^\varepsilon)^{-1}-(\mu+\Lambda^\varepsilon)^{-1}\overset{s}{\to}0$. Set $S=(\mu+A)^{-1+\frac{1}{\alpha}}b\cdot\nabla(\mu+A)^{-\frac{1}{\alpha}}$ and $S_\varepsilon=(\mu+A)^{-1+\frac{1}{\alpha}}b_\varepsilon\cdot\nabla(\mu+A)^{-\frac{1}{\alpha}}$. Then $\sup_\varepsilon\|S_\varepsilon\|,\|S\|<1$ and

$$
(\mu+\Lambda^\varepsilon)^{-1}=(\mu+A)^{-\frac{1}{\alpha}}(1+S_\varepsilon)^{-1}(\mu+A)^{-1+\frac{1}{\alpha}},\quad \mu>0. \tag{$\star$}
$$

Now, let $h\in L^2\cap L^\infty$. Then

$$
\|(\mu+P^\varepsilon)^{-1}h-(\mu+\Lambda^\varepsilon)^{-1}h\|_2=\|(\mu+\Lambda^\varepsilon)^{-1}U_\varepsilon(\mu+P^\varepsilon)^{-1}h\|_2\leq K_1+K_2,
$$

$$
\begin{aligned}
K_1 &= \|(\mu+\Lambda^\varepsilon)^{-1}U_\varepsilon \mathbf{1}_{B(0,1)}(\mu+P^\varepsilon)^{-1}h\|_2 \\
&\leq \|(\mu+\Lambda^\varepsilon)^{-1}|x|^{-\alpha+1}\|\||x|^{\alpha-1}U_\varepsilon \mathbf{1}_{B(0,1)}\|_2\mu^{-1}\|h\|_\infty \\
&\text{(we are using } (\star)) \\
&\leq C\mu^{-1}\|h\|_\infty \|\varepsilon|x|_\varepsilon^{-2}|x|^{-1}\mathbf{1}_{B(0,1)}\|_2 \to 0,
\end{aligned}
$$

$$
K_2 = \|(\mu+\Lambda^\varepsilon)^{-1}U_\varepsilon \mathbf{1}_{B^c(0,1)}(\mu+P^\varepsilon)^{-1}h\|_2 \leq \kappa\alpha\varepsilon(1-\sqrt{\delta})^{-1}\mu^{-2}\|h\|_2 \to 0.
$$

The convergence $e^{-tP^\varepsilon} \overset{s}{\to} e^{-t\Lambda}$ is established. □

Similar arguments show that $e^{-t(P^\varepsilon)^*} \overset{s}{\to} e^{-t\Lambda^*}$.

Remark 2. Above we could have constructed an operator realization Λ of $(-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$ on L^2 for $b(x) := \sqrt{\delta_2}c^{-2}(\frac{\alpha-1}{2},2,d)|x|^{-\alpha}x$, $0<\delta_2<1$, by following the arguments in [KS1, sect. 4]. Note that

$$
c^{-1}(\alpha-1,2,d) < c^{-2}(\frac{\alpha-1}{2},2,d)
$$

(indeed, $\Gamma(\frac{d+2-2\alpha}{4})[\Gamma(\frac{d-1+\alpha}{4})]^2 - \Gamma(\frac{d-2+2\alpha}{4})[\Gamma(\frac{d+1-\alpha}{4})]^2 > 0$), i.e. these assumptions are less restrictive than the ones in the proof of Proposition 7.

Then, in particular,

$$
\|e^{-t\Lambda}f\|_q \leq c_r t^{-j'\left(\frac{1}{r}-\frac{1}{q}\right)}\|f\|_r, \quad f\in L^r\cap L^q, \quad 2\leq r<q\leq\infty
$$

(arguing as in the proof of [KS1, Theorem 4.3]).

8. Operator realizations of $(-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$ in L^r, $r\in]r_c,\infty[$, and in $L^1_{\sqrt{\varphi}}$

Proposition 9. *Let $0<\delta<4$. The following is true:*

(i) There exists an operator realization $\Lambda_r(b)$ of $(-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$ in L^r, $r\in]r_c,\infty[$, $r_c = \frac{2}{2-\sqrt{\delta}}$, as (minus) generator of a contraction C_0 semigroup,

$$
e^{-t\Lambda_r(b)} = s\text{-}L^r\text{-}\lim_n e^{-tP^{\varepsilon_n}} \quad (\textit{loc. uniformly in } t\geq 0),
$$

for a sequence $\{\varepsilon_n\}\downarrow 0$.

(ii) There exists an operator realization $\Lambda_{1,\sqrt{\varphi}}(b)$ of $(-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$ in $L^1_{\sqrt{\varphi}}$, $\varphi\equiv\varphi_s$, as (minus) generator of a contraction C_0 semigroup,

$$
e^{-t\Lambda_{1,\sqrt{\varphi}}(b)} = s\text{-}L^1_{\sqrt{\varphi}}\text{-}\lim_n e^{-tP^{\varepsilon_n}} \quad (\textit{loc. uniformly in } t\geq 0)
$$

for a sequence $\{\varepsilon_n\}\downarrow 0$.

