

STOCHASTIC TRANSPORT EQUATION WITH SINGULAR DRIFT

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ABSTRACT. We prove existence, uniqueness and Sobolev regularity of weak solution of the Cauchy problem of the stochastic transport equation with drift in a large class of singular vector fields containing, in particular, the L^d class, the weak L^d class, as well as some vector fields that are not even in $L_{\text{loc}}^{2+\varepsilon}$ for any $\varepsilon > 0$.

1. INTRODUCTION

Throughout this paper we assume $d \geq 3$. Let B_t be a Brownian motion in \mathbb{R}^d defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a complete and right-continuous filtration \mathcal{F}_t . Let \circ denote the Stratonovich multiplication. Set $L^p \equiv L^p(\mathbb{R}^d) \equiv L^p(\mathbb{R}^d, dx)$, $L_{\text{loc}}^p \equiv L_{\text{loc}}^p(\mathbb{R}^d)$, $W^{1,p} \equiv W^{1,p}(\mathbb{R}^d)$, $W_{\text{loc}}^{1,p} \equiv W_{\text{loc}}^{1,p}(\mathbb{R}^d)$, $C_c^\infty \equiv C_c^\infty(\mathbb{R}^d)$. We denote by $\|\cdot\|_{p \rightarrow q}$ the operator norm $\|\cdot\|_{L^p \rightarrow L^q}$.

The subject of this paper is the problem of existence, uniqueness and Sobolev regularity of weak solution to the Cauchy problem for the stochastic transport equation (STE)

$$\begin{aligned} du + b \cdot \nabla u dt + \sigma \nabla u \circ dB_t &= 0 \quad \text{on } (0, \infty) \times \mathbb{R}^d, \\ u|_{t=0} &= f, \end{aligned} \tag{1}$$

where $u(t, x)$ is a scalar random field, $\sigma \neq 0$, f is in L^p or $W^{1,p}$, and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is in the class of *form-bounded* vector fields (see definition below), a large class of singular vector fields containing, in particular, vector fields b with $|b| \in L^d$, or with $|b|$ in the weak L^d class, as well as some vector fields b with $|b| \notin L_{\text{loc}}^{2+\varepsilon}$ for any $\varepsilon > 0$.

It is well known that the Cauchy problem for the deterministic transport equation $\partial_t u + b \cdot \nabla u = 0$ (corresponding to $\sigma = 0$ in (1)) is in general not well posed already for a bounded but discontinuous b . Moreover, in that case, even if the initial function f is regular, one can not hope that the corresponding solution u will be regular immediately after $t = 0$. This, however, changes if one adds the noise term $\sigma \nabla u \circ dB_t$, $\sigma > 0$. For the stochastic STE (1), a unique weak solution exists and is regular for some discontinuous b . This effect of regularization and well-posedness by noise, demonstrated by the STE, attracted considerable interest in the past few years, as a part of the more general program of establishing well-posedness by noise for SPDEs whose deterministic counterparts arising in fluid dynamics are not well-posed, see [BFGM, GM] for detailed discussions and further references.

In [BFGM], the authors establish existence, uniqueness and Sobolev $W^{1,p}$ -regularity (up to the initial time $t = 0$, with p large) for weak solutions of (1) with time-dependent drift b satisfying

$$|b(\cdot, \cdot)| \in L^q([0, \infty), L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} \leq 1$$

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(actually, [BFGM] allows $b = b_1 + b_2$ with b_1 satisfying the condition above and b_2 being continuously differentiable with at most linear growth at infinity; their uniqueness result imposes additional assumptions on $\operatorname{div} b$). They apply this result to study the SDE

$$X_t = x - \int_s^t b(r, X_r) dr + \sigma(B_t - B_s), \quad (2)$$

constructing, in particular, a unique, $W^{1,p}$ -regular stochastic Lagrangian flow that solves (2) for a.e. $x \in \mathbb{R}^d$. The STE can be viewed as the equation behind both the SDE (via path-wise interpretation of the STE and the SDE, see [BFGM]) and the parabolic equation $(\partial_t - \frac{\sigma^2}{2}\Delta + b \cdot \nabla)v = 0$ (arising from (1) upon taking expectation, i.e. $v = \mathbb{E}[u]$, see, if needed, (8) below).

In this paper, we show that the regularity and well-posedness for (1) hold for a much larger class of drifts b , at least in the time-independent case $b = b(x)$ (see, however, Remark 2 below concerning time-dependent b).

DEFINITION 1. A Borel vector field $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to be form-bounded with relative bound $\delta > 0$, written as $b \in \mathbf{F}_\delta$, if $|b| \in L^2_{\text{loc}}$ and there exists a constant $\lambda = \lambda_\delta \geq 0$ such that

$$\| |b|(\lambda - \Delta)^{-\frac{1}{2}} \|_{2 \rightarrow 2} \leq \sqrt{\delta}.$$

It is easily seen that the condition $b \in \mathbf{F}_\delta$ can be stated equivalently as a quadratic form inequality

$$\|b\varphi\|_2^2 \leq \delta \|\nabla\varphi\|_2^2 + c_\delta \|\varphi\|_2^2, \quad \varphi \in W^{1,2},$$

for a constant $c_\delta (= \lambda\delta)$. Let us also note that

$$b_1 \in \mathbf{F}_{\delta_1}, b_2 \in \mathbf{F}_{\delta_2} \quad \Rightarrow \quad b_1 + b_2 \in \mathbf{F}_\delta, \quad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.$$

Examples. 1. Any vector field

$$b \in L^d(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)$$

is in \mathbf{F}_δ for $\delta > 0$ that can be chosen arbitrarily small. Indeed, for any $\varepsilon > 0$ we can write $b = f + h$ with $\|f\|_d < \varepsilon$, $h \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$. It follows from Hölder's inequality and the Sobolev embedding theorem that for any $g \in L^2$,

$$\begin{aligned} \| |b|(\lambda - \Delta)^{-\frac{1}{2}} g \|_2 &\leq \|f\|_d \|(\lambda - \Delta)^{-\frac{1}{2}} g\|_{\frac{2d}{d-2}} + \|h\|_\infty \lambda^{-\frac{1}{2}} \|g\|_2 \\ &\leq c \|f\|_d \|g\|_2 + \|h\|_\infty \lambda^{-\frac{1}{2}} \|g\|_2 \leq (c+1)\varepsilon \|g\|_2 \quad \text{for } \lambda = \varepsilon^{-2} \|h\|_\infty^{-2}. \end{aligned}$$

2. The class \mathbf{F}_δ also contains vector fields having critical-order singularities, such as

$$b(x) = \pm \sqrt{\delta} \frac{d-2}{2} |x|^{-2} x$$

(by Hardy's inequality $\frac{(d-2)^2}{4} \| |x|^{-1} \varphi \|_2^2 \leq \|\nabla\varphi\|_2^2$, $\varphi \in W^{1,2}$).

3. More generally, the class \mathbf{F}_δ contains vector fields b with $|b|$ in $L^{d,w}$ (the weak L^d space). Recall that a Borel function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is in $L^{d,w}$ if

$$\|h\|_{d,w} := \sup_{s>0} s |\{x \in \mathbb{R}^d : |h(x)| > s\}|^{1/d} < \infty.$$

By the Strichartz inequality with sharp constant [KPS, Prop. 2.5, 2.6, Cor. 2.9], if $|b|$ in $L^{d,w}$, then $b \in \mathbf{F}_{\delta_1}$ with

$$\begin{aligned} \sqrt{\delta_1} &= \| |b|(\lambda - \Delta)^{-\frac{1}{2}} \|_{2 \rightarrow 2} \\ &\leq \| b \|_{d,w} \Omega_d^{-\frac{1}{d}} \| |x|^{-1}(\lambda - \Delta)^{-\frac{1}{2}} \|_{2 \rightarrow 2} \\ &\leq \| b \|_{d,w} \Omega_d^{-\frac{1}{d}} \frac{2}{d-2}, \end{aligned}$$

where $\Omega_d = \pi^{\frac{d}{2}} \Gamma(\frac{d}{2} + 1)$ is the volume of the unit ball in \mathbb{R}^d .

We also note that if $h \in L^2(\mathbb{R})$, $T : \mathbb{R}^d \rightarrow \mathbb{R}$ is a linear map, then the vector field $b(x) = h(Tx)e$, where $e \in \mathbb{R}^d$, is in \mathbf{F}_δ with appropriate δ , but $|b|$ may not be in $L_{\text{loc}}^{d,w}$.

4. More generally, the class \mathbf{F}_δ contains vector fields in the Campanato-Morrey class and the Chang-Wilson-Wolff class, with δ depending on the respective norms of the vector field in these classes, see [CWW].

5. We note that there exists $b \in \mathbf{F}_\delta$ such that $|b| \notin L_{\text{loc}}^{2+\varepsilon}(\mathbb{R}^d, \mathbb{R}^d)$ for any $\varepsilon > 0$, e.g., consider

$$|b(x)|^2 = C \frac{\mathbf{1}_{B(0,1+\alpha)} - \mathbf{1}_{B(0,1-\alpha)}}{||x| - 1|^{-1} (-\ln ||x| - 1|)^\beta}, \quad \beta > 1, \quad 0 < \alpha < 1.$$

We emphasize that the condition $b \in \mathbf{F}_\delta$ is not a refinement of $|b| \in L^d + L^\infty$ in the sense that \mathbf{F}_δ is not situated between $L^d + L^\infty$ and $L^p + L^\infty$, $p < d$. In contrast to the elementary sub-classes of \mathbf{F}_δ listed above, the class \mathbf{F}_δ is defined in terms of the operators that, essentially, constitute the equation in (1).

The key result of this paper is the Sobolev regularity of solutions u to the Cauchy problem for the STE (1):

$$\sup_{t \in [0, T]} \| \mathbb{E} |\nabla u|^{2q} \|_2 \leq C \| \nabla f \|_{4q}^{2q}, \quad q = 1, 2, \dots, \quad (3)$$

provided that b is in \mathbf{F}_δ with δ smaller than a certain explicit constant, see Theorem 2. This is a stochastic (parabolic) counterpart of the Sobolev regularity estimates for solutions of the corresponding deterministic elliptic equation established in [KS]. More precisely, in [KS] the authors consider the operator $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_\delta$ with $0 < \delta < 1 \wedge (\frac{2}{d-2})^2$, $d \geq 3$ and establish the following Sobolev regularity of solutions v to the elliptic equation $(\mu - \Delta + b \cdot \nabla)v = f$ in L^q for $2 \vee (d-2) \leq q < \frac{2}{\delta}$:

$$\| \nabla v \|_{\frac{qd}{d-2}} \leq K \| f \|_q, \quad (4)$$

with K depending only on d , q , the relative bound δ and c_δ . The estimate (4) is needed in [KS] to run a Moser-type iteration procedure that yields the Feller semigroup corresponding to $-\Delta + b \cdot \nabla$. It was established in [KiS2] that, given $b \in \mathbf{F}_\delta$ with $\delta < 1 \wedge (\frac{2}{d-2})^2$, this Feller semigroup determines, for every starting point $x \in \mathbb{R}^d$, a weak solution to the SDE

$$X_t = x - \int_0^t b(X_r) dr + \sqrt{2} B_t \quad (5)$$

(see also [KiS] where the authors consider drifts in a larger class).

