REGULARITY FOR PARABOLIC EQUATIONS WITH SINGULAR NON-ZERO DIVERGENCE VECTOR FIELDS

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ABSTRACT. We establish two-sided Gaussian bounds on the heat kernel of divergence-form parabolic equation with singular time-inhomogeneous vector field satisfying some minimal assumptions.

1. INTRODUCTION

In this paper we study regularity properties of solutions of parabolic equation

$$(\partial_t - \nabla \cdot a \cdot \nabla + b \cdot \nabla)u = 0, \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \quad d \geq 3, \tag{1}$$

under broad assumptions on a singular (that is, locally unbounded) time-inhomogeneous vector field $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$. Here and everywhere below, $\mathbb{R}_+ := [0, \infty[$,

$$\nabla \cdot a \cdot \nabla u(x) := \sum_{i,j=1}^d \nabla_i (a_{ij} \nabla_j u(x)), \quad b \cdot \nabla u(x) := \sum_{k=1}^d b_k \nabla_k u(x).$$

The matrix $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is symmetric, uniformly elliptic, i.e.

$$\sigma I \leq a(t, x) \leq \xi I \quad \text{for some } \xi, \sigma > 0 \text{ for a.e. } (t, x), \tag{$H_{\sigma,\xi}$}$$

and, other than that, is assumed to be only measurable.

In the absence of drift term $b \cdot \nabla$, already the hypothesis $a \in H_{\sigma,\xi}$ provides local Hölder continuity of solutions to (1). This is known since the works of E. De Giorgi [DG] and J. Nash [N]. Moreover, the corresponding heat kernel satisfies two-sided Gaussian bounds, see Aronson [Ar], or see [N] and Fabes-Stroock [FS].

However, with non-zero $b \cdot \nabla$, the regularity theory of (1) is not yet complete even in the case $a = I$. Some particularly important classes of vector fields b are the divergence-free vector fields (i.e. $\operatorname{div} b = 0$) and, more generally, vector fields that have singular divergence $\operatorname{div} b \in L^1_{\text{loc}}$. Parabolic equations with such vector fields, which are in the focus of this paper, arise in hydrodynamics and other physical applications. We study one of the central questions in the regularity theory of (1): what are the minimal assumptions on b so that the heat kernel of (1) admits upper and/or lower Gaussian bounds? Such bounds, once established, open up the way to proving a number of other regularity results for (1).

The present paper continues [KS1] where we dealt with time-homogeneous a, b and investigated how the lower and the upper Gaussian heat kernel bounds depend on the divergence of b. This

paper also continues [S] where two-sided Gaussian bounds for divergence-free time-inhomogeneous b were established.

In [KS1] and [S], the proof of two-sided Gaussian bounds uses a condition on $|b| := (\sum_{k=1}^d b_k^2)^{1/2}$ (see Definition 1.2). This condition, although quite general, does not allow to take into account possible oscillations in vector field b. In the present paper we handle oscillations in b and also allow b to have time-inhomogeneous singular divergence. This required a comprehensive modification of the argument from the two cited papers. Possible oscillations of vector fields are taken into account using the class $L^\infty\mathrm{BMO}^{-1}$ of divergence-free vector fields whose components are distributional derivatives of functions of bounded mean oscillation (we recall the definition in the next section). The results on the regularity theory of equation (1) with $b \in L^\infty\mathrm{BMO}^{-1}$ include, in particular, the parabolic Harnack inequality established, among other results, in Seregin-Silvestre-Šverak-Zlatos [SSSZ], and global in time two-sided Gaussian bounds proved by Qian-Xi in [QX1]. Their Gaussian bounds in the a priori form follow as a special case of our Theorems 2.2, 2.3. Earlier, Osada [O] proved two-sided Gaussian heat kernel bounds for b in $L^\infty(L_\infty^{-1}) \subset L^\infty\mathrm{BMO}^{-1}$, where L_∞^{-1} is the class of divergence-free vector fields whose components are distributional derivatives of functions in L^∞.

Let us now state our hypothesis on b and describe our results. For notations, see Section 1.1.

DEFINITION 1.1. We say that a distributional vector field $b : \mathbb{R}_+ \to [\mathcal{S}']^d$ belongs to class $\mathbf{M}_\delta$ if for a.e. $t \in \mathbb{R}_+$

$$|\langle b(t)\psi, \psi\rangle| \le \delta\|\nabla\psi\|_2\|\psi\|_2 + g(t)\|\psi\|_2^2 \tag{2}$$

for all ψ in the L. Schwartz class $\mathcal{S}$, for a non-negative function g that satisfies, on every finite interval $[0, T]$,

$$\int_s^t g(\tau)d\tau \le c\sqrt{t - s}, \quad s, t \in [0, T] \tag{3}$$

for a constant $c = c_{\delta,T}$ (written as $b \in \mathbf{M}_\delta$, "multiplicative class").

The multiplicative class $\mathbf{M}_\delta$ will also appear in another, more restrictive form:

DEFINITION 1.2. A vector field $b \in [L_{\mathrm{loc}}^1(\mathbb{R}_+ \times \mathbb{R}^d)]^d$ belongs to class $\mathbf{MF}_\delta$ if for a.e. $t \in \mathbb{R}_+$

$$\langle |b(t)|\psi, \psi\rangle \le \delta\|\nabla\psi\|_2\|\psi\|_2 + g(t)\|\psi\|_2^2, \quad \psi \in W^{1,2} \tag{4}$$

under the same assumption on g as in the previous definition (written as $b \in \mathbf{MF}_\delta$, multiplicatively form-bounded vector fields).

Note that both classes $\mathbf{M}_\delta$ and $\mathbf{MF}_\delta$ are closed with respect to addition (up to change of δ).

EXAMPLES. The following vector fields b are in $\mathbf{MF}_\delta$ for appropriate δ.

1. Vector fields in the critical Ladyzhenskaya-Prodi-Serrin class:

$$|b| \in L^s(\mathbb{R}_+, L^r), \quad \frac{d}{r} + \frac{2}{s} \le 1, \quad 2 \le s \le \infty, \ d \le r \le \infty. \tag{LPS$_c$}$$

2. Vector fields b that belong, uniformly in $t \in \mathbb{R}$, to the scaling-invariant Morrey class $M_{1+\varepsilon}$ with $\varepsilon > 0$ fixed arbitrarily small:

$$\sup_{t\in\mathbb{R}}\|b(t)\|_{M_{1+\varepsilon}} = \sup_{t\in\mathbb{R}}\sup_{r>0,x\in\mathbb{R}^d} r\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|b(t,y)|^{1+\varepsilon}dy\right)^{\frac{1}{1+\varepsilon}} < \infty \tag{5}$$

where $B_r(x)$ is the ball of radius r centered at x, and ε is fixed arbitrarily small.

See proofs of inclusions in Appendix C.

3. If $g \in L^2_{\text{loc}}(\mathbb{R}_+)$, then condition (3) obviously holds. Another example is

$$g(t) = |t|^{-\frac{1}{2}},$$

which still satisfies (3) but is not in $L^2_{\text{loc}}(\mathbb{R}_+)$.

The function g describes singularities of the vector field b in the time variable. In particular, if b is bounded in the spatial variables, then condition (3) says that $|b|$ can have at most $t^{-\frac{1}{2}}$ singularities in time, so the speed at which the streaming induced by b displaces the solution is comparable to the speed at which the viscosity diffuses it. In this regard, see Carlen-Loss [CL] who showed that the velocity fields in the Burgers and 2D Navier-Stokes vorticity equations have at most $t^{-1/2}$ time singularities. Let us add that if one is dealing with conditions on b that are invariant with respect to the parabolic scaling, such as $b \in \mathbf{M}_\delta$ (so that there is a hope of arriving at the regularity results for parabolic equation (1) comparable to those for the heat equation, including two-sided Gaussian bounds), then one cannot extend $\|b(t)\|_\infty \leq ct^{-1/2}$ to $\|b(t)\|_\infty \leq ct^{-1/2-\varepsilon}$ regardless of how small one fixes $\varepsilon > 0$ without destroying the parabolic scaling. See also [QX2] in this regard.

In Theorem 2.2, we prove a priori upper Gaussian bound on the heat kernel of (1) assuming that $b \in \mathbf{M}_\delta$ for some finite δ and that the positive part $(\operatorname{div} b)_+$ of divergence $\operatorname{div} b$ has sufficiently small Kato norm (cf. (19)). Here "a priori" refers to the fact that Theorem 2.2 is proved for smooth a, b, but the constants in the upper Gaussian bound do not depend on the smoothness of a and b.

In Theorem 2.3, we prove lower Gaussian bound on a heat kernel of (1) under more restrictive assumption on the vector field:

$$b = \hat{b} + \tilde{b}$$

where $\hat{b} \in \mathbf{MF}_\delta$ for some $\delta < \infty$ and $g \in L^2(\mathbb{R}_+)$, with $\operatorname{div} \hat{b}$ having sufficiently small Kato norm, and divergence-free $\tilde{b}$ is in the class $L^\infty\text{BMO}^{-1}$.

The fact that the assumptions of Theorem 2.2 are broader than the assumptions of Theorem 2.3 is seen from the straightforward inclusion $\mathbf{MF}_\delta \subset \mathbf{M}_\delta$ and the following proposition.

Proposition 1.1. *If $\tilde{b} \in L^\infty\text{BMO}^{-1}$, then $\tilde{b} \in \mathbf{M}_\delta$.*

The class $\mathbf{M}_\delta$ is the largest class of vector fields considered in this paper.

The question of uniqueness of the heat kernel of (1) under the assumption $b \in \mathbf{M}_\delta$ is non-trivial. It is known that the heat kernel is unique if vector field b belongs to $L^\infty\text{BMO}^{-1}$, a particular sub-class of $\mathbf{M}_\delta$. This is a result of Qian-Xi [QX1], which they proved using the Lions' approach in the standard for equation (1) triple of Hilbert spaces $W^{1,2} \subset L^2 \subset W^{-1,2}$. See detailed statement below. In this paper we single out another important sub-class of $\mathbf{M}_\delta$ for which one can prove uniqueness of the heat kernel, although only in the case $a = I$.

DEFINITION 1.3. A vector field $b \in [L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d$ is said to be weakly form-bounded (written as $b \in L^\infty \mathbf{F}^{1/2}_\delta$) if, for a.e. $t \in \mathbb{R}_+$,

$$\||b(t)|^{\frac{1}{2}}\psi\|_2 \leq \sqrt{\delta}\|(\lambda - \Delta)^{\frac{1}{4}}\psi\|_2 \tag{6}$$

for all $\psi \in \mathcal{W}^{\frac{1}{2},2} := (\lambda - \Delta)^{-\frac{1}{4}}L^2$ (Bessel potential space) for some constant $\lambda = \lambda_\delta \geq 0$.

We have

$$L^\infty \, \mathbf{F}_\delta^{1/2} \subset \mathbf{MF}_\delta. \tag{7}$$

Indeed, if $b \in L^\infty \, \mathbf{F}_\delta^{1/2}$ and $\psi \in W^{1,2}$ then

$$\langle |b(t)| \psi, \psi \rangle \le \delta \langle (\lambda - \Delta)^{\frac{1}{4}} \psi, (\lambda - \Delta)^{\frac{1}{4}} \psi \rangle = \delta \langle (\lambda - \Delta)^{\frac{1}{2}} \psi, \psi \rangle$$

$$\le \delta \|(\lambda - \Delta)^{\frac{1}{2}} \psi\|_2 \|\psi\|_2 = \delta \sqrt{\|\nabla \psi\|_2^2 + \lambda \|\psi\|_2^2} \|\psi\|_2 \le \delta \|\nabla \psi\|_2 \|\psi\|_2 + \delta \sqrt{\lambda} \|\psi\|_2,$$

so $b \in \mathbf{MF}_\delta$ with $g(t) = \delta \sqrt{\lambda}$.

The classes of vector fields mentioned in Examples 1 and 2 above are all contained in $L^\infty \, \mathbf{F}_\delta^{1/2}$.

The class $L^\infty \, \mathbf{F}_\delta^{1/2}$ is quite different from $L^\infty \mathrm{BMO}^{-1}$. Indeed, the proof of uniqueness of the heat kernel of the parabolic equation

$$(\partial_t - \Delta + b \cdot \nabla) u = 0 \tag{8}$$

with $b \in L^\infty \, \mathbf{F}_\delta^{1/2}$ requires one to run Lions' approach in a non-standard triple of Hilbert spaces

$$\mathcal{W}^{\frac{3}{2}, 2} \subset \mathcal{W}^{\frac{1}{2}, 2} \subset \mathcal{W}^{-\frac{1}{2}, 2}.$$

In particular, the solution of a Cauchy problem for (8) satisfies different energy inequalities depending on whether b is in $L^\infty \, \mathbf{F}_\delta^{1/2}$ or in $L^\infty \mathrm{BMO}^{-1}$, cf. (14) and (15). Put another way, the Sobolev embedding properties of $-\Delta + b \cdot \nabla$ change drastically as one transitions from $L^\infty \mathrm{BMO}^{-1}$ to $L^\infty \mathbf{F}_\delta^{1/2}$. This allows us to conclude that these two subclasses of $\mathbf{M}_\delta$ are quite far apart.

Let us emphasize that the lower bound in Theorem 2.3 is new even in the case $a = I$, $b = b(x)$.

Concerning the lower and upper Gaussian bounds for time-homogeneous singular drifts having singular divergence, we refer, in addition to [KS1], to earlier results by Liskevich-Zhang [LZ] who also considered form-boundedness and Kato class conditions on $|b|$ and $\mathrm{div}\, b$. See detailed comparison of the results in [KS1, Sect. 1].

The proof of the upper bound in Theorem 2.2 uses the Moser iteration method. The proof of the lower bound in Theorem 2.3 uses a rather non-standard version of Nash's method [N]. More precisely, the assumption $b \in L^\infty \mathbf{F}_\delta^{1/2}$ prohibits, even if $\mathrm{div}\, b = 0$, the use of quadratic inequalities in the analysis of the Nash G-functions in the proof of the a priori lower Gaussian bound (Theorem 6.1). As a consequence, one needs to use a relatively sophisticated regularization of the Nash G-functions. Arguably, $L^\infty \mathbf{F}_\delta^{1/2}$ is the "hard part" of $\mathbf{M}_\delta$.

In this paper, the proof of the lower bound uses the upper bound. We mention, however, [KS1, Theorem 1], where it was demonstrated that these bounds are, in principle, independent. Namely, a lower Gaussian bound on the heat kernel of (1) (with time-homogeneous coefficients) holds provided that b is form-bounded with $\delta < 4\sigma^2$ and $\mathrm{div}\, b \ge 0$ (in the sense of distributions), in which case an upper Gaussian bound is in general invalid. Conversely, there are situations where an upper Gaussian bound holds but there are no lower Gaussian bounds, see [KS1] for details.

It should be added that we prove global in time heat kernel bounds, so our conditions on g are global, but it is straightforward to make them local in time with local conditions on g as above.

Let us note that the classes of singular vector fields discussed in this paper are scaling-invariant. It is known that one can venture beyond the scaling-invariance, considering (8) with $b \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\operatorname{div} b = 0$ such that, for some $0 < a \leq 1$ and $\delta < \infty$,

$$\int_0^\infty \langle |b(t, \cdot)|^{1+a} \varphi^2(t, \cdot) \rangle dt \leq \delta \int_0^\infty \|\nabla \varphi(t, \cdot)\|_2^2 dt, \quad \forall \varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d),$$

containing e.g. zero-divergence $b = b(x)$ with $|b| \in L^p$, $p > \frac{d}{2}$ (essentially, twice more singular than (8)), see [Z1]. See also [QX2]. Although in this case one has to sacrifice much of the regularity theory of (8) and (1), some results can be salvaged. This includes boundedness of weak solutions, a non-Gaussian upper bound, see cited papers for details.

1.1. Notations and auxiliary results.
Let $L^p = L^p(\mathbb{R}^d)$, $p \geq 1$ denote the standard Lebesgue space with norm $\| \cdot \|_p$, $W^{1,p} = W^{1,p}(\mathbb{R}^d)$ the Sobolev spaces, $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ the space of Schwartz distributions.

Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators between Banach spaces $X \to Y$ with operator norm $\| \cdot \|_{X \to Y}$. Let $\mathcal{B}(X) := \mathcal{B}(X, X)$. Set $\| \cdot \|_{p \to q} := \| \cdot \|_{L^p \to L^q}$.

Given a $d \times d$ matrix $P = (P_{ij})_{i,j=1}^d$ with entries in X, we set $\|P\|_X := \left(\sum_{i,j=1}^d \|P_{ij}\|_X^2 \right)^{\frac{1}{2}}$.

Put

$$\langle f \rangle := \int_{\mathbb{R}^d} f dx, \quad \langle f, g \rangle := \langle f \bar{g} \rangle.$$

The following class is well known:

DEFINITION. A vector field $b : \mathbb{R}_+ \to [\mathcal{S}']^d$ with $\operatorname{div} b = 0$ is said to be in the class $L^\infty \mathrm{BMO}^{-1}$ if

$$b_k(t) = \sum_{i=1}^d \nabla_i B_{ik}(t), \quad t \in \mathbb{R}_+, \tag{9}$$

for some skew-symmetric matrix B with entries $B_{ik} \in L^\infty \mathrm{BMO} \equiv L^\infty(\mathbb{R}_+, \mathrm{BMO})$.

The class $L^\infty \mathrm{BMO}^{-1}$ is endowed with semi-norm

$$\|b\|_{L^\infty \mathrm{BMO}^{-1}} = \|B\|_{L^\infty \mathrm{BMO}} := \sup_{t \in \mathbb{R}_+} \|B(t)\|_{\mathrm{BMO}}.$$

Here BMO is the space of functions of bounded mean oscillation on $\mathbb{R}^d$; recall that a function $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is of bounded mean oscillation if

$$\|F\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |F - \bar{F}| dx < \infty, \quad \text{where } \bar{F} := \frac{1}{|Q|} \int_Q F dx.$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^d$ with sides parallel to the axes, $|Q|$ is the volume of Q.

If b is time independent, then we write simply $b \in \mathrm{BMO}^{-1}$. (Occasionally, we will be adding "$\operatorname{div} b = 0$" to make the paper easier to follow although this is redundant.)

As it was demonstrated in [QX1], the functions B_{ik} can always be modified to be in $L^q_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ for all $1 \leq q < \infty$ (by adding functions that only depend on t). We assume in what follows that this modification has been made.

Proposition 1.2 ([CLMS, Theorem 4]). *Let* $b \in \mathrm{BMO}^{-1}$, $\operatorname{div} b = 0$. *Then, for all* $u, v \in W^{1,2}$,

$$|\langle b \cdot \nabla u, v \rangle| \leq \|B\|_{\mathrm{BMO}} \|\nabla u\|_2 \|\nabla v\|_2.$$

Proposition 1.3 ([QX1, Prop. 3.2])**.** *Given a* $f \in W^{1,2}$*, one has* $|f|\nabla_i|f| \in \mathcal{H}_1$*, where* $\mathcal{H}_1$ *is the real Hardy space, and*

$$\||f|\nabla_i|f|\|_{\mathcal{H}_1} \leq C\|\nabla f\|_2\|f\|_2, \quad C = C(d).$$

DEFINITION. $\operatorname{div} b \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ is said to be form-bounded (with form-bound $\nu < \infty$) if

$$\big\||\operatorname{div} b(t)|^{\frac{1}{2}}\psi\big\|_2^2 \leq \nu\|\nabla\psi\|_2^2 + h(t)\|\psi\|_2^2 \quad \forall\psi \in W^{1,2} \tag{10}$$

for a.e. $t \in \mathbb{R}_+$, for some function $0 \leq h \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

2. MAIN RESULTS

We start with the basic results of the well-posedness of Cauchy problem for equations (8) and (1) without any assumptions on $\operatorname{div} b$.

Instead of (8), it will be convenient to work with equation

$$(\partial_t + \lambda - \Delta + b \cdot \nabla)u = 0, \tag{11}$$

where λ is from the condition $b \in L^\infty \, \mathbf{F}_\delta^{1/2}$. In this regard, we introduce the scale of Bessel potential spaces $\mathcal{W}^{\alpha,2}$ endowed with the norm

$$\|v\|_{\mathcal{W}^{\alpha,2}} := \|(\lambda - \Delta)^{\frac{\alpha}{2}}v\|_2.$$

Assertion (*ii*) in theorem below in the case case $b \in L^\infty \mathrm{BMO}^{-1}$ is due to [QX1, Theorem 5.2]. We included it for the sake of completeness. Regarding the elliptic setting, see [Zh, Theorem 3.1].

