# ON DIVERGENCE-FREE (FORM-BOUNDED TYPE) DRIFTS 

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#### Abstract

We develop regularity theory for elliptic Kolmogorov operator with divergence-free drift in a large class (or, more generally, drift having singular divergence). A key step in our proofs is "Caccioppoli's iterations", used in addition to the classical De Giorgi's iterations and Moser's method.


## 1. Introduction

1. This paper is motivated by the following question: what minimal assumptions on a vector field $b$ on $\mathbb{R}^{d}(d \geq 3)$ ensure that the "classical" regularity theory of equations

$$
\begin{equation*}
(-\Delta+b \cdot \nabla) u=0 \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
(-\nabla \cdot a \cdot \nabla+b \cdot \nabla) u=0 \tag{2}
\end{equation*}
$$

is still valid? The matrix $a$ is assumed to be measurable and uniformly elliptic, i.e.

$$
\sigma I \leq a \leq \xi I \text { a.e. on } \mathbb{R}^{d}, \quad 0<\sigma \leq \xi<\infty,
$$

$$
\left(H_{\sigma, \xi}\right)
$$

and the vector field $b$ is assumed to be divergence-free or, more generally, to have divergence in $L_{\text {loc }}^{1} \equiv L_{\text {loc }}^{1}\left(\mathbb{R}^{d}\right)$. In the former case, the elliptic equation (11) can be viewed as a proxy to the corresponding parabolic equation with a time-inhomogeneous divergence-free vector field. This equation plays an important role in hydrodynamics, e.g. as the equation behind the passive tracer SDE where the drift $b$ is the velocity field obtained by solving 3D Navier-Stokes equations, see e.g. MK].

One can prove e.g. local boundedness of weak solutions to the corresponding parabolic equation and a Harnack-type inequality requiring from a divergence-free $b$ only $\|b\|_{p}<\infty$ for a $p>\frac{d}{2}$, see Zhang [Z], Nazarov-Uraltseva [NU]. (Here and below $\|f\|_{p}:=\left(\int_{\mathbb{R}^{d}}|f|^{p} d x\right)^{1 / p}$.) However, to have a classical regularity theory, including Hölder continuity of solutions to (2), one needs $p=d$. Informally, one arrives at $p=d$ by requiring that rescaling the operator leaves invariant the norm of $b$ (neither decreases it, otherwise the singularities of $b$ would be too weak and easy to deal with, nor increases it, otherwise this can destroy the continuity of even bounded solutions to (1), see Filonov [F]). The present paper deals with scaling-invariant conditions on $b$.

[^0]Speaking of the choice of $\|b\|_{d}$ as a measure of singularity of $b$, it is convenient as long as one agrees that the task of verifying $\|b\|_{d}<\infty$ is, in principle, elementary. Nevertheless, this choice is somewhat arbitrary since it largely ignores, beyond the scaling considerations, the operator behind the equation. Much more broad conditions on $b$ are possible:

Definition 1. A vector field $b: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is said to be multiplicatively form-bounded if $|b| \in L_{\text {loc }}^{1}$ and there exists a constant $0<\delta<\infty$ such that

$$
\langle | b|\varphi, \varphi\rangle \leq \delta\|\nabla \varphi\|_{2}\|\varphi\|_{2}+c_{\delta}\|\varphi\|_{2}^{2} \quad \forall \varphi \in W^{1,2}
$$

for some $c_{\delta}<\infty$ (written as $b \in \mathbf{M F}_{\delta}$ ). The constant $\delta$ is called a weak form-bound of $b$.
Here and below

$$
\langle g\rangle:=\int_{\mathbb{R}^{d}} g d x, \quad\langle f, g\rangle:=\langle f g\rangle
$$

(all functions considered below are real-valued), and $W^{1, p}=W^{1, p}\left(\mathbb{R}^{d}\right)$ denotes the usual Sobolev space.

There is a well developed machinery that allows to verify inclusion $b \in \mathbf{M F}_{\delta}$. For instance, the following classes of vector fields $b$, defined in elementary terms, are contained in $\mathbf{M F}_{\delta}$ :

1) $|b| \in L^{d}$, in which case $\delta$ can be chosen arbitrarily small;
2) $|b|$ in the weak $L^{d}$ class;
3) More generally, $|b|^{2}$ is in the Chang-Wilson-Wolff class [CWW], i.e.

$$
\sup _{Q} \frac{1}{|Q|} \int_{Q}|b|^{2} l(Q)^{2} \gamma\left(|b|^{2} l(Q)^{2}\right) d x<\infty
$$

where $\gamma:\left[0, \infty\left[\rightarrow\left[1, \infty\left[\right.\right.\right.\right.$ is an increasing function such that $\int_{1}^{\infty} \frac{d t}{t \gamma(t)}<\infty$. The Chang-WilsonWolff class contains the Campanato-Morrey class

$$
|b| \in L_{\mathrm{loc}}^{2 s} \text { for some } s>1 \text { and }\left(\frac{1}{|Q|} \int_{Q}|b(x)|^{2 s} d x\right)^{\frac{1}{2 s}} \leq c_{s} l(Q)^{-1} \text { for all cubes } Q \text {. }
$$