The approach to studying SDEs via regularity theory of the STE, developed in [BFGM], can be combined with Theorem 2 to obtain strong existence and uniqueness for (2) with $b \in \mathbf{F}_\delta$ (cf. Remark 1 below), albeit potentially excluding a measure zero set of starting points $x \in \mathbb{R}^d$. For results on

strong existence and uniqueness for any $x \in \mathbb{R}^d$, with b satisfying (in the time-independent case) $|b| \in L^p + L^\infty$ with $p > d$ or $p = d$, see [Kr1, Kr2, KrR].

We conclude this introduction with a few remarks concerning the criticality of the singularities of form-bounded drifts.

1. In [BFGM, Sect. 7], the authors show that the SDE (5) with drift $b(x) = \beta|x|^{-2}x$ and starting point $x = 0$ does not have a weak solution if $\beta > d - 2$. In view of Example 2 above, this drift b belongs to \mathbf{F}_δ with $\sqrt{\delta} = \beta \frac{2}{d-2}$, so by the result of [KiS2] cited above, the weak solution to (5) with $x = 0$ exists as long as $\beta > 0$ satisfies $\beta < \frac{1}{2}$ if $d = 3$, $\beta < 1$ if $d \geq 4$ (in fact, for $d \geq 5$ it suffices to require $\beta < \frac{d-3}{2}$ using [KiS3, Corollary 4.10]). Thus, the weak well-posedness of (5) is sensitive to changes in the value of the constant multiple β of b (equivalently, changes in the value of the relative bound δ). In this sense, the singularities of $b \in \mathbf{F}_\delta$ are critical.

Let us note that the diffusion process with drift $b(x) = c|x|^{-2}x$, $c \in \mathbb{R}$, was studied earlier in [W].

2. Let $b \in \mathbf{F}_\delta$. There is a quantitative dependence between the value of the relative bound δ and the regularity properties of solutions to the corresponding equations (PDEs or STEs). Indeed, the admissible values of q in (4), as well as in (3), depend on the value of δ . This dependence is lost if one considers b with $|b| \in L^d + L^\infty$ since any such b has arbitrarily small relative bound, cf. Example 1.

3. Concerning the difference between classes \mathbf{F}_δ and its subclass $L^d + L^\infty$, let us also note the following: if v is a weak solution of the elliptic equation $(\lambda - \Delta + b \cdot \nabla)v = f$, $\lambda > 0$, $f \in C_c^\infty$ with $|b| \in L^d + L^\infty$ and $v \in W^{1,r}$ for r large (e.g. by (4)), then, by Hölder's inequality,

$$\Delta v \in L_{\text{loc}}^{\frac{rd}{d+r}}.$$

However, for $b \in \mathbf{F}_\delta$, one can only say that (cf. Example 5 above)

$$\Delta v \in L_{\text{loc}}^{\frac{2d}{d+2}}$$

(one can in fact show that $v \in W^{2,2}$). That is, in case $b \in \mathbf{F}_\delta$, there are no $W^{2,p}$ estimates on solution v for p large.

See [KiS3] for detailed discussions of remarks 2 and 3 above.

Notations. Denote

$$\langle f, g \rangle = \langle fg \rangle := \int_{\mathbb{R}^d} fg dx$$

(all functions considered below are assumed to be real-valued).

Set

$$\rho(x) \equiv \rho_{\kappa,\theta}(x) := (1 + \kappa|x|^2)^{-\theta}, \quad \kappa > 0, \quad \theta > \frac{d}{2}, \quad x \in \mathbb{R}^d.$$

It is easily seen that

$$|\nabla \rho(x)| \leq \theta \sqrt{\kappa} \rho(x), \quad x \in \mathbb{R}^d. \quad (6)$$

Below we will be applying (6) to ρ with κ chosen sufficiently small.

For any $p > 1$, we use p' to denote its conjugate $p/(p-1)$. Let $L_\rho^p \equiv L^p(\mathbb{R}^d, \rho dx)$. Denote by $\|\cdot\|_{p,\rho}$ the norm in L_ρ^p , and by $\langle \cdot, \cdot \rangle_\rho$ the inner product in L_ρ^2 .

Set $W_\rho^{1,2} := \{g \in W_{\text{loc}}^{1,2} \mid \|g\|_{W_\rho^{1,2}} := \|g\|_{2,\rho} + \|\nabla g\|_{2,\rho} < \infty\}$.

Define constants

$$\beta_{2q} := 1 + 4qd, \quad q = 1, 2, \dots$$

Put $J_T := [0, T]$.

2. MAIN RESULTS

Below we consider the Cauchy problem for the STE

$$\begin{aligned} du + \mu u dt + b \cdot \nabla u dt + \sigma \nabla u \circ dB_t &= 0 \quad \text{on } (0, \infty) \times \mathbb{R}^d, \\ u|_{t=0} &= f \in L^p, \quad p \geq 2, \end{aligned} \tag{CP}$$

where $\mu \geq 0$. Since solutions of the Cauchy problems (1) and (CP) will differ by a multiple $e^{-\mu t}$, it suffices to prove the well-posedness of (CP).

Let us first make a few preliminary remarks.

1. We can rewrite the equation in (CP), using the identity relating Stratonovich and Itô integrals

$$\int_0^t \nabla u \circ dB_s = \int_0^t \nabla u dB_s - \frac{1}{2} \sum_{k=1}^d [\partial_{x_k} u, B^k]_t, \quad B_t = (B_t^k)_{k=1}^d, \tag{7}$$

as

$$du + \mu u dt + b \cdot \nabla u dt + \sigma \nabla u dB_t - \frac{\sigma^2}{2} \Delta u = 0. \tag{8}$$

2. If $b \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $f \in C_c^\infty$, then (see [Ku, Theorem 6.1.9]) there exists a unique adapted strong solution of (CP)

$$u(t) - f + \mu \int_0^t u ds + \int_0^t b \cdot \nabla u ds + \sigma \int_0^t \nabla u \circ dB_s = 0 \text{ a.s.}, \quad t \in J_T,$$

given by

$$e^{-\mu t} u(t) = f(\Psi_t^{-1}), \quad t \geq 0, \tag{9}$$

where $\Psi_t : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ is the stochastic flow for the SDE

$$X_t = x - \int_0^t b(X_r) dr + \sigma B_t, \tag{10}$$

i.e. there exists $\Omega_0 \subset \Omega$, $\mathbb{P}(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, $\Psi_t(\cdot, \omega) \Psi_s(\cdot, \omega) = \Psi_{t+s}(\cdot, \omega)$, $\Psi_0(x, \omega) = x$, and

- 1) for every $x \in \mathbb{R}^d$, the process $t \mapsto \Psi_t(x, \omega)$ is a strong solution of (10),
- 2) $\Psi_t(x, \omega)$ is continuous in (t, x) , $\Psi_t(\cdot, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are homeomorphisms, and $\Psi_t(\cdot, \omega), \Psi_t^{-1}(\cdot, \omega) \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

We first state our basic existence result. Recall that $b \in \mathbf{F}_\delta$ if

$$\|b\varphi\|_2^2 \leq \delta \|\nabla \varphi\|_2^2 + c_\delta \|\varphi\|_2^2, \quad \varphi \in W^{1,2},$$

for some constant $c_\delta \geq 0$.

Theorem 1. *Assume that $d \geq 3$, $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma^2}{2\beta_2}$. Let $T > 0$, $p \geq 2$. Provided that κ is chosen sufficiently small, there are constants $\mu_1(\delta, c_\delta, p) \geq 0$, $C_1 = C_1(\delta, c_\delta, p) > 0$ and $C_2 = C_2(\delta, c_\delta, p, T) > 0$ such that for any $\mu \geq \mu_1(\delta, c_\delta, p)$, for every $f \in L^{2p}$ there exists a function $u \in L^\infty(J_T, L^2(\Omega, L^2_\rho))$ for which the following are true.*

(i)

$$\sup_{t \in J_T} \|\mathbb{E}u^2(t)\|_p \leq \|f\|_{2p}^2, \quad \int_{J_T} \|\nabla v_p\|_2^2 ds \leq C_1 \|f\|_{2p}^p, \quad (11)$$

$$\mathbb{E}\langle \rho |\nabla \int_{J_T} u ds|^2 \rangle \leq C_2 \|f\|_{2p}^2, \quad (12)$$

where $v := \mathbb{E}u^2$ and $v_p := v|v|^{\frac{p}{2}-1}$, so, in particular, for a.e. $\omega \in \Omega$, $\nabla \int_0^T u(s, \cdot, \omega) ds \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ and hence

$$b \cdot \nabla \int_{J_T} u(s, \cdot, \omega) ds \in L^1_{\text{loc}},$$

and, for every test function $\varphi \in C_c^\infty$, we have a.s. for all $t \in J_T$,

$$\begin{aligned} & \langle u(t), \varphi \rangle - \langle f, \varphi \rangle \\ & + \mu \langle \int_0^t u ds, \varphi \rangle + \langle b \cdot \nabla \int_0^t u ds, \varphi \rangle - \sigma \langle \int_0^t u dB_s, \nabla \varphi \rangle + \frac{\sigma^2}{2} \langle \nabla \int_0^t u ds, \nabla \varphi \rangle = 0. \end{aligned} \quad (13)$$

(ii) *For any sequence of smooth vector fields $b_m \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $m = 1, 2, \dots$, that are uniformly form-bounded in the sense that $b_m \in \mathbf{F}_\delta$ with c_δ independent of m , and are such that*

$$b_m \rightarrow b \text{ in } L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d) \text{ as } m \rightarrow \infty,$$

we have for initial functions $f \in C_c^\infty$,

$$u_m(t) \rightarrow u(t) \quad \text{in } L^2(\Omega, L^2_\rho) \quad \text{uniformly in } t \in J_T,$$

where u_m is the unique strong solution to (CP) (with $b = b_m$).

An example of such smooth approximating vector fields $\{b_m\}$ is given in the next section.

The next theorem establishes the Sobolev regularity of u up to the initial time $t = 0$.