The semigroups in (i), (ii) are consistent: $e^{-t\Lambda_r(b)} \upharpoonright L^r\cap L^1_{\sqrt{\varphi}} = e^{-t\Lambda_{1,\sqrt{\varphi}}(b)} \upharpoonright L^r\cap L^1_{\sqrt{\varphi}}$.

(iii) For every $u\in D(\Lambda_r(b))$, $r\in]r_c,\infty[$,

$$
\langle \Lambda_r(b)u,h\rangle = \langle u,(-\Delta)^{\frac{\alpha}{2}}h\rangle - \langle u,b\cdot\nabla h\rangle - \langle u,(\operatorname{div} b)h\rangle, \qquad h\in C_c^\infty.
$$

Proof of Proposition 9. Proof of (i), (ii). 1. Set $0\leq v\equiv v_\varepsilon := (\mu+P^\varepsilon)^{-1}f$, $\mu>0$, $0\leq f\in C_c^\infty$.

Multiplying the equation $(\mu+P^\varepsilon)v=f$ by v^{r-1}, integrating, and then arguing as in the proof of (S_1), we obtain

$$
\mu\|v\|_r^r + \frac{2}{r}\left(\frac{2}{r'}-\sqrt{\delta}\right)\|A^{\frac{1}{2}}v^{\frac{r}{2}}\|_2^2 \leq \|f\|_r\|v\|_r^{r-1},
$$

where $\frac{2}{r'} - \sqrt{\delta} > 0$, and so

$$\mu^{r-1}\frac{2}{r}\left(\frac{2}{r'} - \sqrt{\delta}\right)\|A^{\frac{1}{2}}v^{\frac{r}{2}}\|_2^2 \leq \|f\|_r^r, \tag{$*$}$$

$$\|\mu(\mu+P^\varepsilon)^{-1}f\|_r \leq \|f\|_r, \quad r \in]r_c, \infty[. \tag{$**$}$$

Multiplying $(\mu+P^\varepsilon)v = f$ by $\frac{v}{|v|}\varphi$, integrating, and using $\langle P^\varepsilon v, \frac{v}{|v|}\varphi\rangle \geq -C\|v\|_1$, $C \neq C(\varepsilon)$ (see the proof of Proposition 2), we obtain

$$\|\mu(\mu+P^\varepsilon)^{-1}f\|_{1,\sqrt{\varphi}} \leq \|f\|_{1,\sqrt{\varphi}} \tag{$***$}$$

By density, $(**)$ and $(***)$ hold for all $f \in L^r$ and $f \in L^1_{\sqrt{\varphi}}$, respectively.

2. We will need

Claim 5. (1) *$\mu(\mu+P^\varepsilon)^{-1} \to 1$ in $L^1_{\sqrt{\varphi}}$ as $\mu \to \infty$ uniformly in $\varepsilon > 0$.*

(2) *$\mu(\mu+P^\varepsilon)^{-1} \to 1$ in L^r as $\mu \to \infty$ uniformly in $\varepsilon > 0$, for every $r \in]r_c, \infty[$.*

Proof of Claim 5. Proof of (1). In view of $(***)$, it suffices to prove the convergence only for $f \in C_c^\infty$. We write

$$(\mu+P^\varepsilon)^{-1}f = (\mu+A)^{-1}f - (\mu+P^\varepsilon)^{-1}(b_\varepsilon\cdot\nabla + U_\varepsilon)(\mu+A)^{-1}f.$$

We have $\mu(\mu+A)^{-1}f \to f$ in $L^1_{\sqrt{\varphi}}$ (as $\mu \to \infty$ uniformly in $\varepsilon > 0$). Indeed,

$$\begin{aligned}\big\langle \varphi|\mu(\mu+A)^{-1}f - f|\big\rangle &= \big\langle \mathbf{1}_{B(0,2)}\varphi|\mu(\mu+A)^{-1}f - f|\big\rangle \\ &\quad + \big\langle \mathbf{1}_{B^c(0,2)}\frac{1}{2}|\mu(\mu+A)^{-1}f - f|\big\rangle \to 0,\end{aligned}$$

where the first term tends to 0 since $\mu(\mu+A)^{-1}f \to f$ in C_u, while the second term tends to 0 since $\mu(\mu+A)^{-1}f \to f$ in L^1.

Thus, it remains to show that $\mu(\mu+P^\varepsilon)^{-1}(b_\varepsilon\cdot\nabla + U_\varepsilon)(\mu+A)^{-1}f \to 0$ in $L^1_{\sqrt{\varphi}}$.

First, we prove that $I_{1,\varepsilon}f := \mu(\mu+P^\varepsilon)^{-1}b_\varepsilon\cdot\nabla(\mu+A)^{-1}f \to 0$. Indeed,

$$\begin{aligned}\|I_{1,\varepsilon}f\|_{1,\sqrt{\varphi}} &\leq \|(\mu+P^\varepsilon)^{-1}\mathbf{1}_{B(0,2)}|b_\varepsilon|\|_{1,\sqrt{\varphi}}\|\mu(\mu+A)^{-1}|\nabla f|\|_\infty \\ &\quad + \|(\mu+P^\varepsilon)^{-1}\mathbf{1}_{B^c(0,2)}C_1\mu(\mu+A)^{-1}|\nabla f|\|_{1,\sqrt{\varphi}} \qquad (C_1 := \|\mathbf{1}_{B^c(0,2)}|b|\|_\infty < \infty) \\ &=: J_1 + J_2\end{aligned}$$