Theorem 2.1. *Let* $d \geq 3$*,* $T > 0$*. The following is true:*
(*i*) *Let* b *be weakly form-bounded:*

$$b \in L^\infty \, \mathbf{F}_\delta^{1/2} \quad \text{with } \delta < 1 \quad (\text{see } (6)).$$

Then for every $f \in \mathcal{W}^{\frac{1}{2},2}$ *there exists a unique weak solution to Cauchy problem for* (11) *with initial condition* $u(s+) = f$*, i.e. a unique in* $L^\infty_{\mathrm{loc}}(]s,T[,\mathcal{W}^{\frac{1}{2},2}) \cap L^2_{\mathrm{loc}}(]s,T[,\mathcal{W}^{\frac{3}{2},2})$ *function* u *satisfying*

$$\int_s^T \langle(\lambda - \Delta)^{\frac{1}{4}}u, \partial_t(\lambda - \Delta)^{\frac{1}{4}}\varphi\rangle dt = \int_s^T \langle(\lambda - \Delta)^{\frac{3}{4}}u, (\lambda - \Delta)^{\frac{3}{4}}\varphi\rangle dt$$

$$+ \int_s^T \langle b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}}\varphi\rangle \tag{12}$$

for all $\varphi \in C_c^\infty(]s,T[,\mathcal{S})$ *and*

$$w\text{-}\mathcal{W}^{\frac{1}{2},2}\text{-}\lim_{t\downarrow s} u(t) = f. \tag{13}$$

Furthermore, $u \in C([s,T],\mathcal{W}^{\frac{1}{2},2})$*, and the following energy inequality holds:*

$$\|u(t)\|_{\mathcal{W}^{\frac{1}{2},2}}^2 + 2(1-\delta)\int_s^t \|u(\tau)\|_{\mathcal{W}^{\frac{3}{2},2}}^2 d\tau \leq \|f\|_{\mathcal{W}^{\frac{1}{2},2}}^2, \quad 0 \leq s < t \leq T. \tag{14}$$

The operators $T^{t,s}f(x) := u(t,s,x)$ *constitute a contraction strongly continuous Markov evolution family in* $\mathcal{W}^{\frac{1}{2},2}$*. If* $\{b_\varepsilon\}_{\varepsilon>0}$ *is a family of bounded smooth vector fields such that* $b_\varepsilon \in L^\infty \, \mathbf{F}_\delta^{1/2}$ *with*

the same λ as b, $b_\varepsilon \to b$ in $[L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d$ as $\varepsilon \to 0$ (see example of such vector fields in Proposition 4.3), and if u_ε denotes the solution to Cauchy problem (11), (13) with the vector field b_ε, then

$$u_\varepsilon \to u \quad \text{weakly in } L^2([s,T], \mathcal{W}^{\frac{3}{2},2}) \text{ as } \varepsilon \to 0.$$

(ii) If either ($|b| \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\operatorname{div} b$ is form-bounded with form-bound $\nu < 1$) or ($b \in L^\infty \mathrm{BMO}^{-1}$, $\operatorname{div} b = 0$), then for every $f \in L^2$ there exists a unique weak solution u to the corresponding Cauchy problem for (1) (in the standard triple $W^{1,2} \subset L^2 \subset W^{-1,2}$), which satisfies the classical energy inequality

$$\|u(t)\|_2^2 + c\int_s^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|f\|_2^2, \quad c > 0, \quad 0 \leq s < t. \tag{15}$$

The notion of weak solution to equation (11) in assertion (i) is obtained by formally multiplying (11) by test function $(\lambda - \Delta)^{\frac{1}{2}}\varphi$ and integrating. The last term in (12) is well defined by Proposition 4.1.

The proof of Theorem 2.1(i) uses the Lions variational approach in the triple of Bessel potential spaces $\mathcal{W}^{\frac{3}{2},2} \subset \mathcal{W}^{\frac{1}{2},2} \subset \mathcal{W}^{-\frac{1}{2},2}$. Theorem 2.1$(i)$ can be viewed as the first step towards a regularity theory of (8) with weakly form-bounded b. In the time-homogeneous case $b = b(x)$, the class (6) provides sharp $L^p \to L^q$ bounds on the corresponding to (8) semigroup [S], a detailed Sobolev regularity theory of elliptic operator $-\Delta + b \cdot \nabla$ in L^p for p large and the corresponding Feller semigroup [K], see discussion in Section 8. The latter determines, for every initial point $x \in \mathbb{R}^d$, a "sequentially unique" weak solution to the SDE

$$X_t = x - \int_0^t b(X_s)ds + \sqrt{2}B_t, \quad t \geq 0, \tag{16}$$

where B_t is the standard d-dimensional Brownian motion, $x \in \mathbb{R}^d$ is the initial point, see [KS3]. (See, however, recent developments in [K2] regarding time-inhomogeneous b.)

Remark 2.1. One can extend the definition of the class $L^\infty \mathbf{F}_\delta^{1/2}$ by considering $b \in [L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d$ such that

$$\||b(t)|^{\frac{1}{2}}\psi\|_2^2 \leq \delta\|(\lambda - \Delta)^{\frac{1}{4}}\psi\|_2^2 + g(t)\|\psi\|_2^2, \quad \psi \in (\lambda - \Delta)^{-\frac{1}{4}}L^2, \tag{17}$$

where $0 \leq g \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, see Appendix A for details.

Next, we turn to the question of what assumptions on locally unbounded b provide upper and lower Gaussian bounds on the heat kernel of equation (1). Compared to the previous theorem, we will weaken the assumption on the vector field b even further to $b \in \mathbf{M}_\delta$, in particular taking into account possible cancellations, but requiring the existence of $\operatorname{div} b \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and, moreover, the smallness of the Kato norms of its positive and/or negative parts $(\operatorname{div} b)_+$, $(\operatorname{div} b)_-$. These assumptions allow b and $\operatorname{div} b$ to be quite singular.

Put

$$k_c(t,x,y) \equiv k(ct,x,y) := (4\pi ct)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4ct}}, \quad c > 0.$$

Theorem 2.2. *Let $d \geq 3$. Let $a \in H_{\sigma,\xi}$, $b \in \mathbf{M}_\delta$ with multiplicative bound $\delta < \infty$ and function g satisfying*

$$\int_s^t g(\tau)d\tau \leq c_\delta\sqrt{t-s}, \quad 0 \leq s < t < \infty \tag{18}$$

for some constant $c_\delta \geq 0$. Let μ_+ denote the global in time Kato norm of "potential" $(\operatorname{div} b)_+$, i.e. the maximum between

$$\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^t \langle k(t - \tau, x, \cdot)(\operatorname{div} b)_+(\tau, \cdot)\rangle d\tau, \qquad \sup_{s \geq 0, x \in \mathbb{R}^d} \int_s^\infty \langle k(\tau - s, x, \cdot)(\operatorname{div} b)_+(\tau, \cdot)\rangle d\tau. \qquad (19)$$

Also, assume that a, b, $\operatorname{div} b$ are bounded smooth.

The following is true. If μ_+ is smaller than a certain generic constant (i.e. a constant that depends only on d, σ, ξ, δ, c_δ, but not on the smoothness of a or the boundedness and smoothness of b, $\operatorname{div} b$), then the heat kernel $u(t, x; s, y)$ of equation (1) satisfies a global in time upper Gaussian bound

$$u(t, x; s, y) \leq c_3 k_{c_4}(t - s; x - y) \quad \text{for all } 0 \leq s < t < \infty, \ x, y \in \mathbb{R}^d$$

with generic constants c_i, $i = 3, 4$ that can also depend on μ_+.

If the Kato norm μ_+ of $(\operatorname{div} \hat{b})_+$ is finite, as in the theorem above, then we say that $(\operatorname{div} \hat{b})_+$ belongs to the Kato class.

Remark 2.2. Regarding the above condition on the smallness of the Kato norm μ_+ of $(\operatorname{div} b)_+$, recall that if $(\operatorname{div} b)_+ \in \mathbf{1}_{\{0 \leq t \leq 1\}} L^\infty(\mathbb{R}_+, L^p)$, $p > \frac{d}{2}$, then μ_+ can be chosen arbitrarily small. There also exist $(\operatorname{div} b)_+(x)$ that are not even in L^p_{loc} for any $p > 1$ yet have μ_+ finite (or sufficiently small upon multiplying b by a small constant).

In the next theorem we assume that $a \in H_{\sigma, \xi}$ and b are only measurable, but b satisfies a more restrictive condition:

$$b = \tilde{b} + \hat{b}, \qquad (20)$$

where

$$\tilde{b} \in L^\infty \operatorname{BMO}^{-1}, \quad \operatorname{div} \tilde{b} = 0, \qquad (21)$$

$$\hat{b} \in \mathbf{MF}_\delta \text{ with } g \in L^2(\mathbb{R}_+), \qquad (22)$$

$(\operatorname{div} \hat{b})_\pm \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ belong to the Kato class with Kato norms $\mu_\pm$, respectively, and   (23)

$|(\operatorname{div} \hat{b})_\pm|$ are form-bounded: $\langle |(\operatorname{div} \hat{b})_\pm(t)|\psi, \psi\rangle \leq \nu_\pm \|\nabla \psi\|_2^2 + h_\pm(t)\|\psi\|_2^2, \quad \forall \psi \in W^{1,2}$
for a.e. $t \in \mathbb{R}_+$ for some $0 \leq h_\pm \in L^1(\mathbb{R}_+)$.   (24)

DEFINITION 2.1. Assume that (20)-(24) hold. We say that $v \in L^2([s, T], W^{1,2})$ is an approximation solution to Cauchy problem for equation (1) with initial condition

$$s\text{-}L^2\text{-}\lim_{t \downarrow s} v(s) = f \in L^2 \qquad (25)$$

if

$$v = w\text{-}L^2_{\text{loc}}([s, \infty[, W^{1,2})\text{-}\lim_{\varepsilon \downarrow 0} \lim_{\varepsilon_1 \downarrow 0} v_{\varepsilon_1, \varepsilon},$$

where $v_{\varepsilon_1, \varepsilon}$ solve

$$(\partial_t - \nabla \cdot a_{\varepsilon_1} \cdot \nabla + b_\varepsilon \cdot \nabla)v_{\varepsilon_1, \varepsilon} = 0, \quad v_{\varepsilon_1, \varepsilon}(s) = f$$

for some bounded smooth $a_{\varepsilon_1} \in H_{\sigma,\xi}$, $\hat{b}_\varepsilon$ and $(\operatorname{div} \hat{b})_{\pm,\varepsilon} \geq 0$ that have the same multiplicative bound δ, g and the form-bounds $\nu_\pm$, the Kato norms $\mu_\pm$, respectively, and bounded smooth $\tilde{b}_\varepsilon = \nabla B_\varepsilon$ with the property

$$\|\tilde{b}_\varepsilon\|_{L^\infty \operatorname{BMO}^{-1}} \leq C \|\tilde{b}\|_{L^\infty \operatorname{BMO}^{-1}}$$

for a constant C independent of ε and B_ε bounded smooth skew-symmetric matrices with entries in $L^\infty \operatorname{BMO}$, such that

$$
\begin{aligned}
a_{\varepsilon_1} &\to a & \text{in} \quad & [L^2_{\operatorname{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^{d \times d}, \\
\hat{b}_\varepsilon &\to \hat{b} & \text{in} \quad & [L^1_{\operatorname{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d, \\
B_\varepsilon &\to B & \text{in} \quad & [L^2_{\operatorname{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^{d \times d}, \\
(\operatorname{div} \hat{b})_{\pm,\varepsilon} &\to (\operatorname{div} \hat{b})_\pm & \text{in} \quad & L^1_{\operatorname{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)
\end{aligned}
\tag{26}
$$

as $\varepsilon_1, \varepsilon \downarrow 0$.

It is easy to see that if v is an approximation solution to (1), then it is also a weak solution to (1):

$$-\int_s^t \langle v, \partial_r \varphi \rangle dr + \int_s^t \langle (a+B) \cdot \nabla v, \nabla \varphi \rangle dr - \int_s^t \langle \hat{b}v, \nabla \varphi \rangle dr - \int_s^t \langle (\operatorname{div} \hat{b})v, \varphi \rangle dr = 0 \tag{27}$$

for all $\varphi \in C_c^\infty(]s,t[, \mathcal{S})$. Here $\tilde{b} = \nabla B$ where B is skew-symmetric in $L^\infty \operatorname{BMO} \cap L^2_{\operatorname{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$, see Section 1.1.

DEFINITION 2.2. We call a constant generic* if it depends only on

$$d, \sigma, \xi, \delta, \|g\|_{L^2(\mathbb{R}_+)}, \nu_\pm, \|h_\pm\|_{L^1(\mathbb{R}_+)} \text{ and } \|\tilde{b}\|_{L^\infty \operatorname{BMO}^{-1}}.$$

Theorem 2.3. *Let $d \geq 3$, $a \in H_{\sigma,\xi}$. There exist generic* constants $\mu_\pm^*$ such that if (20)-(24) hold with*

$$\delta < \infty, \quad \nu_\pm < \sigma \quad \text{and with Kato norms } \mu_\pm < \mu_\pm^*,$$

then there exists a Hölder continuous heat kernel $u(t,x;s,y)$ of equation (1) satisfying:

(a) *A global in time lower Gaussian bound*

$$c_1 k_{c_2}(t-s; x-y) \leq u(t,x;s,y)$$

holds, in addition to the upper bound in Theorem 2.2, for all $0 \leq s < t < \infty$, x, $y \in \mathbb{R}^d$, with generic constants c_1-c_4 that can also depend on $\mu_\pm$.*

(b) *The function*

$$v(t,x) := \langle u(t,x;s,\cdot)f(\cdot) \rangle, \quad f \in L^2,$$

is an approximation solution to Cauchy problem for (1).

(c) *The operators $T^{t,s}f := \langle u(t,x;s,\cdot)f(\cdot) \rangle$ determine a quasi bounded strongly continuous Feller evolution family of integral operators in $\mathcal{B}(X)$, $X = L^p$, $1 \leq p < \infty$ or $X = C_u$. The heat kernel $u(t,x;s,y)$ is defined as the integral kernel of these operators, possibly after a modification on a measure zero set.*

(d) *If either*

$$f \in L^2, \quad a \in H_{\sigma,\xi}, \quad b = \tilde{b} \tag{d_1}$$

or

$$f \in \mathcal{W}^{\frac{1}{2},2}, \quad a = I, \quad b = \hat{b} = \hat{b}^{(1)} + \hat{b}^{(2)} \tag{d_2}$$

such that, for a.e. $t \in \mathbb{R}_+$,

$$\||\hat{b}^{(1)}(t)|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \le \sqrt{\delta_1}, \quad \|(\lambda - \Delta)^{-\frac{1}{2}}|\hat{b}^{(2)}(t)|\|_{\infty} \le \sqrt{\delta_2}$$

with $\sqrt{\delta_1} + \sqrt{\delta_2} < 1$, then an approximation solution to Cauchy problem (1), (25) *is unique.*

The last assertion in the case (d_1) is due to [QX1] (in fact, valid for weak solutions in the standard triple, not just approximation solutions). The case (d_2) is a consequence of Theorem 2.1(ii).

If $f \in L^2$ and $|b| \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\mathrm{div}\, b$ is form-bounded with form-bound $\nu < 1$, then the corresponding Cauchy problem for (1) has a unique weak solution, cf. Theorem 2.1(ii). Theorem 2.3(d) has the advantage that it does not require $|b| \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$.

The constants $\mu_\pm^*$ that bound the admissible values of the Kato norms in Theorem 2.3 are given by, in principle, explicit but rather complicated expressions (e.g. they will depend on the constants in the upper Gaussian bound of Theorem 2.2), so we will not attempt writing them down here.

Remark 2.3. The upper bound in Theorem 2.2 becomes local in time if g satisfies only (3), μ_+ by the local Kato norm of $(\mathrm{div}\, b)_+$, i.e. the maximum between

$$\inf_{\vartheta > 0} \sup_{t \ge \vartheta, x \in \mathbb{R}^d} \int_{t-\vartheta}^t \langle k(t-r, x, \cdot)(\mathrm{div}\, b)_+(r, \cdot)\rangle dr$$

and

$$\inf_{\vartheta > 0} \sup_{s \ge 0, x \in \mathbb{R}^d} \int_s^{s+\vartheta} \langle k(r-s, x, \cdot)(\mathrm{div}\, b)_+(r, \cdot)\rangle dr.$$

The two-sided bound in (a) becomes local in time if one requires $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, $h_\pm \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and replaces the global in time Kato norms of $(\mathrm{div}\, b)_\pm$ by their local counterparts.

Remark 2.4. Concerning the divergence form equation (1), the authors obtained in [KS4] an L^1 strong solution theory of (1) with measurable uniformly elliptic $a = a(x)$ and $b = b(x)$ in the elliptic Nash class

$$|b| \in L^2_{\mathrm{loc}} \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} \int_0^h \langle k(t, x, \cdot)|b(\cdot)|^2\rangle^{\frac{1}{2}} \frac{dt}{\sqrt{t}} \quad \text{is sufficiently small, for some } h > 0,$$

without any assumptions on $\mathrm{div}\, b$. These assumptions, moreover, provide two-sided Gaussian bounds on the heat kernel of (1).

The elliptic Nash class contains e.g. $b = b(x)$ with $|b| \in L^p + L^\infty$, $p > d$, but it also contains some b with $|b| \notin L^{2+\varepsilon}_{\mathrm{loc}}$, $\varepsilon > 0$. Despite the fact that equation (1) with b in the elliptic Nash class admits L^1 strong solution theory, it does not seem to admit even an L^2 weak solution theory, see discussion in [KS4].

Remark 2.5. Parabolic equation (1) admits weak solution theory in the standard tripe of Hilbert spaces $W^{1,2} \subset L^2 \subset W^{-1,2}$ if either $b \in L^\infty \mathrm{BMO}^{-1}$ [QX1] or $|b|$ is form-bounded with form-bound $\delta < 1$:

DEFINITION. A vector field $b \in [L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d$ is said to be form-bounded (with form-bound $\delta > 0$) if for a.e. $t \in \mathbb{R}_+$

$$\|b(t,\cdot)\psi\|_2^2 \leq \delta\|\nabla\psi\|_2^2 + g(t)\|\psi\|_2^2, \quad \forall\psi \in W^{1,2}, \tag{28}$$

for a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

For time-homogeneous a and b, Mazya-Verbitsky [MV] proved that the general second order elliptic operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla + V$ with time-homogeneous coefficients is $W^{1,2} \to W^{-1,2}$ bounded if and only if $b = \hat{b} + \tilde{b}$, where $\hat{b}$ is form-bounded with some δ, and $\tilde{b} \in \mathrm{BMO}^{-1}$, and distributional V is form-bounded in the sense that

$$|\langle V\psi, \psi\rangle| \leq \nu\|\nabla\psi\|_2^2 + c(\nu)\|\psi\|_2^2, \quad \psi \in C_c^\infty$$

for suitable $\nu > 0$, $c(\nu) < \infty$.

The results of the present paper show that if one deals with such aspects of the regularity theory of $-\Delta + b \cdot \nabla$, $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ as the existence and uniqueness of weak solution, upper and lower Gaussian bounds, one can consider even less restrictive assumptions on the vector field b. Let us also add that the operator $-\Delta + b \cdot \nabla + V$ also admits an L^2 theory (but not $W^{1,2} \to W^{-1,2}$ boundedness) if b is only weakly form-bounded, i.e. $b \in \mathbf{F}_{\delta_b}^{1/2}$, but the potential V satisfies a somewhat more restrictive condition than the form-boundedness:

$$\||V|^{\frac{3}{4}}(\lambda - \Delta)^{-\frac{3}{4}}\|_{2\to 2} \leq \delta_V^{\frac{3}{4}},$$

where $\delta_b + \delta_V < 1$. See [KS2, Sect. 5.4].

See also further discussion and examples in Section 8.

3. PROOF OF PROPOSITION 1.1

Given a $b \in L^\infty \mathrm{BMO}^{-1}$, we write

$$\langle b_k(t)\psi, \psi\rangle = -2\sum_{i=1}^d \langle B_{ik}(t)|\psi|\nabla_i|\psi|\rangle.$$

By the $\mathcal{H}_1$-BMO duality,

$$|\langle b_k(t)\psi, \psi\rangle| \leq 2\sum_{i=1}^d \|B_{ik}(t)\|_{\mathrm{BMO}}\||\psi|\nabla_i|\psi|\|_{\mathcal{H}_1}.$$

Therefore, by Proposition 1.3,

$$|\langle b(t)\psi, \psi\rangle| \equiv \left(\sum_{k=1}^d \langle b_k(t)\psi, \psi\rangle^2\right)^{1/2} \leq 2C\|B(t)\|_{\mathrm{BMO}}\|\nabla\psi\|_2\|\psi\|_2,$$

i.e. $b \in \mathbf{M}_\delta$ with $\delta = 2C\|B\|_{L^\infty \mathrm{BMO}}$ and $g = 0$.