Theorem 2. *Assume that $d \geq 3$, $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma^2}{2\beta_2}$ and $f \in W^{1,4}$. Let κ be sufficiently small and $\mu_1(\delta, c_\delta, 2)$ be the constant in Theorem 1 with $p = 2$. For $\mu \geq \mu_1(\delta, c_\delta, 2)$, let u be the process constructed in Theorem 1. There exists $\mu_2(\delta, c_\delta) \geq \mu_1(\delta, c_\delta, 2)$ such that for $\mu \geq \mu_2(\delta, c_\delta)$, the following are true.*

(a) $\mathbb{E}u^2, \mathbb{E}|\nabla u|^2 \in L^\infty(J_T, L^2)$, so $u \in L^\infty(J_T, L^2(\Omega, W_\rho^{1,2}))$;

(b) *for any test function $\varphi \in C_c^\infty$, the process $t \mapsto \langle u(t), \varphi \rangle$ is (\mathcal{F}_t) -progressively measurable and has a continuous (\mathcal{F}_t) -semi-martingale modification that satisfies a.s. for every $t \in J_T$,*

$$\begin{aligned} & \langle u(t), \varphi \rangle - \langle f, \varphi \rangle \\ & + \mu \int_0^t \langle u, \varphi \rangle ds + \int_0^t \langle b \cdot \nabla u, \varphi \rangle ds - \sigma \int_0^t \langle u, \nabla \varphi \rangle dB_s + \frac{\sigma^2}{2} \int_0^t \langle u, \Delta \varphi \rangle ds = 0. \end{aligned} \quad (14)$$

Moreover, if $\sqrt{\delta} < \frac{\sigma^2}{2\beta_{2q}}$ for some $q = 1, 2, \dots$, then there exists constants $\mu_2(\delta, c_\delta, q) \geq \mu_1(\delta, c_\delta, 2q)$ (with $\mu_2(\delta, c_\delta, 1)$ equal to the $\mu_2(\delta, c_\delta)$ above) and $C_1 = C_1(\delta, c_\delta, q) > 0$ such that when $\mu \geq \mu_2(\delta, c_\delta, q)$

and $f \in W^{1,4q}$, we have

$$\sup_{0 \leq \alpha \leq 1} \left\| \mathbb{E} |\nabla u|^{2q} \right\|_{L^{\frac{2}{1-\alpha}}(J_T, L^{\frac{2d}{d-2+2\alpha}})} \leq C_1 \|\nabla f\|_{4q}^{2q}. \quad (15)$$

In particular, there exists $C_2 > 0$ such that

$$\sup_{t \in J_T} \mathbb{E} \langle \rho |\nabla u|^{2q} \rangle \leq C_2 \|\nabla f\|_{4q}^{2q}. \quad (16)$$

If $2q > d$, then for a.e. $\omega \in \Omega$, $t \in J_T$, the function $x \mapsto u(t, x, \omega)$ is Hölder continuous, possibly after modification on a set of measure zero in \mathbb{R}^d (in general, depending on ω).

Theorem 3. Assume that $d \geq 3$, $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma^2}{2\beta_2}$ and $f \in W^{1,4}$. Provided κ is sufficiently small, there exists $\mu_3 = \mu_3(\delta, c_\delta) \geq 0$ such that for $\mu \geq \mu_3(\delta, c_\delta)$, (CP) has a unique solution in the class of functions satisfying (a), (b) of Theorem 2.

A function satisfying (a), (b) of Theorem 2 will be called a weak solution of (CP). This definition of weak solution is close to [BFGM, Definition 2.13]. It should be noted however that the authors in [BFGM] prove their uniqueness result, in the time-dependent case, in a larger class of weak solutions (not requiring any differentiability, see [BFGM, Definition 3.3]) but under additional assumptions on b . Specialized to the time-dependent case, they assume that b satisfies

$$\operatorname{div} b \in L^d + L^\infty \quad (17)$$

in addition to $|b| \in L^d + L^\infty$. The latter is needed to establish (15) for solutions of the adjoint equation to the STE, i.e. the stochastic continuity equation (which allows to prove an even stronger result: the uniqueness of weak solution to the corresponding random transport equation), see [BFGM, Sect. 3].

We expect that an analogue of (17) for $b \in \mathbf{F}_\delta$ can be found with some additional effort. However, we will not address this matter in this paper. Of course, in the case $b \in \mathbf{F}_\delta$, $\operatorname{div} b = 0$, one has (15) for solutions to the stochastic continuity equation, so one can prove the uniqueness for (CP) by repeating the argument in [BFGM, Sect. 3].

The proof of the uniqueness result in Theorem 3 (see Section 6) adopts the method of [BFGM, Sect. 3].

Remark 1 (On applications to SDEs). Armed with Theorems 1 and 2, one can repeat the argument in [BFGM, Sect. 4] to prove the following result. Assuming that $b \in \mathbf{F}_\delta$ with δ sufficiently small, there exists a stochastic Lagrangian flow for SDE (10), i.e. a measurable map $\Phi : J_T \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ such that, for a.e. $x \in \mathbb{R}^d$, the process $t \mapsto \Phi_t(x, \omega)$ is a strong solution of the SDE (10):

$$\Phi_t(x, \omega) = x - \int_0^t b(s, \Phi_r(x, \omega)) dr + \sigma B_t(\omega), \quad \text{a.s., } t \in J_T, \quad (18)$$

and $\Phi_t(x, \cdot)$ is \mathcal{F}_t -progressively measurable. If also $\sqrt{\delta} < \frac{\sigma^2}{2\beta_{2q}}$, $q = 1, 2, \dots$, then $\Phi_t(\cdot, \omega) \in W_{\text{loc}}^{1,2q}$ ($t \in J_T$) for a.e. $\omega \in \Omega$. Moreover, Φ_t is unique, i.e. any two such stochastic flows coincide a.s. for every $t > 0$ for a.e. x .

Remark 2 (STE with time-dependent b). The proof of the key result of this paper (Proposition 2 below, i.e. a priori Sobolev regularity of solutions of the STE) carries over, without change, to the time-dependent form-bounded vector fields:

DEFINITION 2. A vector field $b \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^d, \mathbb{R}^d)$ is said to be form-bounded with relative bound $\delta > 0$, written as $b \in \widetilde{\mathbf{F}}_\delta$, if $|b| \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^d)$ and

$$\int_0^\infty \|b(t, \cdot)\phi(t, \cdot)\|_2^2 dt \leq \delta \int_0^\infty \|\nabla\phi(t, \cdot)\|_2^2 dt + \int_0^\infty g(t)\|\phi(t, \cdot)\|_2^2 dt$$

for some $g = g_\delta \in L^1_{\text{loc}}[0, \infty)$, for all $\phi \in C_c^\infty([0, \infty) \times \mathbb{R}^d)$.

The class $\widetilde{\mathbf{F}}_\delta$ contains vector fields

$$|b(\cdot, \cdot)| \in L^q([0, \infty), L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} \leq 1,$$

with δ that can be chosen arbitrarily small (using Hölder's inequality and the Sobolev embedding theorem). Another example is

$$|b(t, x)|^2 \leq c_1|x - x_0|^{-2} + c_2|t - t_0|^{-1}(\log(e + |t - t_0|^{-1}))^{-1-\varepsilon}, \quad \varepsilon > 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d,$$

which belongs to the class $\widetilde{\mathbf{F}}_\delta$ with $\delta = c_1(2/(d-2))^2$ (using Hardy's inequality).

We plan to address the regularity theory of the STE with $b \in \widetilde{\mathbf{F}}_\delta$ elsewhere.

3. A PRIORI ESTIMATES

Assume $b \in \mathbf{F}_\delta$. In the remainder of this paper, we fix some $b_m \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$b_m \rightarrow b \text{ in } L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d) \text{ as } m \rightarrow \infty$$

and for every $m = 1, 2, \dots$

$$\|b_m\varphi\|_2^2 \leq \delta\|\nabla\varphi\|_2^2 + c_\delta\|\varphi\|_2^2, \quad \varphi \in W^{1,2}$$

with c_δ independent of m (see example of such b_m below). Let $f \in C_c^\infty$. Let u_m be the unique strong solution to

$$u_m(t) - f + \mu \int_0^t u_m ds + \int_0^t b_m \cdot \nabla u_m ds + \sigma \int_0^t \nabla u_m \circ dB_s = 0 \text{ a.s.}, \quad t \in J_T = [0, T]. \quad (19)$$

Then, by [Ku, Section 6.1], for any $p, r \geq 1$ and any multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ of non-negative integers,

$$\mathbb{E}(|D^\alpha u_m|^p) \in L^\infty(J_T \times \mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d} (1 + |x|^r)(\mathbb{E}|u_m|^p + \mathbb{E}|\nabla u_m|^p) dx \in L^\infty(J_T).$$

Remark 3 (Example of $\{b_m\}$). Denote by $\mathbf{1}_m$ the indicator of $\{|x| \leq m, |b(x)| \leq m\}$, and by $\eta_m \in C_c^\infty$ a $[0, 1]$ -valued function such that $\eta_m = 1$ on $B(0, m)$. Consider

$$b_m := \eta_m e^{\varepsilon_m \Delta}(\mathbf{1}_m b), \quad (*)$$

where $\varepsilon_m \downarrow 0$ is to be chosen.

First, let us show that, for any $\{\gamma_m\} \downarrow 0$ we can select $\{\varepsilon_m\} \downarrow 0$ in the definition of b_m so that

$$b_m \in \mathbf{F}_{\delta_m} \quad \text{with } \delta_m = (\sqrt{\delta} + \sqrt{\gamma_m})^2 \downarrow \delta \text{ and } c_{\delta_m} \leq 2c_\delta \text{ starting from some } m \text{ on.}$$

Since $b \in \mathbf{F}_\delta$, there exists $\lambda \geq 0$ such that $\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \rightarrow 2} \leq \sqrt{\delta}$. Then $c_\delta = \lambda\delta$. We claim that, we can select $\{\epsilon_m\} \downarrow 0$ fast enough so that

$$\|b_m(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \rightarrow 2} \leq \sqrt{\delta_m}. \quad (**)$$

Once this claim is proven, we will have $c_{\delta_m} = \lambda\delta_m \leq 2c_\delta$ starting from some m on, which implies the required. Now we prove the claim. We have

$$b_m = \mathbf{1}_m b + (b_m - \mathbf{1}_m b),$$

where, clearly, $\|\mathbf{1}_m b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \rightarrow 2} \leq \sqrt{\delta}$ for every m , while $b_m - \mathbf{1}_m b \in L^d$. It follows from Hölder's inequality and the Sobolev embedding theorem that for any $g \in L^2$,

$$\|b_m - \mathbf{1}_m b(\lambda - \Delta)^{-\frac{1}{2}}g\|_2 \leq \|b_m - \mathbf{1}_m b\|_d \|(\lambda - \Delta)^{-\frac{1}{2}}g\|_{\frac{2d}{d-2}} \leq c\|b_m - \mathbf{1}_m b\|_d \|g\|_2.$$

It is easily seen that, for every m , the norm $\|b_m - \mathbf{1}_m b\|_d$ can be made smaller than $c^{-1}\gamma_m$ by selecting $\{\epsilon_m\} \downarrow 0$ sufficiently rapidly. Thus

$$\|(b_m - \mathbf{1}_m b)(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \rightarrow 2} \leq \gamma_m.$$

Now (**) follows.

Finally, to have b_m form-bounded with the original relative bound δ , it suffices to multiply b_m in (*) by $\frac{\delta}{\delta_m}$. (Although, to carry out the proofs of Theorems 1-3, the last step is not necessary since all our assumptions on δ are strict inequalities.)