We have

$$\begin{aligned}J_1 &\leq \|\varphi(\mu+P^\varepsilon)^{-1}\varphi^{-1}\|_{1\to1}\|\varphi\mathbf{1}_{B(0,2}|b|\|_1\|\nabla f\|_\infty \\ &\quad \text{(we apply } (***)) \\ &\leq \mu^{-1}\|\varphi\mathbf{1}_{B(0,2)}|b|\|_1 < \infty.\end{aligned}$$

(Indeed, $\varphi|b| \in L^1_{\rm loc}$ since on $B(0,1)$ $\varphi|b| = \kappa|x|^{-d+\beta}|x|^{-\alpha+1}$, where $\beta > \alpha$.)

$$\begin{aligned}J_2 &\leq C_1\|\varphi(\mu+P^\varepsilon)^{-1}\varphi^{-1}\|_{1\to1}\|\varphi\mathbf{1}_{B^c(0,2)}\mu(\mu+A)^{-1}|\nabla f|\|_1 \\ &\quad \text{(we apply } (***)) \\ &\leq C_1\mu^{-1}\|\varphi\mathbf{1}_{B^c(0,2)}\|_\infty\|\nabla f\|_1.\end{aligned}$$

Thus, $I_{1,\varepsilon}f \to 0$ (as $\mu \to \infty$ uniformly in $\varepsilon > 0$).

Second, we prove that $I_{2,\varepsilon} f := \mu(\mu + P^\varepsilon)^{-1} U_\varepsilon (\mu + A)^{-1} f \to 0$. The proof is analogous. We only note that $\|\mathbf{1}_{B(0,1)} \varphi U_\varepsilon\|_1 \leq C$, $C \neq C(\varepsilon)$. (In fact,

$$\begin{aligned}
\mathbf{1}_{B(0,1)} \varphi U_\varepsilon &= C \mathbf{1}_{B(0,1)} |x|^{-d+\beta} \varepsilon |x|_\varepsilon^{-\alpha-2} \\
&\leq C \mathbf{1}_{B(0,1)} |x|^{-d+\beta-\frac{\beta+\alpha}{2}} \varepsilon |x|_\varepsilon^{\frac{\beta-\alpha}{2}-2} \quad \text{(we use that } \beta > \alpha) \\
&= C|x|^{-d+\frac{\beta-\alpha}{2}} J_\varepsilon, \quad \text{ where } J_\varepsilon = \mathbf{1}_{B(0,1)} \varepsilon (|x|^2 + \varepsilon)^{\frac{\beta-\alpha}{4}-1} \to 0 \quad \text{ in } L^\infty.
\end{aligned}$$

Thus, $\langle \mathbf{1}_{B(0,1)} \varphi U_\varepsilon \rangle \to 0$ as $\varepsilon \downarrow 0$.)

The proof of (1) is completed.

Proof of (2). In view of $(**)$, it suffices to prove convergence on C_c^∞. The latter follows from (1), combined with $\varphi \geq \frac{1}{2}$ and $\|\mu(\mu + P^\varepsilon)^{-1} f - f\|_\infty \leq 2\|f\|_\infty$.

The proof of Claim 5 is completed. □

Claim 6. *Fix $\mu > 0$, $r \in]r_c, \infty[$. Then there exists a sequence $\varepsilon_k \downarrow 0$ such that the following is true:*

(1) $\|(\mu + P^{\varepsilon_n})^{-1} f - (\mu + P^{\varepsilon_k})^{-1} f\|_{1,\sqrt{\varphi}} \to 0$ *as $n, k \to \infty$ for every $f \in L^1_{\sqrt{\varphi}}$.*

(2) $\|(\mu + P^{\varepsilon_n})^{-1} f - (\mu + P^{\varepsilon_k})^{-1} f\|_r \to 0$ *as $n, k \to \infty$ for every $f \in L^r$.*

Proof of Claim 6. By $(**)$, $(***)$, it suffices to prove (1) and (2) for every f from a countable subset F of C_c^∞-functions such that F is dense in $L^1_{\sqrt{\varphi}}$ and L^r, respectively.

Without loss of generality, we prove (1), (2) for $0 \leq f \in F$. Set $0 \leq v_n := (\mu + P^{\varepsilon_n})^{-1} f$, for some $\varepsilon_n \downarrow 0$.

Proof of (2). First, note that by the upper bound $(\mu + P^\varepsilon)^{-1}(x, y) \leq C(|x - y|^{-d-\alpha} \wedge |x - y|^{-d+\alpha})\varphi(y)$, for all $R > 0$ sufficiently large (depending on the compact support of f), we have

$$v_n(x) \leq C \langle |x - y|^{-d-\alpha} \varphi(y) f(y) \rangle \quad \text{ for all } x \in B^c(0, R),$$

and so $v_n(x) \leq C_f |x|^{-d-\alpha}$ for every $x \in B^c(0, R)$. Thus,

$$\|\mathbf{1}_{B^c(0,R)} v_n\|_1 \downarrow 0 \quad \text{ as } R \uparrow \infty \text{ uniformly in } n.$$

Since $\|v_n\|_\infty \leq \mu^{-1} \|f\|_\infty$, we have

$$\|\mathbf{1}_{B^c(0,R)} v_n\|_r \downarrow 0 \quad \text{ as } R \uparrow \infty \text{ uniformly in } n,$$

and so, for a given $\gamma > 0$, there exists $R = R_\gamma$ such that

$$\|\mathbf{1}_{B^c(0,R)} (v_n - v_k)\|_r < \gamma \quad \text{ for all } n, k, \tag{$\bullet$}$$

i.e. we can control $\{v_n - v_k\}$ outside of the finite ball $B(0, R)$.