4. Proof of Theorem 2.1(i)

Put

$$H_+ := \mathcal{W}^{\frac{3}{2},2}, \quad H := \mathcal{W}^{\frac{1}{2},2}, \quad H_- := \mathcal{W}^{-\frac{1}{2},2}.$$

Clearly, H_- is the dual of H_+ with respect to the inner product in H.

Proposition 4.1. *Let* $u \in L^2_{\mathrm{loc}}(]s,T[,H_+)$. *Then, for every* $\varphi \in L^2_{com}(]s,T[,H_+)$, *for all* $T > t_1 > s_1 > s$,

$$\int_{s_1}^{t_1} \langle |b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} \varphi| \rangle dt \leq \delta \|u\|_{L^2([s_1,t_1],H_+)} \|\varphi\|_{L^2([s_1,t_1],H_+)}.$$

Proof. It suffices to consider $b = b_n$, where $b_n = \mathbf{1}_{|b| \leq n} b$, and then take $n \to \infty$ using Fatou's Lemma. Thus, without loss of generality, b is bounded. We have, using Hölder's inequality,

$$\langle |b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} \varphi| \rangle = \langle |b(t)^{\frac{1}{2}} \cdot (\lambda - \Delta)^{-\frac{1}{4}} (\lambda - \Delta)^{\frac{1}{4}} \nabla u|, \big||b(t)|^{\frac{1}{2}} (\lambda - \Delta)^{-\frac{1}{4}} (\lambda - \Delta)^{\frac{3}{4}} \varphi| \rangle$$

$$\leq \||b|^{\frac{1}{2}} (\lambda - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} \|(\lambda - \Delta)^{\frac{1}{4}} \nabla u\|_2 \||b|^{\frac{1}{2}} (\lambda - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} \|(\lambda - \Delta)^{\frac{3}{4}} \varphi\|_2.$$

Note that

$$\|(\lambda - \Delta)^{\frac{1}{4}} \nabla u\|_2^2 = \langle \nabla(\lambda - \Delta)^{\frac{1}{4}} u, \nabla(\lambda - \Delta)^{\frac{1}{4}} u \rangle$$

$$\leq \|(\lambda - \Delta)^{\frac{3}{4}} u\|_2^2.$$

The result now follows upon applying condition $b \in L^\infty \mathbf{F}_\delta^{1/2}$. $\square$

Proposition 4.2. *Let* $\delta < 1$. *Let* $u \in L^\infty_{\mathrm{loc}}(]s,T],H) \cap L^2_{\mathrm{loc}}(]s,T],H_+)$ *be a weak solution to equation* (11). *Then the following is true:*

   (i) $\partial_t u \in L^2_{\mathrm{loc}}(]s,T],H_-)$;
   (ii) $u \in C(]s,T],H)$ *(after redefinition on a measure zero set)*;
   (iii) for all $T \geq t_1 \geq s_1 > s$,

$$\|u(t_1)\|_H^2 + 2(1-\delta) \int_{s_1}^{t_1} \|u(t)\|_{H_+}^2 dt \leq \|u(s_1)\|_H^2.$$

We prove Proposition 4.2 in Appendix B by specifying the results in [LM] to tripe $H_+ \subset H \subset H_-$.

Armed with Proposition 4.2, we now prove Theorem 2.1(i) using some standard arguments. We include the details for the sake of completeness, and also because some care need to be taken when discussing approximation involving non-local operators.

1. First, let $f \in \mathcal{S}$. Fix $b_n \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ that have the same weak form-bound δ and λ as b (so, independent of n), and converge to b in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ (see Proposition 4.3). Let u_n denote the strong solution to Cauchy problem

$$(\partial_t + \lambda - \Delta + b_n \cdot \nabla) u_n = 0, \quad u_n(s) = f.$$

Multiplying the last equation by $(\lambda - \Delta)^{\frac{1}{2}} u_n$ and integrating, we obtain a uniform in n bound

$$\sup_{t \in [s,T]} \|u_n(t)\|_H^2 + 2(1-\delta) \int_s^T \|u_n(t)\|_{H_+}^2 dt \leq \|f\|_H^2. \tag{29}$$

Using a weak compactness argument, we can find a subsequence (also denoted by u_n) and a function $u \in L^\infty([s,T],H) \cap L^2([s,T],H_+)$ such that

$$u_n \to u \quad \text{weakly in } L^2([s,T],L^2), L^2([s,T],H_+). \tag{30}$$

2. Next, let us show that u is a weak solution to (11). Let $\varphi \in C_c^\infty(]s,T[,\mathcal{S})$. Let us pass to the limit $n \to \infty$ in

$$-\int_s^T \langle u_n, \partial_t \varphi \rangle_H dt + \int_s^T \langle u_n, \varphi \rangle_{H_+} dt + \int_s^T \langle b_n(t) \cdot \nabla u_n, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle = 0.$$

Only the last term requires a comment. We have

$$\langle b(t) \cdot \nabla u - b_n(t) \cdot \nabla u_n, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle$$
$$= \langle b(t) \cdot (\nabla u - \nabla u_n), (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle + \langle (b(t) - b_n(t)) \cdot \nabla u_n, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle = I_1 + I_2.$$

Let us show that $\int_s^T I_1, \int_s^T I_2 \to 0$ as $n \to \infty$.

We have

$$I_1 = -\langle (\lambda - \Delta)^{\frac{3}{4}} (u - u_n), \nabla(\lambda - \Delta)^{-\frac{3}{4}} \cdot b(t)(\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle,$$

where, clearly,

$$\|\nabla(\lambda - \Delta)^{-\frac{3}{4}} \cdot b(t)(\lambda - \Delta)^{\frac{1}{2}} \varphi\|_2 \le \delta \|\varphi\|_{H_+},$$

so $\nabla(\lambda - \Delta)^{-\frac{3}{4}} \cdot b(\lambda - \Delta)^{\frac{1}{2}} \varphi \in L^2([s,T],L^2)$, and hence $\int_s^T I_1 dt \to 0$ as $n \to \infty$.

Next,

$$I_2 = \langle \nabla(\lambda - \Delta)^{\frac{1}{4}} u_n, (\lambda - \Delta)^{-\frac{1}{4}} (b(t) - b_n(t))(\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle.$$

Here $\nabla(\lambda - \Delta)^{\frac{1}{4}} u_n$ is uniformly in n bounded in $L^2([s,T],L^2)$, while

$$\|(\lambda - \Delta)^{-\frac{1}{4}} (b(t) - b_n(t))(\lambda - \Delta)^{\frac{1}{2}} \varphi\|_2 \le 2\sqrt{\delta} \||b(t) - b_n(t)|^{\frac{1}{2}} (\lambda - \Delta)^{\frac{1}{2}} \varphi\|_2. \tag{31}$$

Thus, we are left to show that

$$\lim_n \int_s^T \||b(r) - b_n(r)|^{\frac{1}{2}} (\lambda - \Delta)^{\frac{1}{2}} \varphi\|_2^2 dr = 0. \tag{32}$$

Fix a smooth function σ on $\mathbb{R}_+$ taking values in $[0,1]$, such that $\sigma(t) = 1$ on $[0,1]$ and 0 on $[2,\infty[$, and put $\zeta(x) = \sigma(\frac{|x|}{R})$, $R > 0$. Then $|\nabla \zeta(x)| \le c_1 R^{-1} \mathbf{1}_{|\nabla \zeta|}$ and $|\Delta \zeta(x)| \le c_2 R^{-2} \mathbf{1}_{|\nabla \zeta|}$, where $\mathbf{1}_{|\nabla \zeta|}$ is the indicator of the set where $|\nabla \zeta| \neq 0$. Put, for brevity, $h := (\lambda - \Delta)^{\frac{1}{2}} \varphi$. We have

$$\lim_n \int_s^T \||b(r) - b_n(r)|^{\frac{1}{2}} h\|_2^2 dr \le \lim_n \left(\int_s^T \||b(r) - b_n(r)|^{\frac{1}{2}} \zeta h\|_2^2 dr + \int_s^T \||b(r) - b_n(r)|^{\frac{1}{2}} (1 - \zeta) h\|_2^2 dr \right)$$

$$\text{(we use that } b_n \to b \text{ in } L_{\text{loc}}^1(\mathbb{R}_+ \times \mathbb{R}^d))$$

$$= \lim_n \int_s^T \||b(r) - b_n(r)|^{\frac{1}{2}} (1 - \zeta) h\|_2^2 dr,$$

and

$$\||b(t) - b_n(t)|^{\frac{1}{2}}(1 - \zeta)h(t)\|_2 \le 2\sqrt{\delta}\|(\lambda - \Delta)^{\frac{1}{4}}(1 - \zeta)h(t)\|_2$$

$$= 2\sqrt{\delta}\|(\lambda - \Delta)^{-\frac{3}{4}}(\lambda - \Delta)(1 - \zeta)h(t)\|_2$$

$$\le 2\sqrt{\delta}\big(\lambda^{\frac{1}{4}}\|(1 - \zeta)h(t)\|_2 + \lambda^{-\frac{3}{4}}\|\Delta[(1 - \zeta)h(t)]\|_2\big)$$

$$= o(R) \text{ as } R \to \infty \text{ due to the choice of } \zeta.$$

Thus (32) is proved.

We obtain $\int_s^T I_2 \to 0$ as $n \to \infty$. Hence

$$-\int_s^T \langle u, \partial_t \varphi \rangle_H dt + \int_s^T \langle u, \varphi \rangle_{H_+} dt + \int_s^T \langle b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}}\varphi \rangle = 0$$

for all $\varphi \in C_c^\infty(]s, T[, \mathcal{S})$, i.e. u is a weak solution to (11).

3. We show that u satisfies the initial condition

$$\lim_{t \downarrow s} \langle u(t), \psi \rangle = \langle f, \psi \rangle \quad \text{for all } \psi \in H_-. \tag{33}$$

First, consider $\psi \in H$. Put $g_n(t) := \langle u_n(t), \psi \rangle$, $t \in [s, T]$. Then, for all $t, t + \Delta t \in [s, T]$,

$$|g_n(t + \Delta t) - g_n(t)| \le \int_t^{t+\Delta t} |\langle (\lambda - \Delta)^{\frac{3}{4}} u_n, (\lambda - \Delta)^{\frac{1}{4}}\psi \rangle| dr + \int_t^{t+\Delta t} |\langle b_n \cdot \nabla u_n, \psi \rangle| dr$$

$$\le \|u_n\|_{L^2([t,t+\Delta t],H_+)}\|\psi\|_{L^2([t,t+\Delta t],H)} + c(d)\delta\|u_n\|_{L^2([t,t+\Delta t],H_+)}\|\psi\|_{L^2([t,t+\Delta t],H)},$$

where, estimating the last term, we argued as in the proof of Proposition 4.1.

Also, $\|u_n\|_{L^\infty([s,T]\times\mathbb{R}^d)} \le \|f\|_\infty$, so we can apply the Arzelà-Ascoli Theorem. The latter, combined with (30), allows to refine the subsequence $\{u_n\}$ found earlier to obtain convergence of continuous functions

$$\langle u_n(t), \psi \rangle \to \langle u(t), \psi \rangle \quad \text{uniformly on } [T, s] - \Omega_\psi,$$

where Ω_ψ is a measure zero set. Moreover, using the separability of H and the bound

$$\sup_{t \in [s,T]} \|u(t)\|_H \le C\|f\|_H, \tag{34}$$

and applying the diagonal argument, we can further refine $\{u_n\}$ to obtain the uniform convergence on $[T, s] - \Omega$ where Ω is a measure zero set independent of ψ. Further, since u is a weak solution to (11) and hence is in $C(]s, T], H)$ by Proposition 4.1, we obtain that $t \mapsto \langle u(t), \psi \rangle$ can be uniquely extended to a continuous function on $[s, T]$ which must coincide at the endpoint $t = s$ with $\langle f, \psi \rangle$, i.e. we have (33) for $\psi \in H$. Finally, using

$$|\langle u(t), \psi \rangle| \le \|u(t)\|_H \|\psi\|_{H_-}$$

and the fact that H is dense in H_-, we obtain (33) for all $\psi \in H_-$.

4. Given a weak solution u to Cauchy problem (11), (13), we note that Proposition 4.2(iii) gives

$$\|u(t_1)\|_H^2 + 2(1 - \delta)\int_s^{t_1} \|u(t)\|_{H_+}^2 dt \le \|f\|_H^2. \tag{35}$$

Indeed, the initial condition $u(t) \to f$ weakly in H as $t \downarrow s$ gives $\|f\|_H \leq \liminf_{t \downarrow s} \|u(t)\|_H$. At the same time, Proposition 4.2(iii) yields that $t \mapsto \|u(t)\|_H$ is a non-increasing function, so

$$\|f\|_H = \lim_{t \downarrow s} \|u(t)\|_H$$

Hence $u(t) \to f$ (strongly) in H as $t \downarrow s$, which yields (35). Combined with Proposition 4.2(ii), this gives $u \in C([s,T], H)$. The uniqueness of weak solution follows from (35). The reproduction property of the evolution family $T^{t,s} f := u(t)$ is a consequence of the uniqueness. The fact that this evolution family is positivity preserving L^∞ contraction (and hence is Markov) is immediate from the construction of the weak solution u via approximation, cf. (30). The convergence result follows from the weak compactness argument carried out above and the uniqueness of weak solution. Finally, the energy inequality (35) and the fact that $\mathcal{S}$ is dense in H allow to extend these results to an arbitrary $f \in H$.

To end the proof of Theorem 2.1(i), we need to address the question of existence of a bounded smooth approximation of vector field b preserving its weak form-bound δ and constant $\lambda = \lambda_\delta$. We put $b = 0$ for $t < 0$.

Proposition 4.3. *Set $\bar{b}_\varepsilon(t) := e^{\gamma_\varepsilon(t)\Delta} \mathbf{1}_\varepsilon b(t)$, $t \in \mathbb{R}$,*

$$b_\varepsilon := c_\varepsilon e^{\varepsilon \Delta_1} \bar{b}_\varepsilon,$$

where $c_\varepsilon > 0$, $\Delta_1 := \frac{\partial^2}{\partial t^2}$, γ_ε is a $]0,1]$-valued measurable function on $\mathbb{R}_+$, b and γ_ε are extended by 0 to $t < 0$, $\mathbf{1}_\varepsilon$ is the indicator function of $\{(t,x) \mid t \in [0, \varepsilon^{-\frac{1}{2}}], |x| \leq \varepsilon^{-1}, |b(t,x)| \leq \varepsilon^{-1}\}$. There exist $c_\varepsilon \uparrow 1$, $\gamma_\varepsilon(t) \downarrow 0$ for every $t \in \mathbb{R}_+$ as $\varepsilon \downarrow 0$ such that

$$b_\varepsilon \in L^\infty \, \mathbf{F}_\delta^{1/2} \quad \text{ with } \lambda \text{ independent of } \varepsilon,$$

and

$$b_\varepsilon \to b \quad \text{in } L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d).$$

Proof. First, we note that the convergence $\bar{b}_\varepsilon \to b$ in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ is straightforward, provided that $\gamma_\varepsilon(t) \downarrow 0$ sufficiently rapidly as $\varepsilon \downarrow 0$. Let us show that for any $\delta_\varepsilon \downarrow \delta$ we can select $\gamma_\varepsilon \downarrow 0$ fast enough so that

$$\bar{b}_\varepsilon \in L^\infty \, \mathbf{F}_{\delta_\varepsilon}^{1/2} \quad \text{ with the same } \lambda.$$

We have

$$\bar{b}_\varepsilon = \mathbf{1}_\varepsilon b + (\bar{b}_\varepsilon - \mathbf{1}_\varepsilon b),$$

where, clearly, $\||\mathbf{1}_\varepsilon b|^{\frac{1}{2}}(t)(\lambda - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} \leq \sqrt{\delta}$ for a.e. $t \in \mathbb{R}_+$, for all ε, while $\bar{b}_\varepsilon - \mathbf{1}_\varepsilon b \in L^\infty(\mathbb{R}_+, L^d)$. It follows from Hölder's inequality and the Sobolev Embedding Theorem that for any $g \in L^2$, for a.e. $t \in \mathbb{R}_+$,

$$\||\bar{b}_\varepsilon(t) - \mathbf{1}_\varepsilon b(t)|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}} g\|_2 \leq \||\bar{b}_\varepsilon(t) - \mathbf{1}_\varepsilon b(t)|^{\frac{1}{2}}\|_{2d} \|(\lambda - \Delta)^{-\frac{1}{4}} g\|_{\frac{2d}{d-1}}$$

$$\leq c \|\bar{b}_\varepsilon(t) - \mathbf{1}_\varepsilon b(t)\|_d^{\frac{1}{2}} \|g\|_2.$$

For every $\varepsilon > 0$ and every $t \in \mathbb{R}_+$, we can select $\gamma_\varepsilon(t)$ sufficiently small so that $\|\bar{b}_\varepsilon(t) - \mathbf{1}_\varepsilon b(t)\|_d^{\frac{1}{2}} \leq c^{-1}\sqrt{\delta_\varepsilon - \delta}$. Thus $\sup_{t \in \mathbb{R}_+} \||\bar{b}_\varepsilon(t) - \mathbf{1}_\varepsilon b(t)|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}} g\|_2 \leq \sqrt{\delta_\varepsilon - \delta}\|g\|_2$. It follows that for a.e. $t \in$

$\mathbb{R}_+$, $\langle |\bar{b}_\varepsilon(t)|[(\lambda - \Delta)^{-\frac{1}{4}}g]^2\rangle \le \delta_\varepsilon \|g\|_2^2$, $g \in L^2$, and hence

$$\||\bar{b}_\varepsilon(t)|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to2} \le \sqrt{\delta_\varepsilon}.$$

Finally, recalling that $b_\varepsilon = c_\varepsilon \bar{b}_\varepsilon$, it is clear now that we can take $c_\varepsilon := \frac{\delta}{\delta_\varepsilon}$ obtaining

$$\||b_\varepsilon|^{\frac{1}{2}}(t)(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to2} \le \sqrt{\delta}.$$

$\square$

5. Proof of Theorem 2.2

Set $A = -\nabla \cdot a \cdot \nabla$. We prove Theorem 2.2 by first establishing an upper Gaussian bound on the heat kernel of the auxiliary operator

$$H^+ = A + b \cdot \nabla + \operatorname{div} b_+.$$

Let $H^{t,s}f$ denote the solution of

$$\begin{cases} -\frac{d}{dt}H^{t,s}f = H^+H^{t,s}f & , \quad 0 \le s < t < \infty \\ 0 \le f \in L^1 \cap L^\infty \end{cases}$$

in $L^p = L^p(\mathbb{R}^d)$, $p \in [1, \infty[$. Let $h(t, x; s, y)$ denote the heat kernel of H^+, that is,

$$H^{t,s}f = \langle h(t, x; s, \cdot)f(\cdot)\rangle.$$

Theorem 5.1. *There exist generic constants $c_3, c_4 > 0$, $\omega \ge 0$ such that*

$$h(t, x; s, y) \le c_3 k_{c_4}(t - s, x - y) \qquad\qquad (\mathrm{UGB}^{h+})$$

for all $0 \le s < t < \infty$.