We prove the next proposition under more general assumptions on δ and p than in Theorem 1.

Proposition 1. *Let $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \sigma^2$. Let $T > 0$, $p \in (p_c, \infty)$, $p_c := (1 - \frac{\sqrt{\delta}}{\sigma^2})^{-1}$. Let $f \in C_c^\infty$, let b_m and u_m be as above. There exist constants $\mu(\delta, c_\delta, p) \geq 0$, $C_1 = C_1(\delta, c_\delta, p) > 0$ and $C_2 = C_2(\delta, c_\delta, p, T) > 0$ independent of m such that for any $\mu \geq \mu(\delta, c_\delta, p)$ and $m = 1, 2, \dots$, the following are true:*

(i)

$$\sup_{t \in J_T} \|\mathbb{E}u_m^2(t)\|_p \leq \|f\|_{2p}^2, \quad \int_{J_T} \|\nabla v_p\|_2^2 ds \leq C_1 \|f\|_{2p}^p, \quad (E_1)$$

where $v := \mathbb{E}u^2$ and $v_p := v|v|^{\frac{p}{2}-1}$;

(ii) if $\sqrt{\delta} < \frac{\sigma^2}{2}$, then

$$\mathbb{E} \left\langle \rho \left(\nabla \int_{J_T} u_m(s) ds \right)^2 \right\rangle \leq C_2 \|f\|_{2p}^2. \quad (E_2)$$

Proposition 2. *Let $b \in \mathbf{F}_\delta$ and $f \in C_c^\infty$, let b_m and u_m be as above. For every $q \geq 1$, there exists constants $\mu(\delta, c_\delta, q) \geq 0$ and $C = C(\delta, c_\delta, q) > 0$ independent of m such that if $\sqrt{\delta} < \frac{\sigma^2}{2\beta_{2q}}$ and $\mu \geq \mu(\delta, c_\delta, q)$, then*

$$\sup_{0 \leq \alpha \leq 1} \|\mathbb{E}|\nabla u_m|^{2q}\|_{L^{\frac{2}{1-\alpha}}([0, T], L^{\frac{2d}{d-2+2\alpha}})} \leq C \|\nabla f\|_{4q}^{2q}. \quad (E_3)$$

Proof of Proposition 1. For brevity, we write u for u_m in this proof. The identity (7) allows us to rewrite (19) as

$$u(t, \cdot) - f + \mu \int_0^t u ds + \int_0^t b_m \cdot \nabla u ds + \sigma \int_0^t \nabla u dB_s - \frac{\sigma^2}{2} \int_0^t \Delta u ds = 0 \quad \text{a.s., } t \in J_T. \quad (20)$$

Below we will be appealing to (20).

We first prove (E_1) . Applying Itô's formula to u^2 , we obtain, in view of (20),

$$u^2(t) - f^2 = -2\mu \int_0^t u^2 ds - \int_0^t b_m \cdot \nabla u^2 ds - \sigma \int_0^t \nabla u^2 dB_s + \frac{\sigma^2}{2} \int_0^t \Delta u^2 ds.$$

Since $t \mapsto \int_0^t \nabla u^2 dB_s$ is a martingale, $v = \mathbb{E}u^2$ satisfies

$$\partial_t v = -2\mu v - b_m \cdot \nabla v + \frac{\sigma^2}{2} \Delta v, \quad v(0) = f^2.$$

We multiply the last equation by $v|v|^{p-2}$ and integrate by parts (recall that $v_p = v|v|^{\frac{p}{2}-1}$),

$$\frac{1}{p} \partial_t \langle |v_p|^2 \rangle + 2\mu \langle |v_p|^2 \rangle + \frac{4}{pp'} \frac{\sigma^2}{2} \langle |\nabla v_p|^2 \rangle - \frac{2}{p} \langle b_m \cdot \nabla v_p, v_p \rangle \leq 0,$$

so applying the quadratic inequality we have (for $\varepsilon > 0$)

$$\partial_t \langle |v|^p \rangle + 2p\mu \langle |v|^p \rangle + \frac{2\sigma^2}{p'} \langle |\nabla v_p|^2 \rangle - 2 \left(\varepsilon \langle |\nabla v_p|^2 \rangle + \frac{1}{4\varepsilon} \langle b_m^2 v_p^2 \rangle \right) \leq 0.$$

Finally, by our assumption on b_m ,

$$\partial_t \langle |v|^p \rangle + 2p\mu \langle |v|^p \rangle + \frac{2\sigma^2}{p'} \langle |\nabla v_p|^2 \rangle - 2 \left(\varepsilon \langle |\nabla v_p|^2 \rangle + \frac{\delta}{4\varepsilon} \langle |\nabla v_p|^2 \rangle + \frac{c_\delta}{4\varepsilon} \langle |v|^p \rangle \right) \leq 0.$$

Taking $\varepsilon = \frac{\sqrt{\delta}}{2}$ in the last inequality and integrating with respect to t , we obtain for $t > 0$

$$\langle |v(t)|^p \rangle + 2 \left(\frac{\sigma^2}{p'} - \sqrt{\delta} \right) \int_0^t \langle |\nabla v_p|^2 \rangle ds + \left[2p\mu - \frac{c_\delta}{2\sqrt{\delta}} \right] \int_0^t \langle |v|^p \rangle ds \leq \|f^2\|_p^p,$$

where $\frac{\sigma^2}{p'} - \sqrt{\delta} > 0$ since $p > p_c$. Taking $\mu \geq \frac{c_\delta}{4\sqrt{\delta}p}$, we arrive at (E_1) .

Now we deal with (E_2) . Let $\mu \geq \frac{c_\delta}{4\sqrt{\delta}p}$ as above. By (E_1) ,

$$\sup_{t \in J_T} \langle \rho \mathbb{E}u^2(t) \rangle \leq \|\rho\|_{p'} \sup_{t \in J_T} \|\mathbb{E}u^2(t)\|_p \leq c_1 \|f\|_{2p}^2, \quad (21)$$

since $\theta > \frac{d}{2}$ in the definition of ρ .

We multiply (20) by $\rho \int_0^t u ds$, integrate, and take expectation, to get

$$\begin{aligned} \mathbb{E} \langle \rho \int_0^t u ds, u(t) \rangle &= \mathbb{E} \langle \rho \int_0^t u ds, f \rangle - \mathbb{E} \langle \rho \int_0^t u ds, b_m \cdot \nabla \int_0^t u ds \rangle \\ &\quad - \sigma \mathbb{E} \langle \rho \int_0^t u ds, \int_0^t \nabla u dB_s \rangle + \frac{\sigma^2}{2} \mathbb{E} \langle \rho \int_0^t u ds, \int_0^t \Delta u ds \rangle + \mu \mathbb{E} \langle \rho \int_0^t u ds, \int_0^t u ds \rangle \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (22)$$

Denote the left-hand side of (22) by I_0 . Set

$$U := \int_0^t u ds.$$

By Hölder's inequality and (21),

$$\mathbb{E} \langle \rho U^2 \rangle \leq t \langle \rho \int_0^t \mathbb{E}u^2 ds \rangle \leq t^2 c_1 \|f\|_{2p}^2. \quad (23)$$

Integrating by parts in I_4 and using the quadratic inequality, we have

$$\begin{aligned}
\frac{2}{\sigma^2}I_4 &= -E\langle\rho|\nabla U|^2\rangle - E\langle U\nabla\rho, \nabla U\rangle \\
&\leq -E\langle\rho|\nabla U|^2\rangle + \alpha E\langle|\nabla\rho|U^2\rangle + \frac{1}{4\alpha}E\langle|\nabla\rho||\nabla U|^2\rangle \quad (\alpha > 0) \\
&\quad \text{(we are applying (6) in the last term, and (6), (23) in the middle term)} \\
&\leq -\left(1 - \frac{\theta\sqrt{\kappa}}{4\alpha}\right)E\langle\rho|\nabla U|^2\rangle + \theta\sqrt{\kappa}\alpha T^2c_1\|f\|_{2p}^2.
\end{aligned}$$

Substituting the last estimate into (22), we obtain

$$\frac{\sigma^2}{2}\left(1 - \frac{\theta\sqrt{\kappa}}{4\alpha}\right)E\langle\rho|\nabla U|^2\rangle \leq \frac{\sigma^2}{2}\theta\sqrt{\kappa}\alpha T^2c_1\|f\|_{2p}^2 + |I_0| + |I_1| + |I_2| + |I_3| + |I_5|. \quad (24)$$

We now estimate $|I_i|$, $i = 0, 1, 2, 3, 5$. By (21) and (23),

$$|I_0| \leq (E\langle\rho U^2\rangle)^{\frac{1}{2}}(E\langle\rho u^2(t)\rangle)^{\frac{1}{2}} \leq c_2\|f\|_{2p}^2.$$

Similarly,

$$|I_1| \leq c_3\|f\|_{2p}^2, \quad |I_5| \leq \mu c_4\|f\|_{2p}^2.$$

Next, applying the quadratic inequality, we get

$$\begin{aligned}
|I_2| &\leq \nu E\langle\rho b_m^2 U^2\rangle + \frac{1}{4\nu}E\langle\rho|\nabla U|^2\rangle \quad (\nu > 0) \\
&\quad \text{(in the first term, we apply } b_m \in \mathbf{F}_\delta \text{ with } \varphi := \sqrt{\rho}U) \\
&\leq \nu(\delta E\langle|\nabla(\sqrt{\rho}U)|^2\rangle + c_\delta E\langle\rho U^2\rangle) + \frac{1}{4\nu}E\langle\rho|\nabla U|^2\rangle \\
&\quad \text{(in the first term, we use } (a+c)^2 \leq (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})c^2, \epsilon > 0) \\
&\leq \nu\delta(1+\epsilon)E\langle|\sqrt{\rho}\nabla U|^2\rangle + \nu\delta(1+\frac{1}{\epsilon})E\langle|U\nabla\sqrt{\rho}|^2\rangle + \nu c_\delta E\langle\rho U^2\rangle + \frac{1}{4\nu}E\langle\rho|\nabla U|^2\rangle \\
&\quad \text{(in the second term, we apply (6) and then use (23); also, we apply (23) in the last term)} \\
&\leq \left(\nu\delta(1+\epsilon) + \frac{1}{4\nu}\right)\langle\rho|\nabla U|^2\rangle + T^2c_5\|f\|_{2p}^2, \quad c_5 = c_5(\nu, \delta, c_\delta, \theta, \kappa, \epsilon).
\end{aligned}$$