By $(*)$,

$$\|A^{\frac{1}{2}} v_n^{\frac{r}{2}}\|_2^2 \leq c\|f\|_r^r, \quad \text{ for all } n, \quad c \neq c(n).$$

Therefore, by the fractional Rellich-Kondrachov Theorem, there is a subsequence v_{n_l} and a function $0 \leq v$ such that $v^{r/2} \in L^2(B(0, R))$,

$$\begin{aligned}
\|\mathbf{1}_{B(0,R)} v_{n_l}^{r/2}\|_2 &\to \|\mathbf{1}_{B(0,R)} v^{r/2}\|_2, \\
v_{n_l} &\to v \quad \text{ a.e. on } B(0, R).
\end{aligned}$$

The latter yields that $v_{n_l} \to v$ strongly in $L^r(B(0,R))$. Thus,

$$\{\mathbf{1}_{B(0,R)} v_{n_l}\} \text{ is a Cauchy sequence in } L^r. \qquad (\bullet\bullet)$$

Now, we consider ($\bullet$) with $\gamma = \gamma_m \downarrow 0$. Combining it with ($\bullet\bullet$) we obtain a subsequence of $\{v_n\}$ (which we again denote by $\{v_{n_m}\}$) that constitutes a Cauchy sequence in L^r.

A priori, the choice of $\{v_{n_m} = (\mu + P^{\varepsilon_{n_m}})^{-1} f\}$ depends on $0 \leq f \in F$. Since F is countable, we can use the standard diagonal argument to pass to a subsequence $\{\varepsilon_{n_{m_k}}\}$ such that $\{(\mu + P^{\varepsilon_{n_{m_k}}})^{-1} f\}$ is a Cauchy sequence for every $0 \leq f \in F$. Thus, re-denoting $\{\varepsilon_{n_{m_k}}\}$ as $\{\varepsilon_k\}$, we arrive at (2).

Proof of (1). Let us show that $\{v_n\}$ is a Cauchy sequence in $L^1_{\sqrt{\varphi}}$. We have

$$\|v_n - v_k\|_{1,\sqrt{\varphi}} = \frac{1}{2}\|\mathbf{1}_{B^c(0,R)}(v_n - v_k)\|_1 + \|\mathbf{1}_{B(0,R)}\varphi(v_n - v_k)\|_1.$$

Since $\|\mathbf{1}_{B^c(0,R)} v_n\|_1 \downarrow 0$ as $R \uparrow \infty$ uniformly in n, see above, the first term can be made arbitrarily small by choosing R (> 2) sufficiently large. The second term converges to 0 since $\mathbf{1}_{B(0,R)}\varphi \in L^{1+\sigma}$ for a $\sigma > 0$ such that $r := \frac{1+\sigma}{\sigma} > r_c$, while $v_n - v_k \to 0$ strongly in $L^r(B(0,R))$.

The proof of Claim 6 is completed. □

By ($**$), ($***$), the semigroups e^{-tP^ε} are contraction semigroups in L^r and $L^1_{\sqrt{\varphi}}$. Thus, Claim 5 and Claim 6 verify the conditions of the Trotter Approximation Theorem, which thus yields Proposition 9(i), (ii).

Proof of (iii). Write $v = (\mu + \Lambda_r(b))^{-1} f$, $f \in L^r$. We need to show

$$\langle f, h\rangle = \mu\langle v, h\rangle + \langle v, (-\Delta)^{\frac{\alpha}{2}} h\rangle - \langle v, b\cdot\nabla h\rangle - \langle v, (\operatorname{div} b) h\rangle, \qquad h \in C_c^\infty.$$

Set $v_n := (\mu + P^{\varepsilon_n})^{-1} f$, then

$$\langle f, h\rangle = \mu\langle v_n, h\rangle + \langle v_n, (-\Delta)^{\frac{\alpha}{2}} h\rangle - \langle v_n, b_{\varepsilon_n}\cdot\nabla h\rangle - \langle v_n, (\operatorname{div} b_{\varepsilon_n}) h\rangle.$$

Since for $\sigma > 0$ sufficiently small, $b_{\varepsilon_n} \to b$, $\operatorname{div} b_{\varepsilon_n} \to \operatorname{div} b$ in $L^{1+\sigma}_{\rm loc}$, $(-\Delta)^{\frac{\alpha}{2}} h \in L^{1+\sigma}$, while $v_n \to v$ in L^r for $r := \frac{1+\sigma}{\sigma}$, we have

$$\langle v_n, b_{\varepsilon_n}\cdot\nabla h\rangle \to \langle v, b\cdot\nabla h\rangle, \qquad \langle v_n, (\operatorname{div} b_{\varepsilon_n}) h\rangle \to \langle v, (\operatorname{div} b) h\rangle,$$

and $\langle v_n, (-\Delta)^{\frac{\alpha}{2}} h\rangle \to \langle v, (-\Delta)^{\frac{\alpha}{2}} h\rangle$, which yields the required.