Proof of Theorem 5.1. We follow [FS]. We consider

$$\begin{cases} -\frac{d}{dt}H_\alpha^{t,s}f = H_\alpha^+H_\alpha^{t,s}f & , \quad 0 \le s < t < \infty, \\ 0 \le f \in L^1 \cap L^\infty \end{cases}$$

in $L^p = L^p(\mathbb{R}^d)$, $p \in [1, \infty[$, where $H_\alpha^{t,s} := e^{\alpha \cdot x}H^{t,s}e^{-\alpha \cdot x}$ and

$$H_\alpha^+ := e^{\alpha \cdot x}H^+e^{-\alpha \cdot x} = H^+ - \alpha \cdot b - \alpha \cdot a \cdot \alpha + \alpha \cdot a \cdot \nabla + \nabla \cdot a \cdot \alpha.$$

Lemma 5.2. *There are generic constants c, c_4 such that, for all $0 \le s < t < \infty$,*

$$\|H_\alpha^{t,s}\|_{2\to\infty}, \|H_\alpha^{t,s}\|_{1\to2} \le c(t - s)^{-d/4}e^{c_4\alpha^2(t-s)}.$$

Proof of Lemma. Set $u_\alpha(t) := H_\alpha^{t,s}f$, $f = \operatorname{Re}f \in C_c^\infty$, $v(t) := u_\alpha^{p/2}(t)$, $p \ge 2$. Noticing that

$$\langle b \cdot \nabla u_\alpha, u_\alpha^{p-1}\rangle = \frac{2}{p}\langle \nabla v, bv\rangle = -\frac{1}{p}\langle v^2, \operatorname{div} b\rangle,$$

we have, using the equation,

$$-\frac{1}{p}\frac{d}{dt}\langle v^2(t)\rangle = \frac{4}{pp'}\|A^{1/2}v(t)\|_2^2 + \frac{1}{p'}\langle v^2(t), \operatorname{div} b_+\rangle + \frac{1}{p}\langle v^2(t), \operatorname{div} b_-\rangle$$

$$- \frac{2(p-2)}{p}\langle \alpha \cdot a \cdot \nabla v(t), v(t)\rangle - \langle \alpha \cdot b, v^2(t)\rangle - \langle \alpha \cdot a \cdot \alpha, v^2(t)\rangle.$$

By $b \in \mathbf{M}_\delta$,

$$|\langle \alpha \cdot b, v^2 \rangle| \leq |\alpha||\langle b, v^2 \rangle| \leq |\alpha|\delta\|\nabla v\|_2\|v\|_2 + |\alpha|g\|v\|_2^2$$
$$\leq \frac{4\gamma}{pp'}\|A^{\frac{1}{2}}v\|_2^2 + \left(|\alpha|g + \frac{1}{4\gamma}\frac{pp'}{4}\frac{\delta^2}{\sigma}|\alpha|^2\right)\|v\|_2^2 \quad (\gamma > 0),$$

and so, applying the quadratic inequality, we have

$$-\frac{d}{dt}\|v\|_2^2 \geq \frac{4}{p'}(1 - 2\gamma)\|A^{1/2}v\|_2^2 - \left(p|\alpha|g + \frac{p^2p'}{4\gamma}\left[\frac{4\gamma}{pp'}\xi + \left(\frac{p-2}{p}\right)^2\xi + \frac{\delta^2}{4\sigma}\right]\alpha^2\right)\|v\|_2^2$$

Putting $p = 2$ and $\gamma = \frac{1}{2}$, we have, taking into our assumption on g, i.e. $\int_s^t g\,ds \leq c_\delta\sqrt{t-s}$,

$$\|u_\alpha(t)\|_2 \leq \|f\|_2 \exp\left[\left(\xi + \frac{\delta^2}{2\sigma}\right)\alpha^2(t-s) + c_\delta|\alpha|\sqrt{t-s}\right] \qquad (\star^a)$$

Also, using Nash's inequality

$$\|\nabla\psi\|_2^2 \geq C_N\|\psi\|_2^{2+4/d}\|\psi\|_1^{-4/d}, \quad \psi \in W^{1,2} \cap L^1, \qquad (36)$$

we have, putting $\gamma = \frac{1}{4}$, $p \geq 4$ and setting $F(t,\alpha) = p|\alpha|g(t) + p^2\left(\xi + \frac{\delta^2}{3\sigma}\right)\alpha^2$,

$$-\frac{d}{dt}\|v\|_2^2 \geq \frac{2}{p'}\|A^{1/2}v\|_2^2 - F(t,\alpha)\|v\|_2^2$$
$$\geq \frac{2}{p'}\sigma C_N\|v\|_2^{2+\frac{4}{d}}\|v\|_1^{-\frac{4}{d}} - F(t,\alpha)\|v\|_2^2,$$

so

$$\frac{d}{dt}\|v\|_2^{-4/d} \geq \frac{4\sigma C_N}{dp'}\|v\|_1^{-\frac{4}{d}} - \frac{2}{d}F(t,\alpha)\|v\|_2^{-4/d}.$$

Setting $c_g = \frac{4\sigma C_N}{dp'}$, $w(t) = \|v\|_2^{-\frac{4}{d}}$ and $\mu(t) = \frac{2}{d}\int_s^t F(r,\alpha)dr$, we have

$$w_p(t) \geq c_g e^{-\mu_p(t)} \int_s^t e^{\mu_p(r)}w_{\frac{p}{2}}(r)dr$$
$$\geq c_g e^{-\mu_p(t)} V_{\frac{p}{2}}(t) \int_s^t e^{\mu_p(r)}(r-s)^q dr,$$

where $q = \frac{p}{2} - 2$ and $V_{\frac{p}{2}}(t) := \left\{\sup\left[(r-s)^{\frac{qd}{2p}}\|u_\alpha(r)\|_{p/2} \mid s \leq r \leq t\right]\right\}^{-\frac{2p}{d}}.$

Set $\kappa = \frac{2}{d}\left(\frac{3}{2}\xi + \frac{\delta^2}{3\sigma}\right)\alpha^2$. Since for $s \leq r \leq t$,

$$
\begin{aligned}
-\mu_p(t) + \mu_p(r) &= -\frac{2}{d}\left[p^2(\xi + \frac{\delta^2}{3\sigma})\alpha^2(t - r) + p|\alpha|\int_r^t g(\tau)d\tau\right] \\
&\geq -\frac{2}{d}\left[p^2(\xi + \frac{\delta^2}{3\sigma})\alpha^2(t - r) + p|\alpha|c_\delta\sqrt{t - r}\right] \\
&\geq -\frac{2}{d}\left[p^2\left(\frac{3}{2}\xi + \frac{\delta^2}{3\sigma}\right)\alpha^2(t - r) + \frac{c_\delta^2}{2\xi}\right] \\
&= -\left[\kappa p^2(t - s) + \frac{c_\delta^2}{d\xi}\right] + \kappa p^2(r - s), \text{ and so}
\end{aligned}
$$

$$
e^{-\mu_p(t)}\int_s^t e^{\mu_p(r)}(r - s)^q dr \geq e^{-\kappa p^2(t-s)-\frac{c_\delta^2}{d\xi}}\int_s^t e^{\kappa p^2(r-s)}(r - s)^q dr \text{ and}
$$

$$
\int_s^t e^{\kappa p^2(r-s)}(r - s)^q dr \geq Kp^{-2}(t - s)^{\frac{p-2}{2}}e^{\kappa(p^2-1)(t-s)},
$$

where $K := 2\inf\left\{p\left[1 - (1 - p^{-2})^{p-1}\right] \mid p \geq 2\right\} > 0$, we obtain

$$
w_p(t) \geq c_g K p^{-2} e^{-\kappa(t-s)-\frac{c_\delta^2}{d\xi}}(t - s)^{\frac{p-2}{2}}V_{\frac{p}{2}}(t),
$$

or, setting $W_p(t) := \sup\left[(r - s)^{\frac{d(p-2)}{4p}}\|u_\alpha(r)\|_p \mid s \leq r \leq t\right]$,

$$
W_p(t) \leq (c_g K e^{-\frac{2c_\delta^2}{d\xi}})^{-\frac{d}{2p}}p^{\frac{d}{p}}e^{(\frac{3}{2}\xi+\frac{\delta^2}{3\sigma})\frac{\alpha^2}{p}(t-s)}W_{p/2}(t), \quad p = 2^k, \ k = 1, 2, \ldots.
$$

Iterating this inequality, starting with $k = 2$, yields

$$
(t - s)^{\frac{d}{4}}\|u_\alpha(t)\|_\infty \leq C_g e^{(\frac{3}{2}\xi+\frac{\delta^2}{3\sigma})\alpha^2(t-s)}W_2(t).
$$

Finally, taking into account $(\star^a)$, we arrive at

$$
\|H_\alpha^{t,s}\|_{2\to\infty} \leq (t - s)^{-d/4}C_g e^{(3\xi+\frac{\delta^2}{\sigma})\alpha^2(t-s)}.
$$

The same bound holds for $\|H_\alpha^{t,s}(b)\|_{1\to2} = \|(H_{-\alpha}^{t,s}(-b))^*\|_{2\to\infty}$. This ends the proof of Lemma 5.2. $\qquad\square$

From Lemma 5.2 we obtain

$$
h(t, x; s, y) \leq C_g^2(t - s)^{-\frac{d}{2}}e^{\alpha\cdot(y-x)+c_4\alpha^2(t-s)}, \quad c_4 = 3\xi + 2\frac{\delta^2}{3\sigma}.
$$

Putting $\alpha = \frac{x-y}{2c_4(t-s)}$, we obtain (UGB^{h+}). This completes the proof of Theorem 5.1. $\qquad\square$

We are in position to complete the proof of Theorem 2.2. We consider operator $A + b \cdot \nabla$ as H^+ perturbed by potential $-\operatorname{div} b_+$. Hence, the sought upper bound on the heat kernel of $A + b \cdot \nabla$ follows from Theorem 5.1 and a standard argument based on the Duhamel formula using that the Kato norm μ_+ of $(\operatorname{div} b)_+$ is sufficiently small. $\qquad\square$

6. A priori lower bound

In this section, $u(t, x; s, y)$ denotes the heat kernel of operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ with matrix $a \in H_{\sigma,\xi}$, vector field b and div b assumed to be bounded C^∞ smooth.

Theorem 6.1 (a priori lower bound). *In the assumptions of Theorem 2.3, a global in time lower Gaussian bound*

$$c_1 k_{c_2}(t - s; x - y) \leq u(t, x; s, y) \tag{LGB}$$

holds for all $0 \leq s < t < \infty$, $x, y \in \mathbb{R}^d$ with generic constants c_i, $i = 1, 2$ that can also depend on the Kato norms $\mu_\pm$.*

Throughout the rest of the proof, the constants that we find are generic* that can also depend on $\mu_\pm$.

The proof of Theorem 6.1 (given in the end of this section) is based on the following estimates of Nash's G-functions.

6.1. $\hat{G}$-function for $-\nabla \cdot a \cdot \nabla + (\hat{b} + \tilde{b}) \cdot \nabla$. Since $\hat{b} + \tilde{b}$ is in $\mathbf{M}_\delta$, Theorem 2.2 applies and gives

$$u(t, x; s, y) \leq \hat{c}_3 k_{c_4}(t - s; x - y) \tag{37}$$

where, recall, $u(t, x; s, y)$ is the heat kernel $\Lambda = -\nabla \cdot a \cdot \nabla + (\hat{b} + \tilde{b}) \cdot \nabla$. The constants in the next proposition depend on the same parameters as the constants in the theorem except for the Kato norms $\mu_\pm$ and div $b_\pm$.

Set

$$\tilde{Q}(t - s) := \frac{d}{2} \log(t - s).$$

Proposition 6.1. *Let $x, y \in \mathbb{R}^d$. Put $o = \frac{x+y}{2}$, $t_s = \frac{t+s}{2}$. There exist constants β and $\mathbb{C}$ such that*

$$G(t_s) := \langle k_\beta(t - t_s, o - \cdot) \log u(t, z; t_s, \cdot) \rangle \geq -\tilde{Q}(t - t_s) - \mathbb{C}$$

for all $z \in B(o, \sqrt{t - t_s})$.

Proof of Proposition 6.1. Fix $\varepsilon > 0$ and define

$$G_\varepsilon(\tau) := \langle k_\beta(t, o; t_s, \cdot) \log [\varepsilon k_\beta(t, o; t_s, \cdot) + u(t, z; \tau, \cdot)] \rangle.$$

Then

$$G(\tau) = \inf_{\varepsilon > 0} G_\varepsilon(\tau), \quad \tau \in \left[t_s, \frac{t + t_s}{2}\right] \text{ and } G(t_s) = \Theta_\beta(t, t_s, z).$$

Below we are using the following shorthand:

$$G_\varepsilon(\tau) \equiv \langle \Gamma \log [\varepsilon \Gamma + U] \rangle \equiv \langle \Gamma \log [\varepsilon \Gamma + U(\tau)] \rangle,$$

where $\Gamma \equiv \Gamma_\beta \equiv k_\beta(t, o; t_s, \cdot)$, $U \equiv U(\tau) \equiv u(t, z; \tau, \cdot)$.

Also set

$$V := c_0(t - t_s)^{d/2} [\varepsilon \Gamma + U], \quad c_0 = (4\pi c_4)^{d/2} e^{-1} \left[\varepsilon + c_3 e^{\frac{1}{4c_4}}\right]^{-1},$$

where c_3, c_4 are the constant from the upper Gaussian bound for U.

If $\beta \geq 2c_4$, then clearly

$$V(\tau, y) \exp \frac{|o - y|^2}{4\beta(t - t_s)} \leq e^{-1} \text{ for all } y \in \mathbb{R}^d, \ \varepsilon \in]0, 1] \text{ and } \tau \in \left[t_s, \frac{t + t_s}{2}\right]. \tag{38}$$

Let us calculate $-\partial_\tau G_\varepsilon(\tau)$:

$$-\partial_\tau G_\varepsilon(\tau) = \left\langle \Gamma \frac{-\partial_\tau U}{\varepsilon\Gamma + U} \right\rangle = \left\langle \frac{\Gamma}{\varepsilon\Gamma + U}(\nabla \cdot a \cdot \nabla + \nabla \cdot (\hat{b} + \tilde{b}))U \right\rangle$$

$$= \left\langle \nabla \log V \cdot a\Gamma \cdot \frac{\nabla U}{\varepsilon\Gamma + U} \right\rangle - \left\langle \nabla\Gamma \cdot a \cdot \frac{\nabla U}{\varepsilon\Gamma + U} \right\rangle + \left\langle \Gamma \frac{(\hat{b} + \tilde{b}) \cdot \nabla U}{\varepsilon\Gamma + U} \right\rangle + \left\langle \Gamma \frac{\mathrm{div}\hat{b}\, U}{\varepsilon\Gamma + U} \right\rangle$$

$$= \left\langle \nabla \log V \cdot a\Gamma \cdot \nabla \log V \right\rangle - \left\langle \nabla \log V \cdot a\Gamma \cdot \frac{\varepsilon\nabla\Gamma}{\varepsilon\Gamma + U} \right\rangle - \left\langle \nabla\Gamma \cdot a \cdot \nabla \log V \right\rangle$$

$$+ \left\langle \nabla\Gamma \cdot a \cdot \frac{\varepsilon\nabla\Gamma}{\varepsilon\Gamma + U} \right\rangle + \left\langle \Gamma(\hat{b} + \tilde{b}) \cdot \nabla \log V \right\rangle - \left\langle \Gamma \frac{(\hat{b} + \tilde{b}) \cdot \varepsilon\nabla\Gamma}{\varepsilon\Gamma + U} \right\rangle + \left\langle \Gamma \frac{U\mathrm{div}\hat{b}}{\varepsilon\Gamma + U} \right\rangle.$$

Setting

$$\mathcal{N}(\tau) := \left\langle \nabla \log V \cdot a\Gamma \cdot \nabla \log V \right\rangle,$$

applying the quadratic inequality and estimating $\left\langle \Gamma \frac{U\mathrm{div}\hat{b}}{\varepsilon\Gamma + U} \right\rangle \geq -\left\langle \Gamma \mathrm{div}\hat{b}_- \right\rangle$, we have

$$-\partial_\tau G_\varepsilon(\tau) \geq \mathcal{N} - 2\mathcal{N}^{1/2} \left\langle \nabla\Gamma \cdot \frac{a}{\Gamma} \cdot \nabla\Gamma \right\rangle^{1/2} + J$$

$$\geq (1 - \gamma)\mathcal{N} - \frac{\xi}{\gamma}\left\langle \frac{(\nabla\Gamma)^2}{\Gamma} \right\rangle + J \quad (0 < \gamma < 1),$$

where

$$J := \left\langle \Gamma\tilde{b} \cdot \nabla \log V \right\rangle - \left\langle \Gamma \frac{\tilde{b} \cdot \varepsilon\nabla\Gamma}{\varepsilon\Gamma + U} \right\rangle + \left\langle \Gamma\hat{b} \cdot \nabla \log V \right\rangle - \left\langle \Gamma \frac{\hat{b} \cdot \varepsilon\nabla\Gamma}{\varepsilon\Gamma + U} \right\rangle - \left\langle \Gamma \mathrm{div}\hat{b}_- \right\rangle.$$

Applying $\left\langle \frac{|\nabla\Gamma|^2}{\Gamma} \right\rangle = \xi \frac{d}{2\beta(t - t_s)}$, we arrive at

$$-\partial_\tau G_\varepsilon(\tau) \geq (1 - \gamma)\mathcal{N}(\tau) - \frac{\xi}{\gamma}\frac{d}{2\beta(t - t_s)} + J. \tag{39}$$

Define

$$Y(\tau) := G_\varepsilon(\tau) + \tilde{Q}(t - \tau).$$

Our goal is to show that Y is bounded from below by a constant. Note that (39) can be rewritten as

$$-\partial_\tau Y(\tau) \geq (1 - \gamma)\mathcal{N}(\tau) - \frac{\xi}{\gamma}\frac{d}{2\beta(t - t_s)} + J. \tag{40}$$

Here we have used that $-\partial_\tau \tilde{Q}(t - \tau) = \frac{d}{2(t - \tau)} > 0$. Multiplying (40) by $e^{\mu(\tau)}$, where

$$\mu(\tau) := -\frac{t - \tau}{t - t_s} - \frac{F(\tau)}{\sqrt{\beta(t - t_s)}} - P(\tau) - H(\tau)$$

where $F(\tau) := \sqrt{2}\int_\tau^t g(r)dr$, $P(\tau) := \int_\tau^t g^2(r)dr$, $H(\tau) := \int_\tau^t h(r)dr$, we obtain

$$-\partial_\tau\left[e^{\mu(\tau)}Y(\tau)\right] \geq e^{\mu(\tau)}\left[-Y(\tau)\partial_\tau\mu(\tau) + (1-\gamma)\mathcal{N}(\tau) - \frac{\xi}{\gamma}\frac{d}{2\beta(t-t_s)} + J\right].$$

Note that, due to our assumptions on g, g^2 and h, the function μ is uniformly bounded in variables $\tau \in [t_s, (t+t_s)/2]$ and $0 \leq s < t < \infty$.

Lemma 6.2. Let $c > \log(1+c_3)$ with c_3 from $u(t,x;\tau,\cdot) \leq c_3(t-\tau)^{-d/2}$. Then, for all $\varepsilon > 0$ sufficiently small,

$$Y(\tau) < c$$

for all $\tau \in [t_s, \frac{t+t_s}{2}]$.

Proof. Indeed, for $\varepsilon \leq (4\pi\beta)^{\frac{d}{2}}$ sufficiently small,

$$\begin{aligned}
G_\varepsilon(\tau) &= \langle\Gamma\log(\varepsilon\Gamma + U)\rangle \\
&\leq \langle\Gamma\rangle\log\left[(1+c_3)(t-\tau)^{-d/2}\right] \\
&< -\tilde{Q}(t-\tau) + \log(1+c_3).
\end{aligned}$$

$\square$

From now on, let c and ε be as in Lemma 6.2. Then we obtain from (40)

$$\partial_\tau\left(e^{\mu(\tau)}(Y(\tau)-c)\right)^{-1} \geq \left[\gamma\mathcal{N}(\tau) + \mathcal{M}(\tau)\right]e^{-\mu(\tau)}(Y(\tau)-c)^{-2}, \tag{41}$$

(division by zero is ruled out by Lemma 6.2), where

$$\mathcal{M}(\tau) := -(Y(\tau)-c)\partial_\tau\mu(\tau) + (1-2\gamma)\mathcal{N}(\tau) - \frac{\xi}{\gamma}\frac{d}{2\beta(t-t_s)} + J.$$

Lemma 6.3. $\mathcal{M}(\tau) \geq 0$ for all $\tau \in \left[t_s, \frac{t+t_s}{2}\right]$, provided that $\nu_- < \sigma$, γ is sufficiently small and c is sufficiently large.

We prove Lemma 6.3 below.