In the current setting, we have $\int_0^t \nabla u dB_s = \nabla \int_0^t u dB_s$ (see, for instance, [HN]). Thus, integrating by parts, we obtain

$$I_3 = \sigma E\langle\rho\nabla U, \int_0^t u dB_s\rangle + \sigma E\langle U\nabla\rho, \int_0^t u dB_s\rangle,$$

so

$$\begin{aligned}
|I_3| &\leq \sigma (\mathbb{E}\langle \rho |\nabla U|^2 \rangle)^{\frac{1}{2}} (\mathbb{E}\langle \rho \left(\int_0^t u dB_s \right)^2 \rangle)^{\frac{1}{2}} \\
&\quad + \sigma (\mathbb{E}\langle |\nabla \rho| U^2 \rangle)^{\frac{1}{2}} (\mathbb{E}\langle |\nabla \rho| \left(\int_0^t u dB_s \right)^2 \rangle)^{\frac{1}{2}} \\
&\quad \text{(we use (6) and apply the Itô isometry)} \\
&\leq \sigma (\mathbb{E}\langle \rho |\nabla U|^2 \rangle)^{\frac{1}{2}} (\mathbb{E}\langle \rho \int_0^t u^2 ds \rangle)^{\frac{1}{2}} \\
&\quad + \theta \sqrt{\kappa} \sigma (\mathbb{E}\langle \rho U^2 \rangle)^{\frac{1}{2}} (\mathbb{E}\langle \rho \int_0^t u^2 ds \rangle)^{\frac{1}{2}} \\
&\quad \text{(we apply the quadratic inequality in the first term and then use (23))} \\
&\leq \sigma \gamma \mathbb{E}\langle \rho |\nabla U|^2 \rangle + \frac{\sigma T^2 c_1}{4\gamma} \|f\|_{2p}^2 + \theta \sqrt{\kappa} \sigma T^2 c_1 \|f\|_{2p}^2 \quad (\gamma > 0).
\end{aligned}$$

Substituting the above estimates on $|I_0|$, $|I_1|$, $|I_2|$, $|I_3|$ and $|I_5|$ in (24), we obtain

$$\left(\frac{\sigma^2}{2} - \nu \delta (1 + \epsilon) - \frac{1}{4\nu} - \sigma \gamma - \frac{\sigma^2 \theta \sqrt{\kappa}}{2 \cdot 4\alpha} \right) \mathbb{E}\langle \rho |\nabla U|^2 \rangle \leq c_6 \|f\|_{2p}^2$$

for an appropriate constant $c_6 = c_6(\alpha, \gamma, \nu, \delta, \theta, \kappa, \epsilon, c_\delta, \mu) < \infty$. Take $\nu = (2\sqrt{\delta})^{-1}$. Since $\sqrt{\delta} < \frac{\sigma^2}{2}$ by assumption, we can select γ, ϵ sufficiently small and α sufficiently large so that

$$\frac{\sigma^2}{2} - \left(\nu \delta + \frac{1}{4\nu} \right) - \nu \delta \epsilon - \sigma \gamma - \frac{\sigma^2 \theta \sqrt{\kappa}}{2 \cdot 4\alpha} > 0,$$

and thus (E_2) follows with constant $C_2 = c_6 \left(\frac{\sigma^2}{2} - \nu \delta (1 + \epsilon) - \frac{1}{4\nu} - \sigma \gamma - \frac{\sigma^2 \theta \sqrt{\kappa}}{2 \cdot 4\alpha} \right)^{-1}$. \square

Remark 4. In Proposition 1, the interval (p_c, ∞) of admissible values of p decreases to the empty set as $\sqrt{\delta} \uparrow \sigma^2$. In fact, one can show that if $b \in \mathbf{F}_\delta$, $\sqrt{\delta} < \sigma^2$ and $b_m \in C_c^\infty$ are as above, then the limit

$$s\text{-}L^p\text{-}\lim_m e^{-t\Lambda_m} \quad (\text{loc. uniformly in } t \geq 0), \quad p > p_c,$$

where $\Lambda_m = -\frac{\sigma^2}{2}\Delta + b_m \cdot \nabla$, $D(\Lambda_m) = W^{2,p}$, exists and determines a L^∞ contraction, quasi contraction holomorphic semigroup in L^p , say, $e^{-t\Lambda}$, see [KiS3, Theorems 4.2, 4.3]. The operator Λ is an appropriate operator realization of the formal operator $-\frac{\sigma^2}{2}\Delta + b \cdot \nabla$ in L^p . One can compare this result with the example in [BFGM, Sect. 7], where the authors show that the SDE

$$X_t = - \int_0^t b(X_s) ds + \sigma B_t, \quad b(x) = \sqrt{\delta} \frac{d-2}{2} |x|^{-2} x \in \mathbf{F}_\delta,$$

corresponding to operator $-\frac{\sigma^2}{2}\Delta + b \cdot \nabla$, does not have a weak solution if $\sqrt{\delta} > \sigma^2$.

Proof of Proposition 2. For any multiindex I with entries in $\{1, \dots, d\}$, i.e., an element of $\{1, \dots, d\} \times \dots \times \{1, \dots, d\}$, say, p times, we write $|I| = p$. For any such multiindex I and $l \in \{1, \dots, d\}$, we denote by $I - l$ the multiindex obtained from I by dropping an index of value l . Let $I - l + k$ be the multiindex I with an index of value l dropped and replaced with an index of value k . It does not matter from which component the value l is dropped.

For brevity, we write u for u_m in this proof. Set

$$w_r := \partial_{x_r} u, \quad 1 \leq r \leq d,$$

where u is the strong solution of (19), and

$$w_I := \prod_{r \in I} \partial_{x_r} u.$$

Step 1. We apply Itô's formula in Stratonovich form to w_I , obtaining

$$w_I(t) - \prod_{r \in I} \partial_{x_r} f = \sum_{r \in I} \int_0^t w_{I-r}(s) \circ dw_r(s).$$

Next, differentiating (20) in x_r and then substituting the resulting expression for dw_r into the previous formula, we obtain

$$w_I(t) - \prod_{r \in I} \partial_{x_r} f = -\mu \int_0^t w_I ds - \sum_{r \in I} \int_0^t w_{I-r} (b_m \cdot \nabla w_r + \partial_{x_r} b \cdot \nabla u) ds - \sigma \sum_{r \in I} \int_0^t w_{I-r} \nabla w_r \circ dB_s.$$

Let b_m^k , $k = 1, \dots, d$, be the components of the vector field b_m . We have

$$\begin{aligned} w_I(t) - \prod_{r \in I} \partial_{x_r} f &= -\mu \int_0^t w_I ds - \sum_{r \in I} \int_0^t w_{I-r} (b_m \cdot \nabla w_r + \partial_{x_r} b_m \cdot \nabla u) ds - \sigma \int_0^t \nabla w_I \circ dB_s \\ &\text{(we use } \int_0^t \nabla w_I \circ dB_s = \int_0^t \nabla w_I dB_s - \frac{1}{2} \sum_{k=1}^d [\partial_{x_k} w_I, B^k]_t) \\ &= -\mu \int_0^t w_I ds - \sum_{r \in I} \int_0^t w_{I-r} (b_m \cdot \nabla w_r + \partial_{x_r} b_m \cdot \nabla u) ds - \sigma \int_0^t \nabla w_I dB_s + \frac{\sigma^2}{2} \int_0^t \Delta w_I ds \\ &= -\mu \int_0^t w_I ds - \int_0^t b_m \cdot \nabla w_I ds - \sum_{r \in I} \sum_{k=1}^d \int_0^t \partial_{x_r} b_m^k w_{I-r+k} ds - \sigma \int_0^t \nabla w_I dB_s + \frac{\sigma^2}{2} \int_0^t \Delta w_I ds. \end{aligned}$$

Put

$$v_I := \mathbb{E}[w_I].$$

Since $t \mapsto \int_0^t \nabla w_I dB_s$ is a martingale, v_I satisfies

$$v_I(t) - \prod_{r \in I} \partial_{x_r} f = -\mu \int_0^t v_I ds - \int_0^t b_m \cdot \nabla v_I ds - \sum_{r \in I} \sum_{k=1}^d \int_0^t \partial_{x_r} b_m^k v_{I-r+k} ds + \frac{\sigma^2}{2} \int_0^t \Delta v_I ds,$$

i.e.,

$$\partial_t v_I = -\mu v_I + \frac{\sigma^2}{2} \Delta v_I - b_m \cdot \nabla v_I - \sum_{r \in I} \sum_{k=1}^d \partial_{x_r} b_m^k v_{I-r+k}, \quad v_I(0) = \prod_{r \in I} \partial_{x_r} f. \quad (25)$$

Step 2. We multiply the equation in (25) by v_I , and integrate:

$$\frac{1}{2} \partial_t \langle v_I^2 \rangle + \mu \langle v_I^2 \rangle + \frac{\sigma^2}{2} \langle (\nabla v_I)^2 \rangle = -\langle v_I, b_m \cdot \nabla v_I \rangle - \langle v_I, \sum_{r \in I} \sum_{k=1}^d \partial_{x_r} b_m^k v_{I-r+k} \rangle.$$

Then, for every $t \in J_T$,

$$\begin{aligned} & \frac{1}{2}\langle v_I^2(t) \rangle - \frac{1}{2}\langle v_I^2(0) \rangle + \mu \int_0^t v_I^2 ds + \frac{\sigma^2}{2} \int_0^t \langle (\nabla v_I)^2 \rangle ds \\ & = - \int_0^t \langle v_I, b_m \cdot \nabla v_I \rangle ds - \int_0^t \langle v_I, \sum_{r \in I} \sum_{k=1}^d \partial_{x_r} b_m^k v_{I-r+k} \rangle ds =: -S_I^1 - S_I^2. \end{aligned} \quad (26)$$

We estimate $|S_I^1|$ and $|S_I^2|$ as follows:

$$\begin{aligned} |S_I^1| & \leq \left| \int_0^t \langle v_I, b_m \cdot \nabla v_I \rangle ds \right| \leq \gamma \int_0^t \langle (\nabla v_I)^2 \rangle ds + \frac{1}{4\gamma} \int_0^t \langle v_I^2 b_m^2 \rangle ds \\ & \quad (\text{we use } b_m \in \mathbf{F}_\delta) \\ & \leq \left(\gamma + \frac{\delta}{4\gamma} \right) \int_0^t \langle (\nabla v_I)^2 \rangle ds + \frac{c_\delta}{4\gamma} \int_0^t \langle v_I^2 \rangle ds. \end{aligned} \quad (27)$$

Next, integrating by parts, and applying the quadratic inequality, we have

$$\begin{aligned} |S_I^2| & = \left| - \int_0^t \sum_{r \in I} \sum_{k=1}^d \langle (v_{I-r+k} \partial_{x_r} v_I + v_I \partial_{x_r} v_{I-r+k}) b_m^k \rangle ds \right| \\ & \leq \alpha \int_0^t \sum_{r \in I} \sum_{k=1}^d \langle (\partial_{x_r} v_I)^2 + (\partial_{x_r} v_{I-r+k})^2 \rangle ds + \frac{1}{4\alpha} \int_0^t \sum_{r \in I} \sum_{k=1}^d \langle v_{I-r+k}^2 (b_m^k)^2 + v_I^2 (b_m^k)^2 \rangle ds. \end{aligned}$$