The proof of Proposition 9 is completed. □

9. Operator realization of $(-\Delta)^{\frac{\alpha}{2}} - \nabla\cdot b$ in $L^{r'}$, $r' \in]1, r_c'[$

Proposition 10. *Let $0 < \delta < 4$. There exists an operator realization $\Lambda_r^*(b)$ of $(-\Delta)^{\frac{\alpha}{2}} - \nabla\cdot b$ in $L^{r'}$, $r' \in]1, r_c'[$, $r_c = \frac{2}{2-\sqrt{\delta}}$, as (minus) generator of a contraction C_0 semigroup,*

$$e^{-t\Lambda_r^*(b)} = s\text{-}L^{r'}\text{-}\lim_n e^{-t(P^{\varepsilon_n})^*} \quad (\textit{loc. uniformly in } t \geq 0),$$

for a sequence $\{\varepsilon_n\} \downarrow 0$.

We have

$$\langle e^{-t\Lambda_r(b)} f, g\rangle = \langle f, e^{-t\Lambda_r^*(b)} g\rangle, \quad t > 0, \quad f \in L^r, \quad g \in L^{r'},$$

where $\Lambda_r(b)$ has been constructed in Proposition 9.

Proof of Proposition 10. Let $0 \leq f \in C_c^\infty$, set $0 \leq v := (\mu + (P^\varepsilon)^*)^{-1} f$. We multiply the equation $(\mu + (-\Delta)^{\frac{\alpha}{2}} - \nabla \cdot b_\varepsilon + U_\varepsilon)v = f$ by $v^{r'-1}$, integrate, and then argue as in the proof of (S_1), to obtain

$$\mu \|v\|_{r'}^{r'} + \frac{4}{rr'}\|A^{\frac{1}{2}} v^{\frac{r'}{2}}\|_2^2 + \langle(-\nabla \cdot b_\varepsilon + U_\varepsilon)v, v^{r'-1}\rangle \leq \|f\|_{r'}\|v\|_{r'}^{r'-1}.$$

We write $-\nabla \cdot b_\varepsilon v = -(\operatorname{div} b_\varepsilon)v - b_\varepsilon \cdot \nabla v$ and use that $-\operatorname{div} b_\varepsilon = -\kappa(d-\alpha)|x|_\varepsilon^{-\alpha} - U_\varepsilon$ to obtain

$$\mu \|v\|_{r'}^{r'} + \frac{4}{rr'}\|A^{\frac{1}{2}} v^{\frac{r'}{2}}\|_2^2 - \kappa(d-\alpha)\langle |x|_\varepsilon^{-\alpha} v^{\frac{r'}{2}}, v^{\frac{r'}{2}}\rangle - \langle b_\varepsilon \cdot \nabla v, v^{r'-1}\rangle \leq \|f\|_{r'}\|v\|_{r'}^{r'-1}.$$

By integration by parts,

$$\begin{aligned} -\langle b_\varepsilon \cdot \nabla v, v^{r'-1}\rangle &\equiv -\frac{2}{r'}\kappa \langle |x|_\varepsilon^{-\alpha} x \cdot \nabla v^{\frac{r'}{2}}, v^{\frac{r'}{2}}\rangle \\ &= \frac{2}{r'}\bigg(\kappa \frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha} v^{\frac{r'}{2}}, v^{\frac{r'}{2}}\rangle + \frac{1}{2}\langle U_\varepsilon v^{\frac{r'}{2}}, v^{\frac{r'}{2}}\rangle\bigg). \end{aligned}$$

Thus,

$$\mu \|v\|_{r'}^{r'} + \frac{4}{rr'}\|A^{\frac{1}{2}} v^{\frac{r'}{2}}\|_2^2 - \kappa(d-\alpha)\frac{1}{r}\langle |x|_\varepsilon^{-\alpha} v^{\frac{r'}{2}}, v^{\frac{r'}{2}}\rangle \leq \|f\|_{r'}\|v\|_{r'}^{r'-1}.$$

so by the Hardy-Rellich inequality,

$$\mu \|v\|_{r'}^{r'} + \frac{2}{r}\left(\frac{2}{r'} - \sqrt{\delta}\right)\|A^{\frac{1}{2}} v^{\frac{r'}{2}}\|_2^2 \leq \|f\|_{r'}\|v\|_{r'}^{r'-1},$$

where $\frac{2}{r'} - \sqrt{\delta} > 0$. Thus, we have proved

$$\mu^{r'-1}\frac{2}{r}\left(\frac{2}{r'} - \sqrt{\delta}\right)\|A^{\frac{1}{2}} v^{\frac{r'}{2}}\|_2^2 \leq \|f\|_{r'}, \qquad r' \in]1, r'_c[, \tag{$*$}$$

$$\|\mu(\mu + (P^\varepsilon)^*)^{-1}\|_{r' \to r'} \leq 1, \qquad r' \in]1, r'_c[, \tag{$**$}$$

$$\|\mu(\mu + (P^\varepsilon)^*)^{-1}\|_{1 \to 1} \leq 1. \tag{$***$}$$

($(***)$ evidently follows from $(**)$ by taking $r' \downarrow 1$.)