Now, taking Lemma 6.3 for granted, we complete the proof of Proposition 6.1. Lemma 6.3 and (41) give

$$\partial_\tau\left(e^{\mu(\tau)}(Y(\tau)-c)\right)^{-1} \geq \gamma\sigma\mathcal{N}_1(\tau)e^{-\mu(\tau)}(Y(\tau)-c)^{-2}, \tag{42}$$

where, recall, $\mathcal{N}_1(\tau) := \langle\Gamma|\nabla\log V|^2\rangle$. By the Spectral Gap Inequality (see e.g. [N, Sect. 2]),

$$\begin{aligned}
\mathcal{N}_1 &\geq \frac{1}{2\beta(t-t_s)}\langle\Gamma|\log V - \langle\Gamma\log V\rangle|^2\rangle \\
&= \frac{1}{2\beta(t-t_s)}\langle\Gamma|\log\left[\varepsilon\Gamma + U\right] - \langle\Gamma\log\left[\varepsilon\Gamma + U\right]\rangle|^2\rangle \\
&\equiv \frac{1}{2\beta(t-t_s)}\langle\Gamma|\log\left[\varepsilon\Gamma + U\right] - G_\varepsilon|^2\rangle.
\end{aligned}$$

In turn, $\Gamma \geq CU$ for a generic* constant C. (Indeed, $\frac{1}{2}|o-\cdot|^2 \leq |z-\cdot|^2 + |o-z|^2$. Clearly, $\frac{1}{t-t_s} \leq \frac{1}{t-\tau} \leq \frac{2}{t-t_s}$ combined with $|z-o| \leq \sqrt{t-t_s}$ implies that $-\frac{|z-\cdot|^2}{4c_4(t-\tau)} \leq -\frac{|o-\cdot|^2}{8c_4(t-t_s)} + \frac{|o-z|^2}{2c_4(t-t_s)}$.

Thus, if $\beta \geq 2c_4$, then

$$k_{c_4}(t, z; \tau, \cdot) \leq \left(\frac{2\beta}{c_4}\right)^{\frac{d}{2}} e^{\frac{1}{2c_4}} k_\beta(t, o; t_s, \cdot).$$

Therefore, by $U \leq c_3 k_{c_4}(t, z; \tau, \cdot)$, see (37), and $\beta = 2c_4$, we have the required inequality $\Gamma \geq CU$ with $C^{-1} = c_3 2^d e^{\frac{1}{\beta}}$.) Hence

$$\mathcal{N}_1(\tau) \geq \frac{C}{2\beta(t - t_s)} \langle U | \log [\varepsilon\Gamma + U] - G_\varepsilon |^2 \rangle,$$

and so, by $\langle U \rangle = 1$,

$$\mathcal{N}_1(\tau) \geq \frac{C}{2\beta(t - t_s)} \langle U | \log [\varepsilon\Gamma + U] - G_\varepsilon | \rangle^2.$$

Now,

$$
\begin{aligned}
\langle U | \log [\varepsilon\Gamma + U] - G_\varepsilon | \rangle &\geq \langle U \log [\varepsilon\Gamma + U] \rangle - G_\varepsilon \langle U \rangle \\
&\geq \langle U \log U \rangle - G_\varepsilon \langle U \rangle \\
&\geq -G_\varepsilon(\tau) - \tilde{Q}(t - \tau) - \mathcal{C} \\
&\equiv -Y(\tau) - \mathcal{C}.
\end{aligned}
$$

Here we again have used identity $\langle U \rangle = 1$ and the Nash entropy estimate $-\langle U \log U \rangle \leq \tilde{Q}(t-\tau)+\mathcal{C}$. (The latter follows from $e^{\frac{Q}{d}} \leq CM$, where $Q := -\langle U \log U \rangle$, and $M(t, \tau) = \langle |z - \cdot| U \rangle \leq C\sqrt{t - \tau}$, which is a consequence of the upper bound $U \leq c_3 k_{c_4}(t, z; \tau, \cdot)$. The inequality $e^{\frac{Q}{d}} \leq CM$, in turn, follows from $\langle u \rangle = 1$ and $u \log u \geq -\alpha u - e^{-1-\alpha}$ for all real α.)

Case (a): For all $\tau \in [t_s, (t + t_s)/2]$, $-Y(\tau) - c - 2\mathcal{C} \geq 0$. Here c is from (42).

Then $-Y(\tau) - \mathcal{C} \geq \frac{1}{2}(-Y(\tau) + c) > c + \mathcal{C} > 0$ and hence

$$\mathcal{N}_1(\tau) \geq \frac{C}{8\beta(t - t_s)} \left(-Y(\tau) + c\right)^2.$$

Thus, by (42),

$$\left(c - Y(t_s)\right)^{-1} \geq \frac{\gamma\sigma C}{8(t - t_s)} e^{\mu(t_s)} \int_{t_s}^{(t+t_s)/2} e^{-\mu(\tau)} d\tau \geq \frac{\gamma\sigma C}{8(t - t_s)} \int_{t_s}^{(t+t_s)/2} d\tau,$$

and so $c - Y(t_s) \leq \frac{16}{\gamma\sigma C} = \frac{2^{d+4} c_3}{\gamma\sigma} e^{\frac{1}{\beta}}$, or

$$G_\varepsilon(t_s) \geq -\tilde{Q}(t - t_s) + c - \frac{2^{d+4} c_3}{\gamma\sigma} e^{\frac{1}{\beta}}.$$

Case (b): For some $\tau \in [t_s, (t + t_s)/2]$, $-Y(\tau) - c - 2\mathcal{C} < 0$.

By (42),

$$\left(e^{\mu(\tau)}(Y(\tau) - c)\right)^{-1} \geq \left(e^{\mu(t_s)}(Y(t_s) - c)\right)^{-1},$$

or $c - Y(t_s) \leq e^{\mu(\tau) - \mu(t_s)}(c - Y(\tau))$. Therefore, $c - Y(t_s) \leq e^{\mu(\tau) - \mu(t_s)} 2(c + \mathcal{C}) \leq e^{-\mu(t_s)} 2(c + \mathcal{C})$, or

$$G_\varepsilon(t_s) \geq -\tilde{Q}(t - t_s) + c - 2(c + \mathcal{C}) e^{1 + \frac{d+2}{4\sqrt{\beta}} c_\delta + \|g\|_{L^2(\mathbb{R}_+)}^2 + \|h\|_{L^1(\mathbb{R}_+)}}.$$

This ends the proof of Proposition 6.1.   □

6.2. Proof of Lemma 6.3. The main task is to estimate $J = \langle \Gamma \tilde{b} \cdot \nabla \log V \rangle - \langle \Gamma \frac{\tilde{b} \cdot \varepsilon \nabla \Gamma}{\varepsilon \Gamma + U} \rangle - \langle \Gamma \frac{\hat{b} \cdot \varepsilon \nabla \Gamma}{\varepsilon \Gamma + U} \rangle - \langle \Gamma (\text{div} \hat{b})_- \rangle + \langle \Gamma \hat{b} \cdot \nabla \log V \rangle$ in the definition of $\mathcal{M}$.

1. Term "$\langle \Gamma \tilde{b} \cdot \nabla \log V \rangle$". There exists a bounded smooth skew-symmetric matrix $B \in L^\infty$ BMO such that $\tilde{b} = \nabla B$. We have

$$\langle \Gamma \tilde{b} \cdot \nabla \log V \rangle = -\langle \tilde{b} \cdot \nabla \Gamma, \log V \rangle$$
$$= -2 \langle \tilde{b} \cdot \nabla \sqrt{\Gamma}, \sqrt{\Gamma} \log V \rangle.$$

Hence, by Proposition 1.2,

$$|\langle \Gamma \tilde{b} \cdot \nabla \log V \rangle| \leq 2 \|B\|_{\text{BMO}} \|\nabla \sqrt{\Gamma}\|_2 \|\nabla (\sqrt{\Gamma} \log V)\|_2,$$

where

$$\|\nabla \sqrt{\Gamma}\|_2 = \sqrt{\frac{d}{8\beta} \frac{1}{t - t_s}},$$

$$\|\nabla (\sqrt{\Gamma} \log V)\|_2^2 = \left\langle \left((\nabla \sqrt{\Gamma}) \log V + \sqrt{\Gamma} \nabla \log V \right)^2 \right\rangle$$
$$=: \left\langle (p + q)^2 \right\rangle = \left\langle p^2 + 2p \cdot q + q^2 \right\rangle.$$

Note that $\langle 2p \cdot q \rangle = \frac{1}{2} \langle \nabla \Gamma \cdot \nabla \log^2 V \rangle = -\frac{1}{2} \langle \Delta \Gamma, \log^2 V \rangle$. Using the equality $\Delta \Gamma = \frac{(\nabla \Gamma)^2}{\Gamma} - \frac{d}{2\beta(t - t_s)} \Gamma$ we have $\langle 2p \cdot q \rangle = -2 \langle p^2 \rangle + \frac{d}{4\beta(t - t_s)} \langle \Gamma \log^2 V \rangle$. Thus,

$$\|\nabla (\sqrt{\Gamma} \log V)\|_2 \leq \|q\|_2 + \sqrt{\frac{d}{4\beta(t - t_s)}} \|\sqrt{\Gamma} \log V\|_2$$

$$= \mathcal{N}_1^{\frac{1}{2}}(\tau) + \sqrt{\frac{d}{4\beta(t - t_s)}} \|\sqrt{\Gamma} \log V\|_2,$$

where $\mathcal{N}_1(\tau) := \langle \Gamma |\nabla \log V|^2 \rangle$ $(\leq \sigma^{-1} \mathcal{N}(\tau))$. Therefore,

$$|\langle \Gamma \tilde{b} \cdot \nabla \log V \rangle| \leq \frac{\gamma}{2} \mathcal{N}_1 + \|B\|_{\text{BMO}}^2 \frac{1}{\gamma} \frac{d}{4\beta(t - t_s)} + \|B\|_{\text{BMO}} \frac{d}{2\sqrt{2}\beta(t - t_s)} \|\sqrt{\Gamma} \log V\|_2.$$

Let us estimate the third term in the RHS of this inequality. We have

$$\frac{1}{t - t_s} \|\sqrt{\Gamma} \log V\|_2^2 = \frac{1}{t - t_s} \langle \Gamma (\log V - \langle \Gamma \log V \rangle)^2 \rangle + \frac{1}{t - t_s} \langle \Gamma \log V \rangle^2$$
$$\text{(we are applying the Spectral Gap Inequality in the first term)}$$
$$\leq 2\beta \mathcal{N}_1(\tau) + \frac{1}{t - t_s} \langle \Gamma \log V \rangle^2,$$

so

$$\frac{1}{\sqrt{t - t_s}} \|\sqrt{\Gamma} \log V\|_2 \leq \sqrt{2\beta} \sqrt{\mathcal{N}_1} + \frac{\langle \Gamma(-\log V) \rangle}{\sqrt{t - t_s}} \qquad \text{(we use } -\log V \geq 1, \text{ see (38))}.$$

Setting $\varphi := \Gamma(-\log V)$, we arrive at

$$|\langle \Gamma \tilde{b} \cdot \nabla \log V \rangle| \leq \gamma \mathcal{N}_1(\tau) + \|B\|_{\text{BMO}}^2 \frac{d^2}{8\gamma} \frac{1}{2\beta(t - t_s)} + \|B\|_{\text{BMO}} \frac{d}{2\sqrt{2}\beta(t - t_s)} \langle \varphi \rangle.$$

2. Term "$\langle \Gamma \frac{\tilde{b} \cdot \varepsilon \nabla \Gamma}{\varepsilon \Gamma + U} \rangle$". By Proposition 1.2,

$$|\left\langle \Gamma \frac{\tilde{b} \cdot \varepsilon \nabla \Gamma}{\varepsilon \Gamma + U} \right\rangle| = 2|\left\langle \tilde{b} \cdot \nabla \sqrt{\Gamma}, \frac{\varepsilon \Gamma^{\frac{3}{2}}}{\varepsilon \Gamma + U} \right\rangle| \leq 2\|B\|_{\mathrm{BMO}} \|\nabla \sqrt{\Gamma}\|_2 \|\nabla \frac{\varepsilon \Gamma^{\frac{3}{2}}}{\varepsilon \Gamma + U}\|_2,$$

and

$$\|\nabla \frac{\varepsilon \Gamma^{\frac{3}{2}}}{\varepsilon \Gamma + U}\|_2 = \|\frac{3\varepsilon \Gamma \nabla \sqrt{\Gamma}}{\varepsilon \Gamma + U} - \frac{\varepsilon \Gamma}{\varepsilon \Gamma + U} \sqrt{\Gamma} \nabla \log V\|_2$$

$$\leq 3\|\nabla \sqrt{\Gamma}\|_2 + \|\sqrt{\Gamma} \nabla \log V\|_2 \leq \frac{3}{2} \frac{\sqrt{d}}{\sqrt{2\beta(t - t_s)}} + \mathcal{N}_1^{\frac{1}{2}}(\tau).$$

Thus, upon applying quadratic inequality, we have

$$|\left\langle \Gamma \frac{\tilde{b} \cdot \varepsilon \nabla \Gamma}{\varepsilon \Gamma + U} \right\rangle| \leq \gamma \mathcal{N}_1(\tau) + \frac{1}{\gamma} \|B\|_{\mathrm{BMO}}^2 \frac{d}{8\beta(t - t_s)} + \|B\|_{\mathrm{BMO}} \frac{d}{2\beta(t - t_s)}.$$

Remark 6.1. Instead of $\tilde{b}$ above we could have considered $\tilde{b} + \check{b}$ with $\check{b} = \nabla P$, $P \in [L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)]^{d \times d}$, with $\mathrm{div}\, \check{b}$ form-bounded and in the Kato class. Indeed, we could modify Step 1 as follows:

$$|\langle \Gamma \check{b} \cdot \nabla \log V \rangle| \leq \langle (\mathrm{div}\, \check{b})_-, -\Gamma \log V \rangle + 2\|P\|_\infty \|\nabla^2 \sqrt{\Gamma}\|_2 \|\sqrt{\Gamma} \log V\|_2$$

$$+ 2\|P\|_\infty \|\nabla \sqrt{\Gamma}\|_2 \|\nabla (\sqrt{\Gamma} \log V)\|_2.$$

We estimate the first term in the same way as $A_{\mathrm{div}}(\tau)$ below, in the last two terms use $\|\nabla^2 \sqrt{\Gamma}\|_2 \leq C(t - t_s)^{-1}$, $\|\nabla \sqrt{\Gamma}\|_2 \leq C(t - t_s)^{-\frac{1}{2}}$ so that we can argue as in Step 1 above. Step 2 is modified similarly.

3. Term "$\langle \varepsilon \Gamma \frac{\hat{b} \cdot \nabla \Gamma}{\varepsilon \Gamma + U} \rangle$". We have

$$\left|\left\langle \varepsilon \Gamma \frac{\hat{b} \cdot \nabla \Gamma}{\varepsilon \Gamma + U} \right\rangle\right| \leq \langle |\hat{b}| |\nabla \Gamma| \rangle \leq \frac{\sqrt{2}}{\sqrt{\beta(t - t_s)}} \langle |\hat{b}| \Gamma_{2\beta} \rangle,$$

where, using that $\hat{b} \in \mathbf{MF}_\delta$ with multiplicative bound $\delta \equiv \delta_{\hat{b}}$ and function $g \equiv g_{\hat{\delta}}$, we estimate:

$$\langle |\hat{b}| \Gamma_{2\beta} \rangle \leq \delta \|\nabla \sqrt{\Gamma_{2\beta}}\|_2 \|\sqrt{\Gamma_{2\beta}}\|_2 + g(\tau)$$

$$= \frac{\delta}{2} \frac{\sqrt{d}}{\sqrt{2\beta(t - t_s)}} + g(\tau).$$

Thus,

$$\left|\left\langle \varepsilon \Gamma \frac{\hat{b} \cdot \nabla \Gamma}{\varepsilon \Gamma + U} \right\rangle\right| \leq \frac{\delta}{2} \frac{\sqrt{d}}{\beta(t - t_s)} + \frac{\sqrt{2} g(\tau)}{\sqrt{\beta(t - t_s)}}.$$

4. Term "$\langle \Gamma (\mathrm{div}\, \hat{b})_- \rangle$". Since $(\mathrm{div}\, \hat{b})_-$ is form-bounded with form-bound $\nu \equiv \nu_-$ and function $h \equiv h_\nu$, we have

$$\langle \Gamma (\mathrm{div}\, \hat{b})_- \rangle \leq \frac{\nu}{4} \frac{d}{2\beta} \frac{1}{t - t_s} + h(\tau).$$

5. Term "$\langle \Gamma \hat{b} \cdot \nabla \log V \rangle$". This is the most difficult term. Using $\hat{b} \in \mathbf{MF}_\delta$ and the form-boundedness of $\operatorname{div} \hat{b}_-$, we will prove that there exists a constant $\tilde{c}$ such that

$$
|\langle \Gamma \hat{b} \cdot \nabla \log V \rangle| \leq (2\gamma + \nu \sigma^{-1}) \mathcal{N}
$$
$$
+ \left(\frac{c_0^*}{t - t_s} + \frac{g(\tau)}{2\sqrt{\beta(t - t_s)}} + \frac{c_2^*}{t - t_s} + \frac{d\, g(\tau)}{4\sqrt{\beta(t - t_s)}} + \frac{\nu d}{4\beta(t - t_s)} + h(\tau) \right)(-Y(\tau) - c)
$$
$$
+ \frac{c_1^* + c_3^* + \frac{\nu d}{16\beta}}{t - t_s} + \frac{d}{4\gamma\sigma} g^2(\tau). \tag{43}
$$

Proof of (43). We estimate

$$
|\langle \Gamma \hat{b} \cdot \nabla \log V \rangle| \leq |\langle \hat{b} \cdot \nabla \Gamma, -\log V \rangle| + \langle \Gamma (\operatorname{div}\hat{b})_-, -\log V \rangle
$$
$$
\leq \frac{1}{\sqrt{\beta(t - t_s)}} \langle |\hat{b}| \Gamma, -\log V \rangle^{1/2} \left\langle |\hat{b}| \frac{|o - \cdot|^2}{4\beta(t - t_s)} \Gamma, -\log V \right\rangle^{1/2} + \langle \Gamma (\operatorname{div}\hat{b})_-, -\log V \rangle
$$
$$
=: \frac{1}{\sqrt{\beta(t - t_s)}} A_0^{1/2} A_2^{1/2} + A_{\operatorname{div}}
$$
$$
\leq \frac{1}{2\sqrt{\beta(t - t_s)}} (A_0 + A_2) + A_{\operatorname{div}},
$$

where

$$
A_0(\tau) := \left\langle |\hat{b}| \Gamma (-\log V) \right\rangle =: \langle |\hat{b}| \varphi \rangle,
$$

$$
A_2(\tau) := \left\langle |\hat{b}| \frac{|o - \cdot|^2}{4\beta(t - t_s)} \Gamma (-\log V) \right\rangle =: \langle |\hat{b}| \psi \rangle,
$$

$$
A_{\operatorname{div}}(\tau) := \langle (\operatorname{div}\hat{b})_-, \Gamma (-\log V) \rangle =: \langle (\operatorname{div}\hat{b})_- \varphi \rangle.
$$

Let us estimate $A_0(\tau)$.

$$
A_0(\tau) = \left\langle |b| \Gamma (-\log V) \right\rangle \equiv \langle |b| \varphi \rangle
$$
$$
\leq \delta \| \nabla \sqrt{\varphi} \|_2 \langle \varphi \rangle^{1/2} + g \langle \varphi \rangle
$$
$$
= \frac{1}{2} \delta \langle (\nabla \varphi)^2 / \varphi \rangle^{1/2} \langle \varphi \rangle^{1/2} + g \langle \varphi \rangle,
$$

$$
\nabla \varphi = \left(\frac{\nabla \Gamma}{\Gamma} + \frac{\nabla \log V}{\log V} \right) \varphi, \qquad \frac{(\nabla \varphi)^2}{\varphi} = \left(\frac{\nabla \Gamma}{\Gamma} + \frac{\nabla (-\log V)}{-\log V} \right)^2 \varphi,
$$

$$
\frac{(\nabla \varphi)^2}{\varphi} \leq 2 \left(\frac{(\nabla \Gamma)^2}{\Gamma} (-\log V) + \frac{(\nabla \log V)^2}{-\log V} \Gamma \right)
$$
$$
\leq 2 \left(\frac{|o - \cdot|^2}{(2\beta(t - t_s))^2} \Gamma (-\log V) + \Gamma (\nabla \log V)^2 \right) \ (\text{ because } -\log V > 1).
$$

By the equality $\frac{|o-\cdot|^2}{4\beta(t-t_s)}\Gamma = \beta(t-t_s)\Delta\Gamma + \frac{d}{2}\Gamma$,

$$\frac{1}{2}\left\langle\frac{(\nabla\varphi)^2}{\varphi}\right\rangle \leq \left\langle\Gamma(\nabla\log V)^2\right\rangle + \frac{1}{\beta(t-t_s)}\left\langle\left(\beta(t-t_s)\Delta\Gamma + \frac{d}{2}\Gamma\right)(-\log V)\right\rangle$$

$$\leq \sigma^{-1}\mathcal{N} + \left\langle\nabla\Gamma, \nabla\log V\right\rangle + \frac{d}{2\beta(t-t_s)}\langle\varphi\rangle$$

$$\leq 2\sigma^{-1}\mathcal{N} + \frac{1}{4}\left\langle\frac{(\nabla\Gamma)^2}{\Gamma}\right\rangle + \frac{d}{2\beta(t-t_s)}\langle\varphi\rangle$$

$$\leq 2\sigma^{-1}\mathcal{N} + \frac{d}{8\beta(t-t_s)} + \frac{d}{2\beta(t-t_s)}\langle\varphi\rangle.$$

Therefore, by inequalities $(B + C + D)^{1/2} \leq (B + C)^{1/2} + D^{1/2}$ and $E^{1/2}(B + C)^{1/2}M^{1/2} \leq (B + C)\varepsilon + (4\varepsilon)^{-1}EM$ for positive numbers with $\varepsilon = \sigma\gamma/2$,

$$\frac{A_0(\tau)}{2\sqrt{\beta(t-t_s)}} \leq \frac{\delta}{4\sqrt{\beta(t-t_s)}}\left(2\sigma^{-1}\mathcal{N}(\tau) + \frac{d}{8\beta(t-t_s)} + \frac{d}{2\beta(t-t_s)}\langle\varphi\rangle\right)^{1/2}\langle\varphi\rangle^{1/2} + \frac{g(\tau)}{2\sqrt{\beta(t-t_s)}}\langle\varphi\rangle$$

$$\leq \gamma\mathcal{N}(\tau) + \left(\frac{c_0^*}{t-t_s} + \frac{g(\tau)}{2\sqrt{\beta(t-t_s)}}\right)\langle\varphi\rangle + \frac{c_1^*}{t-t_s},$$

where $c_i^* = c_i^*(d,\sigma,\xi,\delta,\gamma) > 0$, $i = 0, 1$.