Let $q = 1, 2, \dots$. Summing over all I with $|I| = 2q$ and noticing that every multiindex of length $2q$ is counted $4qd$ times, we obtain

$$\begin{aligned} \sum_I |S_I^2| & \leq 4\alpha qd \sum_I \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{qd}{\alpha} \sum_I \int_0^t \langle v_I^2 b_m^2 \rangle ds \\ & \quad (\text{use } b_m \in \mathbf{F}_\delta \text{ in the second term}) \\ & \leq 4\alpha qd \sum_I \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{qdc_\delta}{\alpha} \sum_I \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{qdc_\delta}{\alpha} \sum_I \int_0^t \langle v_I^2 \rangle ds. \end{aligned}$$

Also, by (27), we have

$$\sum_I |S_I^1| \leq \left(\gamma + \frac{\delta}{4\gamma} \right) \sum_I \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{c_\delta}{4\gamma} \sum_I \int_0^t \langle v_I^2 \rangle ds.$$

Now, armed with the last two estimates, we sum both sides of (26) over all I with $|I| = 2q$ to obtain

$$\begin{aligned} & \frac{1}{2} \sum_I \langle v_I^2(t) \rangle + \mu \int_0^t v_I^2 ds + \varkappa \int_0^t \sum_I \langle |\nabla v_I|^2 \rangle ds \\ & \leq \frac{1}{2} \sum_I \langle v_I^2(0) \rangle + \left[\frac{qdc_\delta}{\alpha} + \frac{c_\delta}{4\gamma} \right] \sum_I \int_0^t \langle v_I^2 \rangle ds, \end{aligned}$$

where

$$\varkappa := \frac{\sigma^2}{2} - \gamma - \frac{\delta}{4\gamma} - 4\alpha qd - \frac{qdc_\delta}{\alpha}.$$

The maximum $\varkappa_* := \max_{\alpha, \gamma > 0} \varkappa = \frac{\sigma^2}{2} - \sqrt{\delta} - 4qd\sqrt{\delta}$ is attained at

$$\alpha = \frac{\sqrt{\delta}}{2}, \quad \gamma = \frac{\sqrt{\delta}}{2}.$$

For this choice of α and γ , we have $\varkappa_* = \frac{\sigma^2}{2} - \beta_{2q}\sqrt{\delta}$. Since $\beta_{2q}\sqrt{\delta} < \frac{\sigma^2}{2}$ by assumption, we have $\varkappa_* > 0$ and

$$\frac{1}{2} \sum_I \langle v_I^2(t) \rangle + (\mu - \hat{c}) \int_0^t v_I^2 ds + \varkappa_* \int_0^t \sum_I \langle |\nabla v_I|^2 \rangle ds \leq \frac{1}{2} \sum_I \langle v_I^2(0) \rangle,$$

where $\hat{c} := \frac{2qdc_\delta}{\sqrt{\delta}} + \frac{c_\delta}{2\sqrt{\delta}}$. Thus, choosing $\mu \geq \hat{c}$, we obtain

$$\frac{1}{2} \sup_{\tau \in [0, t]} \sum_I \langle v_I^2(\tau) \rangle + \varkappa_* \int_0^t \sum_I \langle |\nabla v_I|^2 \rangle ds \leq \frac{1}{2} \sum_I \langle v_I^2(0) \rangle.$$

Step 3. Recalling that $v_I = \mathbb{E}[\prod_{r \in I} \partial_{x_r} u]$, $v_I(0) = \prod_{r \in I} \partial_{x_r} f$, we obtain from the previous estimate:

$$\sup_{t \in J_T} \sum_{1 \leq k \leq d} \langle (\mathbb{E}(\partial_{x_k} u)^{2q})^2 \rangle \leq c_1 \langle |\nabla f|^{2q} \rangle, \quad (28)$$

$$\sum_{1 \leq k \leq d} \int_0^t \langle |\nabla \mathbb{E}(\partial_{x_k} u)^{2q}|^2 \rangle ds \leq c_2 \langle |\nabla f|^{2q} \rangle, \quad (29)$$

for appropriate positive constants c_1, c_2 . By the Sobolev embedding theorem,

$$\int_0^t \langle (\nabla \mathbb{E}|\nabla u|^{2q})^2 \rangle ds \geq c_3 \int_0^t \langle (\mathbb{E}|\nabla u|^{2q})^{\frac{2d}{d-2}} \rangle^{\frac{d-2}{d}} ds,$$

so (29) yields

$$\|\mathbb{E}|\nabla u|^{2q}\|_{L^2(J_T, L^{\frac{2d}{d-2}})}^2 \leq c_4 \|\nabla f\|_{4q}^{4q},$$

for appropriate constant $c_4 > 0$.

Interpolating between the last estimate, and (28), that is, $\|E|\nabla u|^{2q}\|_{L^\infty(J_T, L^2)}^2 \leq c_1 \|\nabla f\|_{4q}^{4q}$, we obtain (E₃). \square

4. PROOF OF THEOREM 1

Recall that $\|\cdot\|_{p, \rho}$ denotes the norm in $L^p(\mathbb{R}^d, \rho dx)$, and $\langle \cdot, \cdot \rangle_\rho$ the inner product in $L^2(\mathbb{R}^d, \rho dx)$. We assume throughout this section that $b \in \mathbf{F}_\delta$ and b_m , $m = 1, 2, \dots$ are as in the beginning of the previous section.

Lemma 1. *Let $b \in \mathbf{F}_\delta$, and let b_m be as above. Then the following are true:*

- (i) $\|b\sqrt{\rho}\|_2 < \infty$.
- (ii) $\|b\sqrt{\rho}\mathbf{1}_{B^c(0, R+1)}\|_2 \downarrow 0$ as $R \rightarrow \infty$.
- (iii) $\langle \rho|b - b_m|^2 \rangle \rightarrow 0$ as $m \rightarrow \infty$.

Proof. (i) Using $b \in \mathbf{F}_\delta$, and applying (6) and $\langle \rho \rangle < \infty$, we have

$$\|b\sqrt{\rho}\|_2^2 \leq \delta \|\nabla \sqrt{\rho}\|_2^2 + c_\delta \langle \rho \rangle < \infty.$$

(ii) For any $R \geq 1$, let η_R be a $[0, 1]$ -valued smooth function such that $\eta_R(x) = 1$ if $|x| > R + 1$; $\eta_R(x) = 0$ if $|x| \leq R$; and $\sup_{R \geq 1} \|\nabla \eta_R\|_\infty \leq C$. Then

$$\|b\sqrt{\rho}\eta_R\|_2^2 \leq \delta \|\nabla[\sqrt{\rho}\eta_R]\|_2^2 + c_\delta \langle \rho \eta_R^2 \rangle.$$

We have $\nabla[\sqrt{\rho}\eta_R] = \frac{1}{2\sqrt{\rho}}(\nabla\rho)\eta_R + \sqrt{\rho}\nabla\eta_R =: S_1 + S_2$. Using (6), we have

$$\|S_1\|_2^2 \leq C \langle \rho \eta_R^2 \rangle \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Next, we use $\sup_{R \geq 1} \|\nabla \eta_R\|_\infty \leq C$ to get

$$\|S_2\|_2^2 \leq C(1 + \kappa R^2)^{-\theta} \langle \mathbf{1}_{B(0, R+1)} - \mathbf{1}_{B(0, R)} \rangle = c_d C(1 + \kappa R^2)^{-\theta} R^d \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

since $\theta > \frac{d}{2}$. This completes the proof of (ii).

(iii) This is a consequence of (ii) and $b_m \rightarrow b$ in $L_{\text{loc}}^2(\mathbb{R}^d)$.

The proof of Lemma 1 is complete. \square

Lemma 2. *Let $\beta_2\sqrt{\delta} < \frac{\sigma^2}{2}$, $f \in C_c^\infty$ and u_m be the strong solution to (19). Provided that $\kappa > 0$ in the definition of ρ is chosen sufficiently small, there exists $\mu(\delta, c_\delta) \geq 0$ such that for any $\mu \geq \mu(\delta, c_\delta)$,*

$$\lim_{n, m \rightarrow \infty} \sup_{t \in J_T} \|\mathbb{E}|u_n(t) - u_m(t)|^2\|_{2, \rho} = 0.$$

Proof. Set

$$g \equiv g_{n, m} := u_n - u_m, \quad n, m = 1, 2, \dots,$$

then

$$g(t) + \mu \int_0^t g^2 ds + \int_0^t b_m \cdot \nabla g ds + \int_0^t (b_n - b_m) \cdot \nabla u_m ds + \sigma \int_0^t \nabla g dB_s - \frac{\sigma^2}{2} \int_0^t \Delta g ds = 0.$$

Applying Itô's formula, we obtain

$$g^2(t) = -2\mu \int_0^t g^2 ds - \int_0^t b_m \cdot \nabla g^2 ds - 2 \int_0^t g(b_n - b_m) \cdot \nabla u_m ds - \sigma \int_0^t \nabla g^2 dB_s + \frac{\sigma^2}{2} \int_0^t \Delta g^2 ds,$$

so denoting $h := \mathbb{E}[g^2]$ we arrive at

$$\partial_t h + 2\mu h - \frac{\sigma^2}{2} \Delta h + b_m \cdot \nabla h + 2(b_n - b_m) \cdot \mathbb{E}[g \nabla u_m] = 0, \quad h(0) = 0.$$

Multiplying this equation by ρh and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2} \|h(t)\|_{2, \rho}^2 + 2\mu \int_0^t \|h\|_{2, \rho}^2 ds + \frac{\sigma^2}{2} \int_0^t \|\nabla h\|_{2, \rho}^2 ds + \frac{\sigma^2}{2} \int_0^t \langle (\nabla \rho) h, \nabla h \rangle \\ + \int_0^t \langle b_m \cdot \nabla h, h \rangle_\rho ds + 2 \int_0^t \langle h(b_n - b_m) \cdot \mathbb{E}[g \nabla u_m] \rangle_\rho ds = 0. \end{aligned} \quad (30)$$

Since our assumption on δ is a strict inequality, using (6) and selecting κ sufficiently small, we can and will ignore in what follows the terms containing $\nabla \rho$.

Applying the quadratic inequality and using $b_m \in \mathbf{F}_\delta$, we obtain (cf. the proof of (E_1))

$$\frac{\sigma^2}{2} \int_0^t \|\nabla h\|_{2, \rho}^2 ds + \int_0^t \langle b_m \cdot \nabla h, h \rangle_\rho ds \geq \left(\frac{\sigma^2}{2} - \sqrt{\delta} \right) \int_0^t \|\nabla h\|_{2, \rho}^2 ds - \frac{c_\delta}{4\sqrt{\delta}} \int_0^t \|h\|_{2, \rho}^2 ds,$$

where $\frac{\sigma^2}{2} - \sqrt{\delta} > 0$ by the assumption on δ .