Claim 7. $\mu(\mu + (P^\varepsilon)^*)^{-1} \to 1$ strongly in $L^{r'}$ as $\mu \to \infty$ uniformly in ε, for every $r' \in [1, r'_c[$.

Proof of Claim 7. 1. First, we consider the case $r' = 1$. By $(***)$, it suffices to prove the convergence only on $f \in C_c^\infty$. We write

$$\begin{aligned} \mu(\mu + (P^\varepsilon)^*)^{-1} f &= \mu(\mu + A)^{-1} f - \mu(\mu + (P^\varepsilon)^*)^{-1}(-\nabla \cdot b_\varepsilon + U_\varepsilon)(\mu + A)^{-1} f \\ &= \mu(\mu + A)^{-1} f + \mu(\mu + (P^\varepsilon)^*)^{-1} b_\varepsilon \cdot \nabla(\mu + A)^{-1} f \\ &\quad + \mu(\mu + (P^\varepsilon)^*)^{-1}(\operatorname{div} b_\varepsilon)(\mu + A)^{-1} f \\ &\quad + \mu(\mu + (P^\varepsilon)^*)^{-1}(-U_\varepsilon)(\mu + A)^{-1} f \\ &=: \mu(\mu + A)^{-1} f + K_1 + K_2 + K_3. \end{aligned}$$

Since $\mu(\mu + A)^{-1} f \to f$ in L^1 as $\mu \to \infty$, we have to show that $K_i \downarrow 0$, $i = 1, 2, 3$, uniformly in $\varepsilon > 0$. Indeed, we have

$$\begin{aligned} \|K_1\|_1 &\leq (\mu + (P^\varepsilon)^*)^{-1}\mathbf{1}_{B(0,2)}|b|\mu(\mu + A)^{-1}|\nabla f| \\ &\quad + \mu(\mu + (P^\varepsilon)^*)^{-1} C_1(\mu + A)^{-1}|\nabla f|, \qquad C_1 := \|\mathbf{1}_{B^c(0,2)}|b|\|_\infty \\ &\leq \|\mathbf{1}_{B(0,2)}|b|\|_1 \mu^{-1}\|\nabla f\|_\infty + C_1 \mu^{-1}\|\nabla f\|_1. \end{aligned}$$

The terms $\|K_2\|_1$ and $\|K_3\|_1$ are estimated similarly (we only note that $\|\mathbf{1}_{B(0,2)}U_\varepsilon\|_1 \leq C$, $C \neq C(\varepsilon)$, cf. the proof of Claim 5). Thus, $K_i \downarrow 0$, $i = 1, 2, 3$, uniformly in $\varepsilon > 0$, as needed.

In the case $r' \in]1, r'_c[$ the required convergence follows from the case $r' = 1$, $(**)$ for $r'_1 \in]1, r'_c[$, and the interpolation inequality.

The proof of Claim 7 is completed. □

Claim 8. *Fix* $\mu > 0$, $r' \in]1, r'_c[$. *Then there exists a sequence* $\varepsilon_m \downarrow 0$ *such that*

$$\|(\mu + (P^{\varepsilon_m})^*)^{-1} f - (\mu + (P^{\varepsilon_k})^*)^{-1} f\|_{r'} \to 0 \qquad \text{as } m, k \to \infty$$

for every $f \in L^{r'}$.

Proof of Claim 8. We argue as in the proof of Claim 6.

We fix a countable subset F of C_c^∞ such that $F \cap \{f \geq 0\}$ is dense in $L^{r'} \cap \{f \geq 0\}$. It suffices to prove the convergence on $0 \leq f \in F$. Set $0 \leq v_n := (\mu + (P^{\varepsilon_n})^*)^{-1} f$, for some $\varepsilon_n \downarrow 0$.

Using the upper bound $(\mu + (P^\varepsilon)^*)^{-1}(x, y) \leq C\varphi(x)(|x - y|^{-d-\alpha} \wedge |x - y|^{-d+\alpha})$, we obtain that for a given $\gamma > 0$, there exists $R = R_\gamma$ such that

$$\|\mathbf{1}_{B^c(0,R)}(v_n - v_k)\|_{r'} < \gamma \qquad \text{for all } n, k. \tag{$\bullet$}$$

By $(*)$, since $r' > 1$, we appeal to the fractional Rellich-Kondrachov Theorem as in the proof of Claim 6 obtaining a subsequence $\{v_{n_l}\}$ such that

$$\{\mathbf{1}_{B(0,R)} v_{n_l}\} \text{ is a Cauchy sequence in } L^{r'}. \tag{$\bullet\bullet$}$$

Combining $(\bullet)$ and $(\bullet\bullet)$ as in the proof of Claim 6, we obtain a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that $\{(\mu + (P^{\varepsilon_{n_k}})^*)^{-1} f\}$ constitutes a Cauchy sequence in $L^{r'}$ for every $0 \leq f \in F$. Re-denoting ε_{n_k} by $\{\varepsilon_k\}$, we complete the proof of Claim 8. □

By $(**)$, $e^{-t(P^\varepsilon)^*}$ is a contraction semigroup in $L^{r'}$. Claims 7 and 8 now verify the conditions of the Trotter Approximation Theorem, which thus yields the first assertion of Proposition 10. The second assertion follows from the first one and Proposition 9.