Analogous calculations show that there are constants $c_i^* = c_i^*(d,\sigma,\xi,\delta,\gamma) > 0$, $i = 2, 3$, such that

$$\frac{A_2(\tau)}{2\sqrt{\beta(t-t_s)}} \leq \gamma\mathcal{N}(\tau) + \left(\frac{c_2^*}{t-t_s} + \frac{d\,g(\tau)}{4\sqrt{\beta(t-t_s)}}\right)\langle\varphi\rangle + \frac{c_3^*}{t-t_s} + \frac{d}{4\gamma\sigma}g^2(\tau). \qquad (\bullet)$$

Indeed, $A_2(\tau) \leq \frac{1}{2}\delta\left\langle\frac{(\nabla\psi)^2}{\psi}\right\rangle^{\frac{1}{2}}\langle\psi\rangle^{\frac{1}{2}} + g(\tau)\langle\psi\rangle$, $\psi := \frac{|o-\cdot|^2}{4\beta(t-t_s)}\Gamma(-\log V)$.

$$\nabla\psi = \frac{\cdot - o}{2\beta(t-t_s)}\Gamma(-\log V) + \frac{|o-\cdot|^2}{4\beta(t-t_s)}\nabla\Gamma(-\log V) + \frac{|o-\cdot|^2}{4\beta(t-t_s)}\Gamma\nabla(-\log V),$$

$$(\nabla\psi)^2 \leq 3\left(\frac{|o-\cdot|^2}{4\beta^2(t-t_s)^2}\Gamma^2(\log V)^2 + \frac{|o-\cdot|^4}{16\beta^2(t-t_s)^2}(\nabla\Gamma)^2(\log V)^2 + \frac{|o-\cdot|^4}{16\beta^2(t-t_s)^2}\Gamma^2(\nabla\log V)^2\right),$$

$$\frac{(\nabla\psi)^2}{\psi} \leq 3\left(\frac{1}{\beta(t-t_s)}\Gamma(-\log V) + \frac{|o-\cdot|^2}{4\beta(t-t_s)}\frac{(\nabla\Gamma)^2}{\Gamma}(-\log V) + \frac{|o-\cdot|^2}{4\beta(t-t_s)}\Gamma\frac{(\nabla\log V)^2}{-\log V}\right),$$

Using inequality $-\log V > \frac{|o-\cdot|^2}{4\beta(t-t_s)}$ and equality $\frac{(\nabla\Gamma)^2}{\Gamma} = \Delta\Gamma + \frac{d}{2\beta(t-t_s)}\Gamma$ we have

$$\left\langle\frac{(\nabla\psi)^2}{\psi}\right\rangle \leq 3\left(\frac{1}{\beta(t-t_s)}\langle\varphi\rangle + \frac{d}{2\beta(t-t_s)}\langle\psi\rangle + \left\langle\frac{|o-\cdot|^2}{4\beta(t-t_s)}\Delta\Gamma, (-\log V)\right\rangle + \left\langle\Gamma(\nabla\log V)^2\right\rangle\right),$$

$$\left\langle\frac{|o-\cdot|^2}{4\beta(t-t_s)}\Delta\Gamma, (-\log V)\right\rangle = \left\langle\frac{o-\cdot}{2\beta(t-t_s)}\cdot\nabla\Gamma, -\log V\right\rangle - \left\langle\frac{|o-\cdot|^2}{4\beta(t-t_s)}\nabla\Gamma, \nabla(-\log V)\right\rangle$$

$$\leq \left\langle\frac{|o-\cdot|^2}{4\beta^2(t-t_s)^2}\Gamma(-\log V)\right\rangle + \left\langle\frac{|o-\cdot|^3}{8\beta^2(t-t_s)^2}\Gamma^{\frac{1}{2}}, \Gamma^{\frac{1}{2}}|\nabla\log V|\right\rangle$$

$$\leq \frac{1}{\beta(t-t_s)}\langle\psi\rangle + \frac{C(d)}{\sqrt{\beta(t-t_s)}}(\sigma^{-1}\mathcal{N})^{\frac{1}{2}}$$

Thus

$$\left\langle \frac{(\nabla \psi)^2}{\psi} \right\rangle \leq 3 \left(\frac{1}{\beta(t-t_s)} \langle \varphi \rangle + \frac{d+2}{2\beta(t-t_s)} \langle \psi \rangle + 2\sigma^{-1}\mathcal{N} + \frac{C^2(d)}{4\beta(t-t_s)} \right),$$

Also $\langle \psi \rangle = \beta(t-t_s) \langle \frac{(\nabla \Gamma)^2}{\Gamma} (-\log V) \rangle = \frac{d}{2} \langle \varphi \rangle + \beta(t-t_s) \langle \Delta \Gamma, -\log V \rangle$ and so

$$\langle \psi \rangle \leq \frac{d}{2} \langle \varphi \rangle + \left(\frac{d}{2} \beta(t-t_s) \right)^{\frac{1}{2}} (\sigma^{-1}\mathcal{N})^{\frac{1}{2}},$$

$$\left\langle \frac{(\nabla \psi)^2}{\psi} \right\rangle \leq 3 \left(3\sigma^{-1}\mathcal{N} + \frac{(d+2)^2}{4\beta(t-t_s)} \langle \varphi \rangle + \frac{C'(d)}{4\beta(t-t_s)} \right),$$

and (•) follows.

Remark 6.2. Estimate (•) requires $g \in L^2(\mathbb{R}_+)$. Everywhere else in the proof it suffices to assume (18).

Finally, since $(\text{div}\,\hat{b})_-$ is form-bounded,

$$A_{\text{div}}(\tau) \leq \frac{\nu}{4} \left\langle \frac{(\nabla \varphi)^2}{\varphi} \right\rangle + h(\tau)\langle \varphi \rangle$$

$$\leq \nu\sigma^{-1}\mathcal{N} + \left(\frac{\nu d}{4\beta(t-t_s)} + h(\tau) \right) \langle \varphi \rangle + \frac{\nu d}{16\beta(t-t_s)}$$

Therefore,

$$\frac{1}{2\sqrt{\beta(t-t_s)}}(A_0 + A_2) + A_{\text{div}}$$

$$\leq (2\gamma + \nu\sigma^{-1})\mathcal{N} + \left(\frac{c_0^*}{t-t_s} + \frac{g(\tau)}{2\sqrt{\beta(t-t_s)}} + \frac{c_2^*}{t-t_s} + \frac{d\,g(\tau)}{4\sqrt{\beta(t-t_s)}} + \frac{\nu d}{4\beta(t-t_s)} + h(\tau) \right) \langle \varphi \rangle$$

$$+ \frac{c_1^* + c_3^* + \frac{\nu d}{16\beta}}{t-t_s} + \frac{d}{4\gamma\sigma} g^2(\tau).$$

The latter gives (43) upon noticing that $\langle \varphi \rangle = -Y(\tau) + \frac{d}{2} \log \frac{t-\tau}{t-t_s} - \log c_0 \leq -Y(\tau) - \tilde{c}$ for $\tilde{c} = \frac{d}{2} \log 2 + \log c_0$. This ends the proof of 5.

We are in position to complete the proof of Lemma 6.3. By estimates 1-5,

$$J \geq -\left[2\gamma + (2\gamma + \nu)\sigma^{-1} \right] \mathcal{N}(\tau) - \frac{C_1}{t-t_s} - \frac{(d+2)g(\tau)}{4\sqrt{\beta(t-t_s)}} - \frac{d}{4\gamma\sigma} g^2(\tau) - h(\tau)$$

$$- \left(\frac{C_2}{t-t_s} + \frac{\sqrt{2}g(\tau)}{\sqrt{\beta(t-t_s)}} + h(\tau) \right)(-Y(\tau) - \tilde{c}).$$

Therefore,

$$\mathcal{M}(\tau) \geq (1 - (4 + 2\sigma^{-1})\gamma - \sigma^{-1}\nu)\mathcal{N}(\tau) - (Y(\tau) - c)\partial_\tau \mu(\tau)$$

$$- \frac{C_1}{t-t_s} - \frac{(d+2)g(\tau)}{4\sqrt{\beta(t-t_s)}} - \frac{d}{4\gamma\sigma} g^2(\tau) - h(\tau) - \left(\frac{C_2}{t-t_s} + \frac{\sqrt{2}g(\tau)}{\sqrt{\beta(t-t_s)}} + h(\tau) \right)(-Y(\tau) - \tilde{c}),$$

where, recalling that $\nu < \sigma$, we select $\gamma > 0$ sufficiently small to keep the coefficient of $\mathcal{N}(\tau)$ non-negative. Next, recall that

$$\partial_\tau \mu(\tau) = \frac{1}{t - t_s} + \frac{\sqrt{2} g(\tau)}{\sqrt{\beta(t - t_s)}} + g^2(\tau) + h(\tau).$$

It is now easily seen that we can select c sufficiently large so that $\mathcal{M}(\tau) \geq 0$. The proof of Lemma 6.3 is completed.

6.3. $\hat{G}$-function for $-\nabla \cdot a \cdot \nabla + \nabla \cdot (\hat{b} + \tilde{b})$. Let $u_*(t, x; s, y)$ denote the heat kernel of

$$\Lambda_* = -\nabla \cdot a \cdot \nabla + \nabla \cdot (\hat{b} + \tilde{b}).$$

By (37), by duality, $u_*(t, x; s, y)$ satisfies the upper Gaussian bound

$$u_*(t, x; s, y) \leq \hat{c}_3 k_{c_4}(t - s; x - y). \tag{44}$$

The constants in the next proposition depend on the same parameters as the constants in the theorem except of the Kato bounds $\mu_\pm$ and $(\operatorname{div} b)_\pm$.

Proposition 6.2. *Let β and $\mathbb{C}$ be constants from Proposition 6.1. Set $o = \frac{x+y}{2}$, $x, y \in \mathbb{R}^d$, $t_s = \frac{t+s}{2}$. Then*

$$\hat{G}(t_s) := \langle k_\beta(t_s - s, o - \cdot) \log u_*(t_s, \cdot; s, z) \geq -\tilde{Q}(t_s - s) - \mathbb{C}, \quad z \in B(o, \sqrt{t_s - s}).$$

Proof. The proof repeats the proof of Proposition 6.1, except that we have to deal with the positive part $\operatorname{div} \hat{b}_+$ of the divergence of $\hat{b}$. $\qquad\square$

6.4. Auxiliary operator $-\nabla \cdot a \cdot \nabla + (\hat{b} + \tilde{b}) \cdot \nabla - (\operatorname{div} \hat{b})_-$. Recall the notation $\Lambda = -\nabla \cdot a \cdot \nabla + (\hat{b} + \tilde{b}) \cdot \nabla$, $\Lambda_* = -\nabla \cdot a \cdot \nabla + \nabla \cdot (\hat{b} + \tilde{b})$. Set

$$\mathcal{H}^- := \Lambda - (\operatorname{div} \hat{b})_-.$$

Let $H^{t,s} f$ denote the solution of

$$\begin{cases} -\frac{d}{dt} H^{t,s} f = \mathcal{H}^- H^{t,s} f \\ 0 \leq f \in L^1 \cap L^\infty. \end{cases},$$

Let $h(t) := H^{t,s} f$. It is seen (for example, using the Duhamel formula) that

$$u(t, x; t_s, y) \leq h(t, x; t_s, y),$$

$$u_*(t_s, x; s, y) \leq h(t_s, x; s, y),$$

where u, u_* are the heat kernels of Λ, Λ_*, respectively. It is seen that

$$h(t, x; s, y) \geq (4\pi\beta(t - t_s))^{d/2} \langle k_\beta(t - t_s, o - \cdot) h(t, x; t_s, \cdot) h(t_s, \cdot; s, y) \rangle,$$

$$k_\beta(t - t_s, o - \cdot) = k_\beta(t_s - s, o - \cdot),$$

and, for all $2|x - y| \leq \sqrt{\beta(t - t_s)}$, due to Proposition 6.1 and Proposition 6.2,

$$
\begin{aligned}
\log h(t, x; s, y) &\geq \log(4\pi\beta)^{d/2} + \tilde{Q}(t - t_s) \\
&\quad + \langle k_\beta(t - t_s, o - \cdot) \log u(t, x; t_s, \cdot)\rangle + \langle k_\beta(t - t_s, o - \cdot) \log u_*(t_s, \cdot; s, y)\rangle \\
&\geq \log(4\pi\beta)^{d/2} - \tilde{Q}(t - t_s) - 2\mathbb{C} \\
&= -\tilde{Q}(t - s) - 2\mathbb{C} + \log(8\pi\beta)^{d/2},
\end{aligned}
$$

i.e. we have proved a lower Gaussian bound for $h(t, x; s, y)$ but only for $2|x - y| \leq \sqrt{\beta(t - t_s)}$. Now, the standard argument ("small gains yield large gain"), see e.g. [D, Theorem 3.3.4], gives

Theorem 6.4. *There exist constants c_1, $c_2 > 0$ such that, for all $x, y \in \mathbb{R}^d$, $0 \leq s < t < \infty$,*

$$
c_1 k_{c_2}(t - s, x - y) \leq h(t, x; s, y) \tag{LGB$^{h_-}$.}
$$

6.5. **Proof of Theorem 6.1 (a priori lower bound).** *Step 1.* First, we establish an upper bound on the heat kernel $h(t, x; s, y)$ of $\mathcal{H}^- = A + (\hat{b} + \tilde{b}) \cdot \nabla - (\operatorname{div}\hat{b})_-$:

$$
h(t, x; s, y) \leq \tilde{c}_3 k_{c_4}(t - s; x - y) \tag{$**$}
$$

for all $x, y \in \mathbb{R}^d$ and $0 \leq s < t < \infty$. Indeed, we can write the Duhamel series for $h(t, x; s, y)$, with $\mathcal{H}^-$ is viewed as $A + (\hat{b} + \tilde{b}) \cdot \nabla$ perturbed by potential $-(\operatorname{div}\hat{b})_-$. Then the upper Gaussian bound on $u(t, x; s, y)$, established in Theorem 2.2, and the hypothesis that the Kato norm μ_- of $(\operatorname{div}\hat{b})_-$ is sufficiently small, yield $(**)$ via a standard argument.

Step 2. Let us consider $A + (\hat{b} + \tilde{b}) \cdot \nabla$ as the perturbation of $\mathcal{H}^-$ by $(\operatorname{div}\hat{b})_-$. Then

$$
u(t, x; s, y) = h(t, x; s, y) - \int_s^t \langle u(t, x; \tau, \cdot)(\operatorname{div}\hat{b})_-(\cdot)h(\tau, \cdot; s, y)\rangle d\tau.
$$

We apply Theorem 6.4 to the first term. In the second term, we can apply the upper Gaussian bound on $u(t, x; \tau, \cdot)$ established in Theorem 2.2, apply $(**)$, and use the hypothesis that the Kato norm μ_- of $\operatorname{div}\hat{b}_-$ is sufficiently small to obtain

$$
u(t, x; s, y) \geq \frac{C_1}{(t - s)^{\frac{d}{2}}} - \frac{C_2\mu_-}{(t - s)^{\frac{d}{2}}}
$$

whenever $|x - y| \leq \sqrt{t - s}$, for all $0 \leq s < t < \infty$ (see [Z3, Sect. 5] for detailed argument). This yields $u(t, x; s, y) \geq C_0(t - s)^{-\frac{d}{2}}$ provided that μ_- is sufficiently small. Now, a standard argument ([D, Theorem 3.3.4]) gives the required lower Gaussian bound on $u(t, x; s, y)$ for all $x, y \in \mathbb{R}^d$, $0 \leq s < t < \infty$.

7. Proof of Theorem 2.3

We fix the following bounded smooth approximation of a, $\hat{b}$, $\tilde{b}$. In what follows, we extend a, $\hat{b}$, $\tilde{b}$ to $t < 0$ by zero. Let E_ε be the De Giorgi mollifier on $\mathbb{R} \times \mathbb{R}^d$,

$$
E_\varepsilon f(t, x) := \int_\mathbb{R} \langle e^{\varepsilon\Delta_{d+1}}(t, x; s, \cdot)f(s, \cdot)\rangle ds, \quad f \in L^1_{\operatorname{loc}}(\mathbb{R} \times \mathbb{R}^d)
$$

(Δ_{d+1} is the Laplacian on $\mathbb{R} \times \mathbb{R}^d$). Let

$$
E_\varepsilon^d f(x) := \langle e^{\varepsilon\Delta}(x, \cdot)f(\cdot)\rangle, \quad f \in L^1_{\operatorname{loc}}.
$$

For a Banach-valued measurable function $h = h(t)$, we define its Steklov averaging

$$[h]_\varepsilon(t) := \frac{1}{\varepsilon} \int_t^{t+\varepsilon} h(r)dr.$$

Put

$$a_{\varepsilon_1} := E_{\varepsilon_1} a.$$

Clearly, a_{ε_1} is C^∞ smooth, symmetric and uniformly elliptic with the same constants as a.

Set

$$\hat{b}_\varepsilon := \left[E_\varepsilon^d \hat{b}\right]_{c(\varepsilon)},$$

$$(\operatorname{div} \hat{b})_{\pm,\varepsilon} := [E_\varepsilon^d (\operatorname{div} \hat{b})_\pm]_{c(\varepsilon)},$$

where $c(\varepsilon) > 0$ is to be chosen. Then, clearly, $(\operatorname{div} \hat{b})_\varepsilon = (\operatorname{div} \hat{b})_{+,\varepsilon} - (\operatorname{div} \hat{b})_{-,\varepsilon}$.

(Note that we can not use the same regularization of $\hat{b}$ as in Proposition 4.3 since the indicator function $\mathbf{1}_\varepsilon$ there would not allow us to control $\operatorname{div} \hat{b}_\varepsilon$.)

Finally, given a vector field $(\tilde{b} \in L^\infty \operatorname{BMO}^{-1}, \operatorname{div} \tilde{b} = 0)$, we define its bounded smooth approximation as in [QX1, Sect. 3]. There exist a skew-symmetric matrix $B \in [L^\infty \operatorname{BMO}]^{d \times d} \cap [L^p_{\operatorname{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d$ for all $1 \leq p < \infty$ such that $b = \nabla B$. Set

$$B_\varepsilon := e^{\varepsilon \Delta_{d+1}}(B \wedge U_\varepsilon \vee V_\varepsilon) \quad \text{(max and min are taken component-wise)},$$

where $U_\varepsilon := (-c \log|x| + \varepsilon^{-1}) \wedge \varepsilon^{-1} \vee 0$, $V_\varepsilon := (c \log|x| - \varepsilon^{-1}) \wedge 0 \vee (-\varepsilon^{-1})$ are BMO functions with compact support. The constant c is chosen so that $\|c \log|x|\|_{\operatorname{BMO}} \leq \|B\|_{L^\infty \operatorname{BMO}}$. Define

$$\tilde{b}_\varepsilon := \nabla B_\varepsilon$$

(since B_ε are skew-symmetric, $\operatorname{div} \tilde{b}_\varepsilon = 0$).