We obtain from (30):

$$\begin{aligned} \frac{1}{2} \sup_{\tau \in [0, t]} \|h(\tau)\|_{2, \rho}^2 + \left(\frac{\sigma^2}{2} - \sqrt{\delta} \right) \int_0^t \|\nabla h(s)\|_{2, \rho}^2 ds + \left[2\mu - \frac{c_\delta}{4\sqrt{\delta}} \right] \int_0^t \|h\|_{2, \rho}^2 ds \\ \leq 2 \int_0^t \langle h | b_n - b_m | \cdot \mathbb{E}[|g \nabla u_m|] \rangle_\rho ds. \end{aligned}$$

Select $\mu \geq \frac{c_\delta}{4\sqrt{\delta}}$. Then the previous estimate yields

$$\frac{1}{2} \sup_{\tau \in [0, t]} \|h(\tau)\|_{2, \rho}^2 \leq 2 \int_0^t \langle h | b_n - b_m | \cdot \mathbb{E}[|g \nabla u_m|] \rangle_\rho ds,$$

so it remains to show that

$$\int_0^t \langle h | b_n - b_m | \cdot \mathbb{E}[|g \nabla u_m|] \rangle_\rho ds \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

We estimate

$$\begin{aligned} \langle h | b_n - b_m | \cdot \mathbb{E}[|g \nabla u_m|] \rangle_\rho &\leq \langle |b_n - b_m| h (\mathbb{E}[g^2])^{\frac{1}{2}} (\mathbb{E}[|\nabla u_m|^2])^{\frac{1}{2}} \rangle_\rho \equiv \langle |b_n - b_m| h^{\frac{3}{2}} (\mathbb{E}[|\nabla u_m|^2])^{\frac{1}{2}} \rangle_\rho \\ &\leq \langle |b_n - b_m|^2 \rangle_\rho^{\frac{1}{2}} \langle h^3 \mathbb{E}[|\nabla u_m|^2] \rangle_\rho^{\frac{1}{2}} \leq \langle |b_n - b_m|^2 \rangle_\rho^{\frac{1}{2}} \langle h^3 \mathbb{E}[|\nabla u_m|^2] \rangle_\rho^{\frac{1}{2}} \\ &\leq \langle |b_n - b_m|^2 \rangle_\rho^{\frac{1}{2}} \langle h^6 \rangle_\rho^{\frac{1}{4}} \langle (\mathbb{E}[|\nabla u_m|^2])^2 \rangle_\rho^{\frac{1}{4}} \\ &\text{(we apply Proposition 1, and (28) with } q = 1) \\ &\leq c \langle |b_n - b_m|^2 \rangle_\rho^{\frac{1}{2}} \|f\|_{12}^3 \|\nabla f\|_4 \\ &\text{(we apply Lemma 1(iii))} \\ &\rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

The proof of Lemma 2 is complete. \square

Lemma 2 allows to prove that $\{u_m\}$ is a Cauchy sequence in $L^\infty(J_T, L^2(\Omega, L_\rho^2))$.

Lemma 3. *Let $\beta_2 \sqrt{\delta} < \frac{\sigma^2}{2}$, $f \in C_c^\infty$ and u_m be the strong solution to (19). Provided that $\kappa > 0$ in the definition of ρ is chosen sufficiently small, it holds that u_m converges in $L^2(\Omega, L_\rho^2)$ to a process u , uniformly in $t \in J_T$.*

Proof. Let κ be small enough and μ greater than or equal to the $\mu(\delta, c_\delta)$. Let $\mu \geq \mu(\delta, c_\delta)$. Then by Lemma 2,

$$\sup_{t \in J_T} \mathbb{E} \| (u_n(t) - u_m(t)) \|_{2, \rho}^2 \leq \langle \rho \rangle^{\frac{1}{2}} \sup_{t \in J_T} \| \mathbb{E} |u_n(t) - u_m(t)|^2 \|_{2, \rho} \rightarrow 0$$

as $m, n \rightarrow \infty$. Thus, we can define

$$u(t) := s\text{-}L^2(\Omega, L_\rho^2)\text{-}\lim_m u_m(t) \quad \text{uniformly in } t \in J_T.$$

The proof is complete \square

We are in position to give the proof of Theorem 1.

Proof of Theorem 1. It suffices to carry out the proof for $f \in C_c^\infty$, and then use a density argument.

It follows from the assumption $\sqrt{\delta} < \frac{\sigma^2}{2\beta_2}$ that $p \geq 2$ is in the interval (p_c, ∞) , $p_c = (1 - \frac{\sqrt{\delta}}{\sigma^2})^{-1}$. (Indeed, $p_c < 2$ if and only if $\sqrt{\delta} < \frac{\sigma^2}{2}$. In particular, $p_c < 2$ if $\sqrt{\delta} < \frac{\sigma^2}{2\beta_2}$ since $\beta_2 > 1$.) Let $\mu(\delta, c_\delta, p)$ be the constant from Proposition 1. Assume that $\mu \geq \mu(\delta, c_\delta, p)$. Then the conclusions of Proposition 1 are valid.

We prove (i) first. We do this in two steps.

Step 1. Selecting κ sufficiently small so that Lemma 3 applies, we obtain that u_m converges in $L^2(\Omega, L_\rho^2)$ to a process u , uniformly in $t \in J_T$. Thus $u \in L^\infty(J_T, L_{\text{loc}}^2(\mathbb{R}^d, L^2(\Omega)))$, and we have for all $t \in J_T$,

$$u_m \rightarrow u \quad \text{in } L^\infty(J_T, L^2(\Omega, L_\rho^2)), \quad (31)$$

which yields

$$\int_0^t u_m ds \rightarrow \int_0^t u ds \quad \text{in } L^2(\Omega, L_\rho^2); \quad (32)$$

the latter, (E_2) and a standard weak compactness argument yield

$$\nabla \int_0^t u_m ds \rightarrow \nabla \int_0^t u ds \quad \text{weakly in } L^2(\Omega, L_\rho^2(\mathbb{R}^d, \mathbb{R}^d)). \quad (33)$$

Step 2. Given a test function $\varphi \in C_c^\infty$, we multiply (19) by $\rho\varphi$, integrate and write (we take $\mu = 0$ to shorten calculations)

$$\begin{aligned} \langle u_m(t) - u(t), \rho\varphi \rangle + \langle u(t), \rho\varphi \rangle - \langle f, \rho\varphi \rangle &= -\langle (b_m - b) \cdot \nabla \int_0^t u_m ds, \rho\varphi \rangle - \langle b \cdot \nabla \int_0^t u_m ds, \rho\varphi \rangle \\ &\quad + \sigma \langle \int_0^t (u_m - u) dB_s, \nabla \rho\varphi \rangle + \sigma \langle \int_0^t u dB_s, \nabla \rho\varphi \rangle \\ &\quad - \frac{\sigma^2}{2} \langle \nabla \int_0^t (u_m - u) ds, \nabla \rho\varphi \rangle - \frac{\sigma^2}{2} \langle \nabla \int_0^t u ds, \nabla \rho\varphi \rangle. \end{aligned} \quad (34)$$

Let us now note the following. In view of (31) and (33), $\langle u_m(t) - u(t), \rho\varphi \rangle \equiv \langle u_m(t) - u(t), \varphi \rangle_\rho \rightarrow 0$ in $L^2(\Omega)$. Similarly, using (33) and (6),

$$\langle \nabla \int_0^t (u_m - u) ds, \nabla \rho\varphi \rangle \rightarrow 0 \quad \text{weakly in } L^2(\Omega), \quad (a)$$

and, since $\varphi|b| \in L_\rho^2$ (using that φ has compact support),

$$\langle b \cdot \nabla \int_0^t u_m ds, \rho\varphi \rangle \rightarrow \langle b \cdot \nabla \int_0^t u ds, \rho\varphi \rangle \quad \text{weakly in } L^2(\Omega). \quad (b)$$

By (E_2) , $\|\nabla \int_0^t u_m ds\|_{L^2(\Omega, L_\rho^2)} \leq c_1$ with $c_1 < \infty$ independent of m , and $\varphi|b_m - b_n| \rightarrow 0$ in L_ρ^2 (in fact, in L^2). Thus

$$\langle (b_m - b) \cdot \nabla \int_0^t u_m ds, \rho\varphi \rangle \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (c)$$

Finally, let us show that

$$\langle \int_0^t (u_m - u) dB_s, \nabla \rho\varphi \rangle \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (d)$$

Indeed, using Itô's isometry, we have using (6)

$$\begin{aligned} \mathbb{E} \left| \left\langle \int_0^t (u_m - u) dB_s, \nabla \rho \varphi \right\rangle \right|^2 &\leq c_2 \mathbb{E} \langle |\int_0^t (u_m - u) dB_s|^2 \rangle_\rho \langle |\varphi|^2 \rangle_\rho \\ &= c_3 \langle \mathbb{E} \int_0^t (u_m - u)^2 ds \rangle_\rho \rightarrow 0 \quad \text{by (31)}. \end{aligned}$$

The convergence (d) follows.

Thus, using (a)-(d), we can pass to the $L^2(\Omega)$ -weak limit in (34) as $m \rightarrow \infty$, obtaining that u satisfies (13) (with test functions $\varphi \rho$ which, clearly, exhaust C_c^∞).

The estimates in (11), (12) now follow from Proposition 1.

The last assertion (ii) is Lemma 3 proved above.

The proof of Theorem 1 is complete. \square

5. PROOF OF THEOREM 2

Proof of Theorem 2. Part (a) follows from Theorem 1(i). The last assertion, (15), follows from Proposition 2 and Lemma 3. So we only need to prove part (b).

Since the weak- $L^2(J_T \times \Omega)$ limit of any sequence of (\mathcal{F}_t) -progressively measurable processes on J_T remains (\mathcal{F}_t) -progressively measurable and $t \mapsto \langle u_m(t), \varphi \rangle$ is (\mathcal{F}_t) -progressively measurable for every m , in view of (32), the process $t \mapsto \langle u(t), \varphi \rangle$ is (\mathcal{F}_t) -progressively measurable as well. The proof of (14) follows closely the proof of (13) above except that now, instead of (E_2) , we appeal to the Sobolev regularity estimate (16) with $q = 1$.

The existence of a continuous (\mathcal{F}_t) -semi-martingale modification of $t \mapsto \langle u(t), \varphi \rangle$ is a consequence of the identity (14).

The proof of Theorem 2 is complete. \square

6. PROOF OF THEOREM 3 (WEAK UNIQUENESS)

The fact that (CP) has at least one weak solution was proved in Theorem 2. We now prove its uniqueness. We adopt the argument of [BFGM, Sect. 3]. We will need the following definitions and results. Let us fix a version of the Brownian motion B_t having continuous trajectories $B_t(\omega)$ for every $\omega \in \Omega$.