The proof of Proposition 10 is completed. □

Appendix A. Extrapolation Theorem

In the text we use the following dual variant of Theorem 1 with $\psi \equiv 1$.

Theorem 4. *Let* $U^{t,s} : L^1 \cap L^\infty \to L^1 + L^\infty$ *be an evolution family of operators. Suppose that, for some* $1 < p < q < r \leq \infty$, $\nu > 0$, M_1 *and* M_2, *the inequalities*

$$\|U^{t,s} f\|_r \leq M_1 \|f\|_r \qquad \text{and} \qquad \|U^{t,s} f\|_q \leq M_2 (t - s)^{-\nu} \|f\|_p$$

are valid for all (t, s) *and* $f \in L^1 \cap L^\infty$. *Then*

$$\|U^{t,s} f\|_r \leq M (t - s)^{-\nu/(1-\beta)} \|f\|_p,$$

where $\beta = \frac{r}{q} \frac{q-p}{r-p}$ *and* $M = 2^{\nu/(1-\beta)^2} M_1 M_2^{1/(1-\beta)}$.

Appendix B.

Set $I_\alpha = (-\Delta)^{-\frac{\alpha}{2}}$, the Riesz potential defined by the formula

$$I_\alpha f(x) := \frac{1}{\gamma(\alpha)}\langle |x-\cdot|^{-d+\alpha} f(\cdot)\rangle, \quad \gamma(\alpha) := \frac{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2}-\frac{\alpha}{2})}.$$

The identity

$$\frac{\gamma(\beta-\alpha)}{\gamma(\beta)}|x|^{-d+\beta} = I_\alpha |x|^{-d+\beta-\alpha}, \quad 0<\alpha<\beta<d,$$

follows e.g. from $I_\beta = I_\alpha I_{\beta-\alpha}$.

It follows that $\tilde{\varphi}_1(x) = |x|^{-d+\beta}$ is a Lyapunov's function to the formal operator $(-\Delta)^{\frac{\alpha}{2}} - V$:

$$(-\Delta)^{\frac{\alpha}{2}}|x|^{-d+\beta} = V(x)|x|^{-d+\beta}, \quad V(x) = \frac{\gamma(\beta)}{\gamma(\beta-\alpha)}|x|^{-\alpha}.$$

Appendix C.

Let P be a closed operator on L^1 such that $\mathrm{Re}\langle(\lambda+P)f, \frac{f}{|f|}\rangle \geq 0$ for all $f \in D(P)$, and $R(\mu+P)$ is dense in L^1 for a $\mu > \lambda$.

Then $R(\mu+P) = L^1$.

Indeed, let $y_n \in R(\mu+P)$, $n=1,2,\dots$, be a Cauchy sequence in L^1; $y_n = (\mu+P)x_n$, $x_n \in D(P)$. Write $[f,g] := \langle f, \frac{g}{|g|}\rangle$. Then

$$\begin{aligned}(\mu-\lambda)\|x_n - x_m\|_1 &= (\mu-\lambda)[x_n-x_m, x_n-x_m] \\ &\leq (\mu-\lambda)[x_n-x_m,x_n-x_m] + [(\lambda+P)(x_n-x_m), x_n-x_m] \\ &= [(\mu+P)(x_n-x_m), x_n-x_m] \leq \|y_n-y_m\|_1.\end{aligned}$$

Thus, $\{x_n\}$ is itself a Cauchy sequence in L^1. Since P is closed, the result follows.

Appendix D.

Let us show that

$$\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \lesssim t^{1-\frac{1}{\alpha}} e^{-tA}(x,y) \quad \text{for all } x,y\in\mathbb{R}^d, \quad t>0.$$

Indeed,

$$\begin{aligned}&e^{-(t-\tau)A}(x,z)E^\tau(z,y) \approx e^{-(t-\tau)A}(x,z)e^{-\tau A}(z,y)(|z-y|^{-1}\wedge \tau^{-\frac{1}{\alpha}}) \\ &\text{(we are applying } (*)) \\ &\lesssim e^{-tA}(x,y)\big(e^{-(t-\tau)A}(x,z) + e^{-\tau A}(z,y)\big)\big(|z-y|^{-1}\wedge\tau^{-\frac{1}{\alpha}}\big).\end{aligned}$$

Therefore, using $e^{-tA}(x,z) \lesssim (t|x-z|^{-d-\alpha}) \wedge t^{-\frac{d}{\alpha}} \lesssim |x-z|^{-d}\wedge t^{-\frac{d}{\alpha}}$, we obtain

$$\begin{aligned}e^{-(t-\tau)A}(x,z)E^\tau(z,y) &\lesssim e^{-tA}(x,y)\big[(|x-z|^{-d}\wedge(t-\tau)^{-\frac{d}{\alpha}}) + (|z-y|^{-d}\wedge\tau^{-\frac{d}{\alpha}})\big](|z-y|^{-1}\wedge\tau^{-\frac{1}{\alpha}}) \\ &=: e^{-tA}(x,y)\, I,\end{aligned}$$

where, it is easily seen using Young's inequality,

$$I \lesssim |x-z|^{-d-1}\wedge(t-\tau)^{-\frac{d+1}{\alpha}} + |z-y|^{-d-1}\wedge\tau^{-\frac{d+1}{\alpha}}, \quad \text{and so } \int_0^t \langle I\rangle_z d\tau \lesssim t^{1-\frac{1}{\alpha}}.$$

Appendix E. Nash initial estimate for $\alpha = 2$

For $\alpha = 2$, the operator $(-\Delta)^{\frac{\alpha}{2}} + \kappa|x|^{-\alpha}x\cdot\nabla$ becomes $-\Delta + \delta\frac{d-2}{2}|x|^{-2}x\cdot\nabla$.