Proposition 7.1. *Let* $\hat{b} \in \mathbf{MF}_\delta$. *Then the following are true:*

(i) For every $t \geq 0$, $x \in \mathbb{R}^d$,

$$|E_\varepsilon^d \hat{b}(t,x)| \leq \sqrt{\frac{d}{8\varepsilon}}\delta + g(t).$$

(ii) $E_\varepsilon^d \hat{b} \in L^2_{\operatorname{loc}}(\mathbb{R}_+, C_b(\mathbb{R}^d))$.

Proof. (i) We have

$$|E_\varepsilon^d \hat{b}(t,x)| = |\langle \hat{b}(t,\cdot)\sqrt{e^{\varepsilon \Delta}(x,\cdot)}, \sqrt{e^{\varepsilon \Delta}(x,\cdot)}\rangle|$$

$$\leq \delta \langle |\nabla \sqrt{e^{\varepsilon \Delta}(x,\cdot)}|^2 \rangle^{\frac{1}{2}} + g(t)$$

$$\left(\text{we use } \langle |\nabla \sqrt{e^{\varepsilon \Delta}(x,\cdot)}|^2 \rangle = \frac{d}{8\varepsilon}\right)$$

$$\leq \sqrt{\frac{d}{8\varepsilon}}\delta + g(t).$$

(ii) Since $E_\varepsilon^d = E_{\varepsilon/2}^d E_{\varepsilon/2}^d$, we have for a.e. $t \in \mathbb{R}_+$ $E_\varepsilon^d \hat{b}(t,\cdot) \in C_b$. Since $g \in L^2(\mathbb{R}_+)$, we have $E_\varepsilon^d \hat{b} \in L^2_{\operatorname{loc}}(\mathbb{R}_+, C_b(\mathbb{R}^d))$. $\qquad \square$

Recall that

$$\hat{b}_\varepsilon = \left[E_\varepsilon^d \hat{b}\right]_{c(\varepsilon)},$$

where the rate of Steklov averaging $c(\varepsilon) \downarrow 0$ is to be chosen.

Proposition 7.2. *Let $\hat{b} \in \mathbf{MF}_\delta$. Then*

(i) $\hat{b}_\varepsilon \in [L^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \cap C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)]^d$.

(ii) $\hat{b}_\varepsilon \in \mathbf{MF}_\delta$ *with* $\|g_\varepsilon\|_{L^2(\mathbb{R}_+)} \leq \|g\|_{L^2(\mathbb{R}_+)}$.

Proof. (i) follows immediately from Proposition 7.1(ii) and the properties of Steklov averaging.

(ii) First, let us note that $\|\nabla\sqrt{E_\varepsilon^d |\varphi|^2}\|_2 \leq \|\nabla\varphi\|_2$. Indeed,

$$\|\nabla\sqrt{E_\varepsilon^d |\varphi|^2}\|_2 = \|\frac{E_\varepsilon^d(|\varphi||\nabla|\varphi|)}{\sqrt{E_\varepsilon^d |\varphi|^2}}\|_2 \leq \|\sqrt{E_\varepsilon^d |\nabla|\varphi|^2}\|_2 = \|E_\varepsilon |\nabla|\varphi|^2\|_1^{\frac{1}{2}} \leq \|\nabla|\varphi|\|_2 \leq \|\nabla\varphi\|_2.$$

By $\hat{b} \in \mathbf{MF}_\delta$, we have for a.e. $t \in \mathbb{R}_+$ and all $\varphi \in \mathcal{S}$,

$$\langle |E_\varepsilon^d \hat{b}(t)|\varphi, \varphi\rangle \leq \langle |b(t)|, E_\varepsilon^d |\varphi|^2\rangle$$

$$\leq \delta\|\nabla\sqrt{E_\varepsilon^d |\varphi|^2}\|_2 \|\sqrt{E_\varepsilon^d |\varphi|^2}\|_2 + g(t)\|\sqrt{E_\varepsilon^d |\varphi|^2}\|_2$$

$$\leq \delta\|\nabla\varphi\|_2 \|\varphi\|_2 + g(t)\|\varphi\|_2^2.$$

Hence for a.e. $t \in \mathbb{R}_+$

$$\langle |\left[E_\varepsilon^d \hat{b}\right]_{c(\varepsilon)}(t)|\varphi, \varphi\rangle \leq \left[\langle |E_\varepsilon^d \hat{b}|\varphi, \varphi\rangle\right]_{c(\varepsilon)}(t)$$

$$\leq \delta\|\nabla\varphi\|_2 \|\varphi\|_2 + [g]_{c(\varepsilon)}(t)\|\varphi\|_2^2.$$

Hence $\hat{b}_\varepsilon \in \mathbf{M}_{\delta, g_\varepsilon}$ with $g_\varepsilon := [g]_{c(\varepsilon)}$, and, clearly, $\|g_\varepsilon\|_2 \leq \|g\|_2$. $\qquad\square$

Recall

$$(\operatorname{div}\hat{b})_{\pm,\varepsilon} := [E_\varepsilon^d(\operatorname{div}\hat{b})_\pm]_{c(\varepsilon)}.$$

Proposition 7.3. *Let $(\operatorname{div}\hat{b})_\pm$ be form-bounded, i.e. $(\operatorname{div}\hat{b})_\pm \in L_{\mathrm{loc}}^1(\mathbb{R}_+ \times \mathbb{R}^d)$, the inequality*

$$\langle(\operatorname{div}\hat{b})_\pm\varphi, \varphi\rangle \leq \nu_\pm\|\nabla\varphi\|_2^2 + h_\pm(t)\|\varphi\|_2^2, \quad \varphi \in W^{1,2}$$

holds for a.e. $t \in \mathbb{R}_+$ and some functions $0 \leq h_\pm \in L_{\mathrm{loc}}^1(\mathbb{R}_+)$.

Let $\mu_\pm$ be the Kato norms of $(\operatorname{div}\hat{b})_\pm$. Then the following is true:

(i) $(\operatorname{div}\hat{b})_{\pm,\varepsilon} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \cap C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$.

(ii) $(\operatorname{div}\hat{b})_{\pm,\varepsilon}$ *is form-bounded with the same form-bounds $\nu_\pm$ and $\|h_{\pm,\varepsilon}\|_{L^1(\mathbb{R}_+)} \leq \|h_\pm\|_{L^1(\mathbb{R}_+)}$.*

(iii) $(\operatorname{div}\hat{b})_{\pm,\varepsilon}$ *have Kato norms $\mu_\pm$, for all $\varepsilon > 0$.*

Proof. The proof of (i), (ii) follows closely the proof of Proposition 7.2.

(iii) Since the translations of $(\operatorname{div} b)_+(s, \cdot)$ in s belong to the Kato class with the same Kato norm, we have

$$\sup_{t \geq 0, x \in \mathbb{R}^d} \int_0^t \langle k(t - s, x, \cdot)(\operatorname{div} b)_{+,\varepsilon}(s, \cdot)\rangle ds$$

$$= \sup_{t \geq 0, x \in \mathbb{R}^d} \frac{1}{c(\varepsilon)} \int_0^{c(\varepsilon)} \int_0^t \langle k(t - s, x, \cdot) E_\varepsilon^d(\operatorname{div} b)_+(s + r, \cdot)\rangle ds dr \leq \frac{1}{c(\varepsilon)} \int_0^{c(\varepsilon)} \mu_+ dr = \mu_+.$$

The second integral in the definition of the Kato norm is treated in the same way.   □

Let us prove the convergence results in (26). The convergence $a_{\varepsilon_1} \to a$ in $[L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^{d \times d}$ is evident. Next,

$$\hat{b}_\varepsilon \equiv [E^d_\varepsilon \hat{b}]_{c(\varepsilon)} \to \hat{b} \quad \text{in} \quad [L^1_{\mathrm{loc}}(\mathbb{R}_+, L^1_{\mathrm{loc}})]^d \quad \text{as } \varepsilon \downarrow 0 \tag{45}$$

provided that $c(\varepsilon) \downarrow 0$ sufficiently rapidly. Indeed, for any $T > 0$, $r > 0$,

$$\mathbf{1}_{[0,T]} \mathbf{1}_{B(0,r)} [E^d_\varepsilon \hat{b}]_{c(\varepsilon)} = \mathbf{1}_{[0,T]} [\mathbf{1}_{B(0,r)} E^d_\varepsilon \hat{b}]_{c(\varepsilon)}.$$

Since $C(\bar{B}(0,r))$ is a separable Banach space, by known properties of the Steklov averaging of Banach-valued functions, we have, for every fixed $\varepsilon > 0$,

$$\mathbf{1}_{[0,T]} [\mathbf{1}_{B(0,r)} E^d_\varepsilon \hat{b}]_{\varepsilon_2} \to \mathbf{1}_{[0,T]} \mathbf{1}_{B(0,r)} E^d_\varepsilon \hat{b} \quad \text{in} \quad [L^1([0,T], C(\bar{B}(0,r)))]^d \quad \text{as } \varepsilon_2 \downarrow 0.$$

Therefore, for every $\varepsilon > 0$ can find ε_2 such that

$$\|\mathbf{1}_{[0,T]} [\mathbf{1}_{B(0,r)} E^d_\varepsilon \hat{b}]_{\varepsilon_2} - \mathbf{1}_{[0,T]} \mathbf{1}_{B(0,r)} E^d_\varepsilon \hat{b}\|_{L^1([0,T],L^1(\bar{B}(0,r)))} < \varepsilon.$$

We put $c(\varepsilon) := \varepsilon_2$. In turn,

$$\mathbf{1}_{[0,T]} \mathbf{1}_{B(0,r)} E^d_\varepsilon \hat{b} \to \mathbf{1}_{[0,T]} \mathbf{1}_{B(0,r)} \hat{b} \quad \text{in } [L^1([0,T], L^1(\bar{B}(0,r)))]^d \quad \text{as } \varepsilon \downarrow 0.$$

Hence

$$\hat{b}_\varepsilon \equiv [E^d_\varepsilon \hat{b}]_{c(\varepsilon)} \to \hat{b} \quad \text{in} \quad [L^1([0,T], L^1(\bar{B}(0,r)))]^d \quad \text{as } \varepsilon \downarrow 0.$$

Our choice of $c(\varepsilon)$ depends on T and r. It is clear however that, using a diagonal argument, we can select $c(\varepsilon)$ even smaller to have (45).

The same argument yields

$$(\mathrm{div}\,\hat{b})_{\pm,\varepsilon} \equiv [E^d_\varepsilon \mathrm{div}\,\hat{b}_\pm]_{c(\varepsilon)} \to \mathrm{div}\,\hat{b}_\pm \quad \text{in } [L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)]^d \quad \text{as } \varepsilon \downarrow 0$$

(we take $c(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ even more rapidly, if needed).

Proposition 7.4 ([QX1, Sect. 3]). *Let $\tilde{b} \in L^\infty\,\mathrm{BMO}^{-1}$, $\mathrm{div}\,\tilde{b} = 0$. Then*

(*i*) $\tilde{b}_\varepsilon \in [L^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \cap C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)]^d$.

(*ii*) $\|B_\varepsilon\|_{L^\infty\,\mathrm{BMO}} \leq C\|B\|_{L^\infty\,\mathrm{BMO}}$ *for a constant C that only depends on the dimension d, and*

$$B_\varepsilon \to B \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d) \text{ for all } 1 \leq p < \infty.$$

By the definition of $\|\cdot\|_{L^\infty\,\mathrm{BMO}^{-1}}$, it follows from assertion (*ii*) that

$$\|\tilde{b}_\varepsilon\|_{L^\infty\,\mathrm{BMO}^{-1}} \leq C\|\tilde{b}\|_{L^\infty\,\mathrm{BMO}^{-1}}.$$

We are in position to end the proof of Theorem 2.3. Arguing as in [FS], we obtain from the a priori two-sided Gaussian heat kernel bounds established in Theorems 2.2, 6.1: for all ε_1, $\varepsilon > 0$, given a solution $v_{\varepsilon_1,\varepsilon} \in C^\infty([r - R^2, r] \times \bar{B}(z, R))$ to $(\partial_t - \nabla \cdot a_{\varepsilon_1} \cdot \nabla + b_\varepsilon \cdot \nabla)v_{\varepsilon_1,\varepsilon} = 0$ in $]r - R^2, r[\times B(z, R)$, where $R \leq 1$, $z \in \mathbb{R}^d$, one have for every $0 < \alpha < 1$

$$|v_{\varepsilon_1,\varepsilon}(t,x) - v_{\varepsilon_1,\varepsilon}(t',x')| \leq C\|v_{\varepsilon_1,\varepsilon}\|_{L^\infty([r-R^2,r]\times\bar{B}(z,R))} \left(\frac{|t - t'|^{\frac{1}{2}} + |x - x'|}{R} \right)^\beta$$

for all (t, x), $(t', x') \in [r - (1 - \alpha^2)R^2, r] \times \bar{B}(z, (1 - \alpha)R)$ for some constants C and $\beta \in]0, 1[$ (i.e. independent of ε_1, ε). This result applies, in particular, to the heat kernel $u_{\varepsilon_1,\varepsilon}(t, x; s, y)$ of $-\nabla \cdot a_{\varepsilon_1} \cdot \nabla + b_\varepsilon \cdot \nabla$ with s and y fixed. Therefore, applying Arzelà-Ascoli Theorem on sets

$\{(t,x) \mid t \geq s + \frac{1}{n}, |x| \leq n\}$ coupled with a diagonal argument, we can extract sequences ε_1, $\varepsilon \downarrow 0$ such that

$$u_{\varepsilon_1, \varepsilon}(\cdot, \cdot; s, y) \to u(\cdot, \cdot; s, y) \quad \text{uniformly on every } U_n, \, n = 1, 2, \ldots$$

to some function $u(\cdot, \cdot; s, y)$. By construction, $u(t, x; s, y)$ satisfies two-sided Gaussian bounds and is Hölder continuous in (t, x). Moreover, using again two-sided Gaussian bounds and a weak compactness argument, we may assume that, for every $m = 1, 2, \ldots$,

$$u_{\varepsilon_1, \varepsilon} \to u \text{ weakly in } L^2(\{\frac{1}{m} \leq t - s \leq m\} \times \{|x - y| \geq \frac{1}{m}\}),$$

so $u(t, x; s, y)$ as a function of variables $(t, x; s, y)$ is measurable.

Furthermore, two-sided Gaussian bounds on u and a standard mollifier argument yield

$$\langle u(t, x; s, \cdot) f(\cdot) \rangle \to f(x) \quad \text{as } t \downarrow s$$

in L^p ($1 \leq p < \infty$) or C_u depending on where f is. We define the sought evolution family $T^{t,s} \equiv T^{t,s}(a, b)$ by

$$T^{t,s} f(x) := \langle u(t, x; ; s, \cdot) f(\cdot) \rangle.$$

The assertions (a), (c) of the theorem now follow. Assertion (b) follows via a standard compactness argument. The integral kernel $u(t, x; s, y)$ is, by definition, a heat kernel of the formal operator $-\nabla a \cdot \nabla + (\hat{b} + \tilde{b}) \cdot \nabla$.

Finally, to prove the second statement in (d), let us regularize $\hat{b} = \hat{b}^{(1)} + \hat{b}^{(2)}$ as above. That is, put $\hat{b}_\varepsilon^{(1)} := [E_\varepsilon \hat{b}^{(1)}]_{c(\varepsilon)}$ and note that it satisfies

$$\||\hat{b}_\varepsilon^{(1)}(t)|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta_1}$$

(by an argument similar to the one in the proof of Proposition 7.2), while $\hat{b}_\varepsilon^{(2)} := [E_\varepsilon \hat{b}^{(2)}]_{c(\varepsilon)}$ satisfies

$$\|(\lambda - \Delta)^{-\frac{1}{2}}|\hat{b}_\varepsilon^{(2)}(t)|\|_\infty \leq \sqrt{\delta_2}$$

(by the integration by parts). Hence $\hat{b}_\varepsilon := \hat{b}_\varepsilon^{(1)} + \hat{b}_\varepsilon^{(2)} \in L^\infty \, \mathbf{F}_\delta^{1/2}$, so we can apply the convergence result in Theorem 2.1, which yields the required.

8. Further discussion and examples

Remark 8.1. The theory of operator $-\Delta + b \cdot \nabla$ is quite different from the theory of $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ with general uniformly elliptic measurable matrix a. This is clear already from the existence of the Kato class of vector fields $\mathbf{K}_\delta^{d+1}$, which is specific to $-\Delta + b \cdot \nabla$. Recall that, in the time-homogeneous case (for brevity),

$$\text{Kato class } \mathbf{K}_\delta^{d+1}: \quad |b| \in L_{\text{loc}}^1 \quad \text{and} \quad \|(\lambda - \Delta)^{-\frac{1}{2}}|b|\|_\infty \leq \sqrt{\delta}$$

for some $\lambda = \lambda_\delta \geq 0$. Also, recall:

$$\text{class of form-bounded vector fields } \mathbf{F}_\delta: \quad |b| \in L_{\text{loc}}^2 \quad \text{and} \quad \||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}.$$

The Kato class $\mathbf{K}_\delta^{d+1}$ provides two-sided Gaussian bounds on the heat kernel of (8) [Z2] and ensures uniqueness in law for the corresponding SDE, at least as long as the relative bound δ can be chosen arbitrarily small [BC]. The Kato class contains vector fields b with $|b| \notin L_{\text{loc}}^p(\mathbb{R}_+ \times \mathbb{R}^d)$ for any

$p > 1$. Clearly, they cannot be form-bounded. On the other hand, the Kato class does not contain even $b = b(x)$ with $|b| \in L^d(\mathbb{R}^d)$, so the two classes are incomparable.

The Kato class condition implies, by duality, that

$$\|b \cdot \nabla(\lambda - \Delta)^{-1}\|_{1\to1} \leq c(d)\sqrt{\delta} \equiv \sqrt{\delta_1},$$

i.e. that $b \cdot \nabla$ is strongly subordinate to $\lambda - \Delta$ in L^1. Then, if $\delta_1 < 1$, the Miyadera Theorem ensures that the algebraic sum $-\Delta + b \cdot \nabla$ of domain $(1 - \Delta)^{-1}L^1$ generates a quasi bounded C_0 semigroup in L^1. (This is one instance where the Miyadera Theorem is indispensable.) On the other hand, if $b \in \mathbf{F}_\delta$ with $\delta < 1$, then the KLMN Theorem ensures that the quadratic form of $-\Delta + b \cdot \nabla$ of domain $W^{1,2}$ determines the (minus) generator of a quasi contraction C_0 semigroup in L^2. The former semigroup cannot be a quasi contraction in L^2, while the latter semigroup cannot be strongly continuous in L^1. The bases of these solution theories of (8) are, essentially, mutually exclusive.

One arrives at the problem of unification of the two solution theories of equation (8), for instance, to treat $b = b^{(1)} + b^{(2)}$, where $b^{(1)}$ is form-bounded and $b^{(2)}$ is from the Kato class. The two classes can be unified: by the Heinz inequality and the interpolation, we have

$$b^{(1)} \in \mathbf{F}_{\delta_1^2}, \quad b^{(2)} \in \mathbf{K}_{\delta_2^2}^{d+1} \quad \Rightarrow \quad b^{(1)} + b^{(2)} \in \mathbf{F}_\delta^{1/2},$$

where $\delta = \delta_1 + \delta_2$ (we used the fact that $b \in \mathbf{F}_\delta^{1/2}$ is equivalent to $\||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to2} \leq \sqrt{\delta}$). However, as Theorem 2.1(ii) shows, one should not be looking for the unification in the scale of L^p solution spaces.

Example 8.1. Speaking of elementary examples of $\hat{b}$ in assertion (iii), we single out the following class of time-homogeneous vector fields:

$$\mathbf{M}_\nu' := \big\{\hat{b} : \mathbb{R}^d \to \mathbb{R}^d \mid \hat{b} = \nabla(-\Delta)^{-1}W \text{ for some } W \in \mathbf{K}_\nu^d\big\},$$

where (time-homogeneous) $W \in \mathbf{K}_\nu^d$ if and only if $\|(-\Delta)^{-1}|W|\|_\infty \leq \nu$.

To see that any $\hat{b}$ from $\mathbf{M}_\nu'$ satisfies the hypothesis of Theorem 2.3, note that, for a given $\hat{b} \in \mathbf{M}_\nu'$, one automatically has $\operatorname{div} \hat{b} \in \mathbf{K}_\nu^d$. It is also clear that $\mathbf{M}_\nu' \subset \mathbf{K}_\nu^{d+1}$.