Lemma 4. *Let $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma^2}{2\beta_2}$ and $f \in W^{1,4}$. Let $u = u(t, x, \omega)$ be a weak solution to (CP). Then for a.e. $\omega \in \Omega$,*

$$\tilde{u}^\omega(t, x) := u(t, x + \sigma B_t(\omega), \omega)$$

is a weak solution to the Cauchy problem

$$\partial_t \tilde{u}^\omega + \mu \tilde{u}^\omega + \tilde{b}^\omega \cdot \nabla \tilde{u}^\omega = 0, \quad \tilde{u}^\omega|_{t=0} = f, \quad \text{where } \tilde{b}^\omega(t, x) := b(x + \sigma B_t(\omega)), \quad (35)$$

that is, the following are true:

- 1) $\tilde{u}^\omega \in L^\infty(J_T, W_\rho^{1,2})$;
- 2) for every $\psi \in C^1(J_T, C_c^\infty)$, the function $t \mapsto \langle \tilde{u}^\omega(t), \psi(t) \rangle$ has a continuous representative, i.e. a continuous function which coincides with $t \mapsto \langle \tilde{u}^\omega(t), \psi(t) \rangle$ for a.e. $t \in J_T$;

3) for every $\psi \in C^1(J_T, C_c^\infty)$, this continuous representative of $t \mapsto \langle \tilde{u}^\omega(t), \psi(t) \rangle$ satisfies for every $t \in J_T$,

$$\langle \tilde{u}^\omega(t), \psi(t) \rangle = \langle f, \psi(0) \rangle + \mu \int_0^t \langle \tilde{u}^\omega(s), \psi(s) \rangle ds + \int_0^t \langle \tilde{u}^\omega(s), \partial_s \psi(s) \rangle ds - \int_0^t \langle \nabla \tilde{u}^\omega(s), \tilde{b}^\omega(s) \psi(s) \rangle ds.$$

The proof of Lemma 4 follows closely the proof of [BFGM, Prop. 3.4] (taking into account the definition of the weak solution to (CP)) and we omit the details.

Consider the terminal value problem

$$dv_m + \mu v_m dt + \nabla \cdot (b_m v_m) dt + \sigma \nabla v_m \circ dB_t = 0, \quad t \in [0, t_*], \quad v_m|_{t=t_*} = v_0 \in C_c^\infty, \quad (36)$$

where $b_m \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ ($m = 1, 2, \dots$) (since b_m are bounded and smooth, we have strong existence and uniqueness for this equation).

The following is an analogue of [BFGM, Cor. 3.8].

Lemma 5. $\tilde{v}_m^\omega(t, x) := v_m(t, x + \sigma B_t(\omega))$ satisfies, for a.e. $\omega \in \Omega$, $\tilde{v}_m^\omega \in C^1([0, t_*], C_c^\infty)$ and

$$\partial_t \tilde{v}_m^\omega + \mu \tilde{v}_m^\omega + \nabla \cdot (b_m^\omega \tilde{v}_m^\omega) = 0, \quad \tilde{v}_m^\omega(t_*, x) = v_0(x + \sigma B_{t_*}(\omega)).$$

We will also need

Lemma 6. Let $\sqrt{\delta} < \frac{\sigma^2}{6}$. There exist a constant $\mu(c_\delta) \geq 0$ and a sufficiently small $\kappa > 0$ (in the definition of ρ) such that

$$\sup_{t \in J_T} \|\rho^{-1} \mathbb{E}[v_m^2(t)]\|_2 \leq \|\rho^{-1} v_0\|_4^2, \quad \mu \geq \mu(c_\delta), m = 1, 2, \dots$$

where v_m is the strong solution to (36).

Proof. Without loss of generality, we will carry out the proof for the forward equation, and will drop the subscript m from b_m . Set $w := \mathbb{E}[v^2]$. Arguing as in the proof of Proposition 1, we obtain that w satisfies

$$\partial_t w + 2\mu w - \frac{\sigma^2}{2} \Delta w - 2\nabla \cdot (bw) + b \cdot \nabla w = 0, \quad w(0) = v_0^2. \quad (37)$$

We first carry out the proof for $\rho \equiv 1$. Multiplying the previous equation by w and integrating, we obtain

$$\frac{1}{2} \partial_t \langle |w|^2 \rangle + 2\mu \langle |w|^2 \rangle + \frac{\sigma^2}{2} \langle |\nabla w|^2 \rangle + 3 \langle \nabla w, bw \rangle = 0.$$

Applying the quadratic inequality and the form-boundedness condition $b \in \mathbf{F}_\delta$, we get that, for any $\gamma > 0$,

$$\frac{1}{2} \partial_t \langle |w|^2 \rangle + (2\mu - 3\gamma c_\delta) \langle |w|^2 \rangle + \left[\frac{\sigma^2}{2} - 3(\gamma\delta + \frac{1}{4\gamma}) \right] \langle |\nabla w|^2 \rangle \leq 0,$$

and so, selecting $\mu(c_\delta) := \frac{3}{2}\gamma c_\delta$ and $\mu \geq \mu(c_\delta)$, we obtain

$$\frac{1}{2} \langle |w(t)|^2 \rangle + \left[\frac{\sigma^2}{2} - 3(\gamma\delta + \frac{1}{4\gamma}) \right] \int_0^t \langle |\nabla w|^2 \rangle ds \leq \frac{1}{2} \langle |v_0|^4 \rangle.$$

Upon maximizing the coefficient in the square brackets in γ (thus, selecting $\gamma = \frac{1}{2\sqrt{\delta}}$), we obtain that the coefficient is positive since $\sqrt{\delta} < \frac{\sigma^2}{6}$. In particular, it follows that $\sup_{t \in J_T} \|\mathbb{E}[v_m^2(t)]\|_2 \leq \|v_0\|_4^2$.

In presence of ρ^{-1} , we argue as above but get new terms containing $\nabla\rho^{-1}$, which we bound appealing to the estimate

$$|\nabla\rho^{-1}| = \left| \frac{\nabla\rho}{\rho^2} \right| \leq \theta\sqrt{\kappa}\rho^{-1} \quad (\text{by (6)}),$$

with κ selected sufficiently small. (Note that to justify $\|\rho^{-1}\mathbb{E}[v_m^2(t)]\|_2 < \infty$ we can appeal to qualitative Gaussian upper bound on the heat kernel of (37).) \square

Let us note that the assumption of the theorem $\beta_2\sqrt{\delta} < \frac{\sigma^2}{2}$ implies $\sqrt{\delta} < \frac{\sigma^2}{6}$.

We are now in position to complete the proof of Theorem 3.

Proof of Theorem 3. Let μ and κ be as in Lemma 6. In view of the linearity of the stochastic transport equation, it suffices to show that a weak solution u to (CP) with initial condition $u(0) = 0$ must be identically zero for all $t \in J_T$. In view of Lemma 4, it suffices to show that \tilde{u}^ω corresponding to u is identically zero a.s.

Let $v_0 \in C_c^\infty$. It follows from Lemma 5 that, for a.e. $\omega \in \Omega$, $\tilde{v}^\omega(s) \in C^1(J_T, C_c^\infty)$. Thus by Lemma 4, for a.e. $\omega \in \Omega$ with $\psi(s) := \tilde{v}^\omega(s)$, for all $0 < t_* \leq T$,

$$\begin{aligned} & \langle \tilde{u}^\omega(t_*), v_0(\cdot + \sigma B_{t_*}(\omega)) \rangle & (\bullet) \\ &= \mu \int_0^{t_*} \langle \tilde{u}^\omega(s), \tilde{v}_m^\omega(s) \rangle ds + \int_0^{t_*} \langle \tilde{u}^\omega(s), \partial_s \tilde{v}_m^\omega(s) \rangle ds - \int_0^{t_*} \langle \nabla \tilde{u}^\omega(s), \tilde{b}^\omega(s) \tilde{v}_m^\omega(s) \rangle ds \\ &= \int_0^{t_*} \langle \nabla \tilde{u}^\omega, (\tilde{b}_m^\omega(s) - \tilde{b}^\omega(s)) \tilde{v}_m^\omega \rangle ds =: I. \end{aligned}$$

Step 1. Let us first show that

$$\mathbb{E} \left| \int_0^{t_*} \langle \nabla u, (b - b_m)v_m \rangle ds \right| \rightarrow 0 \quad \text{as } m \uparrow \infty. \quad (\bullet\bullet)$$

We have

$$\begin{aligned} & \mathbb{E} \left| \int_0^{t_*} \langle \nabla u, (b - b_m)v_m \rangle ds \right| \leq \int_0^{t_*} \langle |b - b_m| \mathbb{E}[|\nabla u|^2]^{\frac{1}{2}} \mathbb{E}[|v_m|^2]^{\frac{1}{2}} \rangle ds \\ & \leq \left(\int_0^{t_*} \langle \rho |b - b_m|^2 \rangle ds \right)^{\frac{1}{2}} \left(\int_0^{t_*} \langle (\mathbb{E}[|\nabla u|^2])^2 \rangle ds \right)^{\frac{1}{4}} \left(\int_0^{t_*} \langle \rho^{-2} (\mathbb{E}[|v_m|^2])^2 \rangle ds \right)^{\frac{1}{4}}. \end{aligned}$$

The first integral converges to 0 as $m \uparrow \infty$ by Lemma 1(iii), the second integral is finite by the definition of weak solution before Theorem 3, and the third integral is bounded from above uniformly in m by $\sqrt{t_*} \|\rho^{-1}v_0\|_4^2 < \infty$, see Lemma 6. Thus, $(\bullet\bullet)$ follows.

Step 2. By Step 1, there exists a subset $\Omega_{t_*, v_0} \subset \Omega$ of probability 1 and a sequence $m_k \uparrow \infty$ such that for every $\omega \in \Omega_{t_*, v_0}$,

$$\int_0^{t_*} \langle \nabla u, (b - b_{m_k})v_{m_k} \rangle ds \rightarrow 0 \quad \text{as } m_k \uparrow \infty.$$

Making the change of variable $x \mapsto x + \sigma B_t(\omega)$ and using the fact that $c_{t_*, w}^{-1} \rho(\cdot) \leq \rho(\cdot + \sigma B_t(\omega)) \leq c_{t_*, w} \rho(\cdot)$ for some constant $c_{t_*, w} > 1$ we obtain that for every $\omega \in \Omega_{t_*, v_0}$,

$$I \rightarrow 0 \quad \text{as } m_k \uparrow \infty.$$

Fix a countable dense subset D of $C_c^\infty(\mathbb{R}^d)$ and define

$$\tilde{\Omega} := \bigcap_{t_* \in [0, T] \cap \mathbb{Q}, v_0 \in D} \Omega_{t_*, v_0},$$

a full measure set in Ω . Applying the diagonal argument (and so passing to a subsequence of $\{\varepsilon_k\}$), we obtain by (\bullet) and Step 2 that for every $\omega \in \tilde{\Omega}$, $\tilde{u}^\omega(t) = 0$ for all $t \in [0, T] \cap \mathbb{Q}$. Since $t \mapsto \langle \tilde{u}^\omega(t), \varphi \rangle$, $\varphi \in C_c^\infty(\mathbb{R}^d)$ is continuous, we obtain that $\tilde{u}^\omega(t) = 0$ for all $t \in [0, T]$ for all $\omega \in \tilde{\Omega}$, as needed.

The proof of Theorem 3 is complete. \square

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