Let $d \geq 3$, $0 < \delta < 4$. For every $r \in]r_c, \infty[$, $r_c := \frac{2}{2-\sqrt{\delta}}$ the limit

$$e^{-t\Lambda} = s\text{-}L^r\text{-}\lim_{\varepsilon\downarrow 0} e^{-t\Lambda^\varepsilon} \quad (\text{loc. uniformly in } t \geq 0) \tag{$*$}$$

exists and determines a C_0 semigroup; here $\Lambda^\varepsilon := -\Delta + b_\varepsilon\cdot\nabla$, $b_\varepsilon(x) = \delta\frac{d-2}{2}|x|_\varepsilon^{-2}x$, $D(\Lambda^\varepsilon) = (1-\Delta)^{-1}L^r$; see e.g. [KS1, Theorem 4.1].

Then $e^{-t\Lambda}$, $t > 0$, is an integral operator, and

$$e^{-t\Lambda}(x,y) \leq Ct^{-\frac{d}{\alpha}}\varphi_t(y), \quad x, y \in \mathbb{R}^d, y \neq 0, \quad t > 0,$$

where $\varphi_t(y) = \eta(t^{-\frac{1}{\alpha}}|y|)$, and η is a $C^2(]0,\infty[)$ function such that

$$\eta(r) = \begin{cases} r^{-\sqrt{\delta}\frac{d-2}{2}}, & 0 < r < 1, \\ \frac{1}{2}, & r \geq 2. \end{cases}$$

Proof. This is a consequence of Theorem A applied to $(e^{-t\Lambda^\varepsilon}, \varphi_s)$, $\Lambda^\varepsilon := -(-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon\cdot\nabla$, $\varepsilon > 0$. To verify (S_1), (S_2), (S_3), (S_4), we argue as in the proof of Theorem 2. We note that in the local case (S_3) can also be proved by direct calculations:

Proof of (S_3). It suffices to prove a priori estimate

$$\|\varphi_s^\varepsilon e^{-t\Lambda^\varepsilon}(\varphi_s^\varepsilon)^{-1}g\|_{1\to 1} \leq e^{c\frac{t}{s}}\|g\|_1, \quad c \neq c(\varepsilon), \quad g \in C_c, \quad 0 < t \leq s, \tag{$**$}$$

where $\varphi_s^\varepsilon(y) = \eta(s^{-\frac{1}{2}}|y|_\varepsilon)$. Then, taking $(**)$ for granted, we obtain $\|\varphi_s e^{-t\Lambda}\varphi_s^{-1}g\|_{1\to 1} \leq e^{c\frac{t}{s}}\|g\|_1$ by $(*)$ and Fatou's Lemma, and so (S_3) follows upon taking $s = t$.

Proof of $(**)$.

1) Set

$$H(\varphi^\varepsilon) = -\Delta + \nabla\cdot(b_\varepsilon + 2\frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon}) + W_\varepsilon, \quad D(H(\varphi^\varepsilon)) = W^{2,2},$$

$$W_\varepsilon = -\frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon}\cdot\big(b_\varepsilon + \frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon}\big) - \mathrm{div}\big(b_\varepsilon + \frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon}\big).$$

In fact, we have $H(\varphi^\varepsilon) = \varphi_s^\varepsilon\Lambda^\varepsilon(\varphi_s^\varepsilon)^{-1}$.

2) $\varphi_s^\varepsilon e^{-t\Lambda^\varepsilon}(\varphi_s^\varepsilon)^{-1} = e^{-tH(\varphi^\varepsilon)}$.

Indeed, put $F^t = \varphi^\varepsilon e^{-t\Lambda^\varepsilon}(\varphi^\varepsilon)^{-1}$. 2) is valid because $s\text{-}L^2\text{-}\lim_{t\downarrow 0} t^{-1}(1-F^t)f$ exists for all $f \in W^{2,2}$ and coincides with $H(\varphi^\varepsilon)f$.

3) We have $b_\varepsilon + \frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon} = 0$ in $B(0,\sqrt{s})$. Moreover, $|b_\varepsilon + \frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon}| \leq \frac{c_1}{\sqrt{s}}$, $|\mathrm{div}\big(b_\varepsilon + \frac{\nabla\varphi^\varepsilon}{\varphi^\varepsilon}| \leq \frac{c_2}{s}$, the potential $|W_\varepsilon| \leq \frac{c}{s}$, $c \neq c(\varepsilon)$. It follows that $e^{-tH(\varphi^\varepsilon)}$ is a quasi contraction on L^1: $\|e^{-tH(\varphi^\varepsilon)}\|_{1\to 1} \leq e^{\frac{c}{s}t}$, $0 < t \leq s$. In view of 2), $(**)$ follows. □

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