In fact, any $\hat{b}$ from $\mathbf{M}_\nu'$ is form-bounded. Indeed,

$$\langle \hat{b}^2|\psi|^2\rangle = \langle(-\Delta)^{-1}W, W|\psi|^2\rangle - 2\langle(-\Delta)^{-1}W, \hat{b}|\psi||\nabla|\psi|\rangle \qquad \psi \in C_c^\infty(\mathbb{R}^d)$$

$$\leq \nu\langle|W||\psi|^2\rangle + 2\nu\langle\hat{b}^2|\psi|^2\rangle^{\frac{1}{2}}\|\nabla|\psi|\|_2$$

$$\leq \nu\langle|W||\psi|^2\rangle + \frac{1}{2}\langle\hat{b}^2|\psi|^2\rangle + 2\nu^2\|\nabla|\psi|\|_2^2.$$

Thus $\langle\hat{b}^2|\psi|^2\rangle \leq 2\nu\langle|W||\psi|^2\rangle + 4\nu^2\|\nabla\psi\|_2^2$. It remains to note that $\mathbf{K}_\nu^d \subset \mathbf{F}_\nu$.

Example 8.2. One can modify the previous example by considering $\hat{b} = \hat{b}^{(1)} + \hat{b}^{(2)}$, where $\hat{b}^{(1)} \in \mathbf{M}_\nu'$ and

$$\hat{b}^{(2)} := (\phi_1(x)|x_2|^{-1+\varepsilon}, \phi_2(x)|x_1|^{-1+\varepsilon}, 0, \ldots, 0), \quad \varepsilon \in]0,1],$$

where $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^d)$. Clearly, $\hat{b}^{(2)} \in \mathbf{K}_\delta^{d+1}$, has divergence in $\mathbf{K}_\nu^d$, but is not in L_{loc}^2 and hence is not form-bounded. This vector field satisfies the assumptions of assertion (d) in Theorem 2.3.

Remark 8.2. In Theorem 2.3 we could consider

$$b = \hat{b} + \tilde{b} + \check{b},$$

where $\hat{b}$ and $\tilde{b}$ are as above, and $\check{b} = \nabla P$, $P \in [L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)]^{d \times d}$, with div $\check{b}$ form-bounded and in the Kato class. The corresponding analysis of Nash's G-functions follows the argument below and requires few modifications, see Remark 6.1. Theorem 2.2 does not require a modification since, as is easily seen, $\check{b} \in \mathbf{M}_\delta$ with $\delta = \|P\|_\infty$.

Remark 8.3. The problem of unification of the class of form-bounded vector fields and the Kato class was addressed earlier in the simpler case of time-homogeneous vector fields $b = b(x)$ in [S, K].

First, it was noticed in [S] that neither the form-boundedness nor the Kato class condition for $b = b(x)$ are responsible for the (L^p, L^q) bound

$$\|u(t)\|_q \leq c_T t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_p, \quad f \in L^p \cap L^q, \quad \frac{2}{2 - \sigma^{-1}\sqrt{\delta}} < p < q \leq \infty. \tag{46}$$

for the semigroup of $-\Delta + b \cdot \nabla$. In fact, it suffices to require that $b \in \mathbf{F}_\delta^{1/2}$, i.e. $\||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} \leq \sqrt{\delta}$. It turned out that this new class of *weakly form-bounded* vector fields contains the sums of vector fields $\mathbf{F}_{\delta^2}$ and $\mathbf{K}_{\delta^2}^{d+1}$. [S] also proposed a way to construct the semigroup generated by $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_\delta^{1/2}$, $\delta < 1$ in L^2 by "guessing" the resolvent of an appropriate operator realization Λ_2 of $-\Delta + b \cdot \nabla$:

$$(\zeta + \Lambda_2)^{-1} := (\zeta - \Delta)^{-\frac{3}{4}}(1 + S)^{-1}(\zeta - \Delta)^{-\frac{1}{4}}, \tag{47}$$

where $\operatorname{Re}\zeta \geq \frac{d}{d-1}\lambda$, the operator $S(\zeta) := (\zeta - \Delta)^{-\frac{1}{4}} b \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}}$ is bounded in L^2 by $b \in \mathbf{F}_\delta^{1/2}$. The RHS of (47) coincides, after expanding $(1 + S)^{-1}$ in the geometric series, with the Neumann series for $-\Delta + b \cdot \nabla$.

Next, it was shown in [K] that equation (8) with $b \in \mathbf{F}_\delta^{1/2}$ has a detailed L^p regularity theory for $p \in I_\delta$, where the open interval I_δ, centered around 2, expands to $]1, \infty[$ as $\delta \downarrow 0$. However one has to guess the resolvent differently:

$$(\zeta + \Lambda_p)^{-1} := (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} Q_p(q)(1 + T_p)^{-1} R_p(r)(\zeta - \Delta)^{-\frac{1}{2r'}}, \tag{48}$$

where $1 \leq r < p < q < \infty$, $T_p(\zeta) := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}} \in \mathcal{B}(L^p)$ and $Q_p(q)$, $R_p(r) \in \mathcal{B}(L^p)$ are such that one obtains again, after expanding $(1 + T_p)^{-1}$ in the geometric series, the Neumann series for $-\Delta + b \cdot \nabla$. (Note that the direct analogue of (47) in L^p requires a much more restrictive condition: $|b|$ is in the weak L^d space.) From (48), one obtains right away L^p regularity of the $1 + \frac{1}{q}$-order spatial derivatives of solutions to the corresponding parabolic equation, hence the corresponding Feller semigroup. See also further developments in [KS2]. The Feller semigroup determines, for every starting point, a weak solution to the corresponding SDE [KS3]. The crucial point here is that one works in L^p while keeping intact the $L^2 \to L^2$ assumption on the vector field b (i.e. weak form-boundedness) and hence the class of its admissible singularities, except for requiring a smaller δ.

The proof that the operator-valued function in the RHS of (47) determines the resolvent of the generator of a C_0 semigroup is delicate. For (48) the situation is more difficult. In both proofs, one needs the Trotter Approximation Theorem and Hille's theory of pseudo-resolvents. Both proofs depend crucially on the holomorphy of the constructed semigroup since the latter are only quasi-bounded.

This paper concerns with the time-inhomogeneous case $b = b(t, x)$, which presents the next level of difficulty. Having at hand the evolution family for (8) in $\mathcal{W}^{\frac{1}{2}, 2}$, provided by Theorem 2.1(ii), we now approach the problem of constructing the evolution family in L^2 as a separate problem.

The construction in [S, K] of the semigroup directly in L^2 and in L^p can be viewed now as solving several problems at the same time. We expect that the spatial $\mathcal{W}^{1+1/q,p}$ regularity of the evolution family constructed in Theorem 2.1, for $q > p$ and p large, can be obtained with additional effort, as well as the ensuing Feller property and a weak well-posedness of the corresponding SDE.

Appendix A.

Below we obtain the energy inequality (14) on a fixed interval $0 \le s < t \le T$ assuming that b satisfies (17) with $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$. We will be working at the a priori level, i.e. with b, f additionally assumed to be bounded and smooth, and $u(s) = f$.

We use notations introduced in the beginning of the proof of Theorem 2.1.

Multiplying (8) by $(\lambda - \Delta)^{\frac{1}{2}} u$ and integrating, we obtain

$$\|u(t)\|_H^2 + 2\int_s^t \|u(r)\|_{H_+}^2\, dr + 2\int_s^r \langle b(r) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} u\rangle dr \le \|f\|_H^2.$$

The term to control:

$$\left|\langle b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} u\rangle\right| \le \left\||b(t)|^{\frac{1}{2}}|\nabla u|\right\|_2 \left\||b(t)|^{\frac{1}{2}}|(\lambda - \Delta)^{\frac{1}{2}} u|\right\|_2.$$

By (17),

$$\left\||b(t)|^{\frac{1}{2}}|(\lambda - \Delta)^{\frac{1}{2}} u|\right\|_2^2 \le \delta\|(\lambda - \Delta)^{\frac{1}{4}}|(\lambda - \Delta)^{\frac{1}{2}} u|\|_2^2 + g(t)\|u\|_{\mathcal{W}^{1,2}}^2$$

$$\text{(we are using the Beurling-Deny inequality)}$$

$$\le \delta\|u\|_{H_+}^2 + g(t)\|u\|_{\mathcal{W}^{1,2}}^2.$$

Similarly, a variant of the Beurling-Deny inequality: $\|(\lambda - \Delta)^{\frac{1}{4}}|\nabla u|\|_2 \le \|(\lambda - \Delta)^{\frac{1}{4}}\nabla u\|_2$ and integration by parts yield

$$\left\||b(t)|^{\frac{1}{2}}|\nabla u|\right\|_2^2 \le \delta\|u\|_{H_+}^2 + g(t)\|u\|_{\mathcal{W}^{1,2}}^2.$$

Finally, we estimate $g(t)\|u\|_{\mathcal{W}^{1,2}}^2 \le g(t)\|u\|_{H_+}\|u\|_H$, so

$$\int_s^t g(r)\|u\|_{\mathcal{W}^{1,2}}^2 dr \le \varepsilon \int_s^t \|u(r)\|_{H_+}^2\, dr + \frac{1}{4\varepsilon}\int_s^t g^2(r)\|u\|_H^2 dr.$$

Hence

$$\|u(t)\|_H^2 + 2(1 - \delta - \varepsilon)\int_s^t \|u(r)\|_{H_+}^2\, dr - \frac{1}{4\varepsilon}\int_s^t g^2(r)\|u\|_H^2 dr \le \|f\|_H^2,$$

where ε is sufficiently small so that $1 - \delta - \varepsilon > 0$. It follows that

$$\sup_{r \in [0,t]} \|u(r)\|_H^2 + 2(1 - \delta - \varepsilon)\int_s^t \|u(r)\|_{H_+}^2\, dr \le \frac{1}{4\varepsilon}\int_s^t g^2(r)\|u\|_H^2 dr + \|f\|_H^2,$$

so, assuming first that s, t are sufficiently close so that $\frac{1}{4\varepsilon}\int_s^t g^2(r)dr < 1$, and then using the reproduction property, we obtain

$$C_1 \sup_{r \in [0,t]} \|u(r)\|_H^2 + C_2 \int_s^t \|u(r)\|_{H_+}^2\, dr \le \|f\|_H^2$$

for appropriate C_1, $C_2 > 0$, as required.

APPENDIX B. PROOF OF PROPOSITION 4.2

1. Fix some $[s_1, t_1] \subset]s, T[$. Let us show that $\partial_t u \in L^2([s_1, t_1], H_-)$. Put

$$F_t(\varphi) := \langle (\lambda - \Delta)^{\frac{3}{4}} u, (\lambda - \Delta)^{\frac{3}{4}} \varphi \rangle + \langle b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle \qquad \varphi \in L^2_{com}(]s, T[, H_+).$$

By Proposition 4.1,

$$|F_t(\varphi)| \leq (1 + \delta) \|u(t)\|_{\mathcal{W}^{\frac{3}{2}2}} \|\varphi(t)\|_{\mathcal{W}^{\frac{3}{2}2}}.$$

Hence by the Riesz Representation Theorem there exists a unique $w(t) \in H_+$ such that

$$F_t(\varphi)) = \langle (\lambda - \Delta)^{\frac{3}{4}} w(t), (\lambda - \Delta)^{\frac{3}{4}} \varphi(t) \rangle,$$

where $\|w(t)\|_{H_+} \leq (1 + \delta) \|u(t)\|_{H_+}$, so $w \in L^2([s_1, t_1], H_+)$.

In terms of $w(t)$, the hypothesis that u is a weak solution becomes

$$-\int_s^T \langle u(t), \partial_t \varphi \rangle_H \, dt + \int_s^T \langle w(t), \varphi(t) \rangle_{H_+} \, dt = 0.$$

In particular, taking $\varphi = \psi \eta$, where $\psi \in H_+$, $\eta \in C_c^\infty(]s_1, t_1[)$, we have

$$-\int_{s_1}^{t_1} \langle u(t), \psi \rangle_H \eta' \, dt + \int_{s_1}^{t_1} \langle w(t), \psi \rangle_{H_+} \eta \, dt = 0.$$

Since

$$\left| \int_{s_1}^{t_1} \langle w(t), \psi \rangle_{H_+} \eta \, dt \right| \leq \|\psi\|_{H_+} \|w\|_{L^2([s_1, t_1], H_+)} \|\eta\|_{L^2[s_1, t_1]},$$

we have

$$\left| \int_{s_1}^{t_1} \langle u(t), \psi \rangle_H \eta' \, dt \right| \leq (1 + \delta) \|\psi\|_{H_+} \|u\|_{L^2([s_1, t_1], H_+)} \|\eta\|_{L^2[s_1, t_1]}.$$

It follows that $\frac{d}{dt} \langle u(t), \psi \rangle_H \in L^2[s_1, t_1]$, $\psi \in H_+$ and, furthermore,

$$\left\| \frac{d}{dt} \langle u(t), \psi \rangle_H \right\|_{L^2[s_1, t_1]} \leq (1 + \delta) \|\psi\|_{H_+} \|u\|_{L^2([s_1, t_1], H_+)}.$$

Hence there exists $\partial_t u \in L^2([s_1, t_1], H_-)$ such that

$$-\int_s^T \langle \partial_t u(t), \psi \rangle_{H_-, H_+} \eta \, dt = \int_s^T \langle u(t), \psi \rangle_H \eta' \, dt,$$

where $\langle \cdot, \cdot \rangle_{H_-, H_+}$ denotes the H_+, H_- pairing, so

$$-\int_s^T \langle \partial_t u(t), \psi \eta \rangle_{H_-, H_+} \, dt = \int_s^T \langle u(t), (\psi \eta)' \rangle_H \, dt$$

$$= \int_s^T \left(\langle (\lambda - \Delta)^{\frac{3}{4}} u, (\lambda - \Delta)^{\frac{3}{4}} \psi \eta \rangle + \langle b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} \psi \eta \rangle \right) dt.$$

Since $\{\psi \eta \mid \psi \in H_+, \eta \in C_c^\infty(]s_1, t_1[)\}$ is dense in $L^2([s_1, t_1], H_+)$, we have

$$-\int_s^T \langle \partial_t u(t), \varphi \rangle_{H_-, H_+} \, dt = \int_s^T \left(\langle (\lambda - \Delta)^{\frac{3}{4}} u, (\lambda - \Delta)^{\frac{3}{4}} \varphi \rangle + \langle b(t) \cdot \nabla u, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle \right) dt$$

for all $\varphi \in L^2([s_1, t_1], H_+)$.

In particular, taking $\varphi = u$ and using $b \in L^\infty \mathbf{F}^{1/2}$ as in the proof of Proposition 4.1, we have

$$\int_{s_1}^{t_1} \langle \partial_t u(t), u \rangle_{H_-, H_+} dt + (1-\delta) \int_{s_1}^{t_1} \|(\lambda - \Delta)^{\frac{3}{4}} u\|_2^2 dt \leq 0. \tag{49}$$

2. We put $u_\varepsilon := \xi_\varepsilon * u$, where ξ_ε is a Friedrichs mollifier of compact support. Since H_-, H_+ are separable spaces, we have $u_\varepsilon(t) \to u(t)$ in H_+ for a.e. $t \in]s, T[$, $u_\varepsilon \to u$ in $L^2([s_1, t_1], H_+]$, $u'_\varepsilon \to u'$ in $L^2([s_1, t_1], H_-]$

Further, we have

$$\frac{d}{dt}\|u_\varepsilon(t) - u_\delta(t)\|_H = \frac{d}{dt}\|(\lambda - \Delta)^{\frac{1}{4}} u_\varepsilon(t) - (\lambda - \Delta)^{\frac{1}{4}} u_\delta(t)\|_2$$

$$= 2\langle (\lambda - \Delta)^{\frac{1}{4}} \partial_t u_\varepsilon(t) - (\lambda - \Delta)^{\frac{1}{4}} \partial_t u_\delta(t), (\lambda - \Delta)^{\frac{1}{4}} u_\varepsilon(t) - (\lambda - \Delta)^{\frac{1}{4}} u_\delta(t) \rangle$$

so

$$\|u_\varepsilon(t_1) - u_\delta(t_1)\|_H - \|u_\varepsilon(s_1) - u_\delta(s_1)\|_H$$

$$= 2\int_{s_1}^{t_1} \langle (\lambda - \Delta)^{-\frac{1}{4}} \partial_r u_\varepsilon - (\lambda - \Delta)^{-\frac{1}{4}} \partial_r u_\delta, (\lambda - \Delta)^{\frac{3}{4}} u_\varepsilon - (\lambda - \Delta)^{\frac{3}{4}} u_\delta \rangle dr.$$

Hence, fixing $s_1 \in]s, T[$ such that $u_\varepsilon(s_1) \to u(s_1)$ in H_+, we obtain

$$\limsup_{\varepsilon, \delta \downarrow 0} \sup_{t \in [s_1, t_1]} \|u_\varepsilon(t) - u_\delta(t)\|_H \leq \int_0^1 \|\partial_r u_\varepsilon - \partial_r u_\delta\|_{H_-}^2 dr + \int_{s_1}^{t_1} \|u_\varepsilon - u_\delta\|_{H_+}^2 dr \to 0$$

as $\varepsilon, \delta \downarrow 0$. It follows that $\{u_\varepsilon\}$ converges in $L^\infty([s_1, t_1], H)$ to u, and so $u \in C([s_1, t_1], H)$. This gives (ii).

3. Finally, we note that $\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u'(t), u(t) \rangle_H$, as follows from

$$\|u_\varepsilon(t_1)\|_H^2 - \|u_\varepsilon(s_1)\|_H^2 = 2\int_{s_1}^{t_1} \langle (u_\varepsilon)', u_\varepsilon \rangle_H dr$$

upon taking the limit $\varepsilon \downarrow 0$. Combining this with (49) we obtain (iii).

Appendix C.

1. Let us first show that if $b \in L^\infty(\mathbb{R}_+, L^d)$, then for a.e. $t \in \mathbb{R}_+$

$$\||b(t)|\psi\|_2^2 \leq \delta\|(-\Delta)^{\frac{1}{2}} \psi\|_2^2, \quad \psi \in W^{1,2}, \tag{50}$$

for $\delta := \sup_{t \in \mathbb{R}_+} \|b(t)\|_d^2 < \infty$. Indeed,

$$\|b(t)\psi\|_2^2 \leq \|b(t)\|_d^2 \|\psi\|_{\frac{2d}{d-2}}^2$$

(we are applying the Sobolev Embedding Theorem)

$$\leq C_S \|b(t)\|_d^2 \|\nabla \psi\|_2^2,$$

as claimed. Now, applying Heinz' inequality in (50), we obtain $b \in L^\infty \mathbf{F}_{\sqrt{\delta}}^{1/2}$, so $b \in \mathbf{MF}_{\sqrt{\delta}}$ by (7).

If b belongs to the critical Ladyzhenskaya-Prodi-Serrin class (LPS$_c$), i.e.

$$b \in L^p([0, \infty[, L^q), \quad \frac{d}{q} + \frac{2}{p} \leq 1, \quad p \geq 2, \quad q \geq d$$

then we estimate

$$|b(t,x)| = \frac{|b(t,x)|}{\langle|b(t,\cdot)|^q\rangle^{\frac{1}{q}}}\langle|b(t,\cdot)|^q\rangle^{\frac{1}{q}} \leq \frac{d}{q}\left(\frac{|b(t,x)|^q}{\langle|b(t,\cdot)|^q\rangle}\right)^{\frac{1}{d}} + \frac{2}{p}\left(\langle|b(t,\cdot)|^q\rangle^{\frac{1}{q}}\right)^{\frac{p}{2}},$$

where the first term is in $L^\infty([0,\infty[, L^d)$ and the second term is in $L^2([0,\infty[, L^\infty)$. In view of the previous example, it is clear that $b \in \mathbf{MF}_\delta$.

2. One has

$$\sup_{t\in\mathbb{R}}\|b(t)\|_{M_{1+\varepsilon}} < \infty \text{ (see (5))} \quad \Rightarrow \quad b \in L^\infty\mathbf{F}_\delta^{1/2}$$

with δ proportional to the Morrey norm $\sup_{t\in\mathbb{R}}\|b(t)\|_{M_{1+\varepsilon}}$, see [A, Theorem 7.3]. So, in particular, if $\sup_{t\in\mathbb{R}}\|b(t)\|_{M_{1+\varepsilon}} < \infty$, then $b \in \mathbf{MF}_\delta$.

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