4) $|b|$ in the Campanato-Morrey class

$$
\begin{equation*}
|b| \in L_{\text {loc }}^{S} \text { for some } s>1 \text { and }\left(\frac{1}{|Q|} \int_{Q}|b(x)|^{s} d x\right)^{\frac{1}{s}} \leq c_{s} l(Q)^{-1} \text { for all cubes } Q . \tag{3}
\end{equation*}
$$

5) $b$ in the Kato class of vector fields $\mathbf{K}_{\delta}^{d+1}$, i.e. $|b| \in L_{\text {loc }}^{1}$ and $\left\|(\lambda-\Delta)^{-\frac{1}{2}}|b|\right\|_{\infty} \leq \delta$ for a $\lambda>0$. In 2)-5) the value of $\delta$ is proportional to the norm of $|b|$ in respective classes.

Below we show that $\mathbf{M F}_{\delta}$ contains the standard class of form-bounded vector fields $\mathbf{F}_{\delta}$ (see Definition (2). In turn, $\mathbf{F}_{\delta}$ contains classes 1)-3) (we note in passing that the Chang-Wilson-Wolff class was born out of attempts by many authors to obtain a necessary and sufficient condition for " $b \in \mathbf{F}_{\delta}$ " in elementary terms, see references in (CWW). The inclusions of 4) and 5) in $\mathbf{M F}_{\boldsymbol{\delta}}$ follow from (11) and the fact that vector fields in the Campanato-Morrey class (3) or the Kato class are weakly form-bounded, see below.

The class $\mathbf{M F}_{\delta}$ neither is contained in, nor contains another well-known class of divergence-free vector fields $\mathbf{B M O}^{-1}$ (i.e. $b=\nabla F$ for skew-symmetric matrix-valued function $F$ with entries in
the space $\mathrm{BMO}=\mathrm{BMO}\left(\mathbb{R}^{d}\right)$ of functions of bounded mean oscillation, see Definition (3). See, however, Remark 1 about combining $\mathbf{M F}_{\delta}$ and $\mathbf{B M O}^{-1}$.

In the present paper we develop De Giorgi's approach to the regularity theory of Kolmogorov operator

$$
\begin{equation*}
-\nabla \cdot a \cdot \nabla+b \cdot \nabla \tag{4}
\end{equation*}
$$

with $b \in \mathbf{M F}_{\delta}, \delta<\infty$ and div $b$ satisfying some broad assumptions (see (5), (6) below). The multiplicative form-boundedness was introduced by Semënov [ $\mathbf{S}$ ] as a condition providing a priori two-sided Gaussian bound on the heat kernel of (4) and hence its a priori Hölder continuity, assuming $b$ is divergence-free. His proof of the upper Gaussian bound used Moser's method. The proof of the lower Gaussian bound in [ $[$ ] required a deep modification of Nash's method. The reason is that when dealing with multiplicative form-boundedness, one cannot use one of the key instruments in the analysis of PDEs: the quadratic inequality. Our motivation, beyond the desire to arrive at a priori Hölder continuity of solutions to (2) using a somewhat simpler argument, and curiosity (in fact, the proof of Caccioppoli's inequality for multiplicatively form-bounded $b$ turned out to be rather interesting, see below), is driven by the following two goals not addressed by the other methods:

- A posteriori solution theory for Dirichlet problem for (2) and a posteriori Harnack inequality.

In Theorem 3] we prove approximation uniqueness of solution to Dirichlet problem for (2) with multiplicatively form-bounded $b$, going beyond the borderline case $|b| \in L_{\text {loc }}^{2}$. (The approximation uniqueness means uniqueness among weak solutions that can be constructed via an approximation procedure.) The proof uses higher integrability of the gradient of solution.

- The minimal assumptions on the divergence of $b$.

Physical applications require one to treat singular div $b$. We allow in this paper $(\operatorname{div} b)_{ \pm}$in the class of form-bounded potentials, i.e. $(\operatorname{div} b)_{ \pm} \in L_{\text {loc }}^{1}$ and

$$
\begin{array}{ll}
\left\langle(\operatorname{div} b)_{+} \varphi, \varphi\right\rangle \leq \nu_{+}\|\nabla \varphi\|_{2}^{2}+c_{\nu_{+}}\|\varphi\|_{2}^{2}, & \nu_{+}<2 \sigma, \\
\left\langle(\operatorname{div} b)_{-} \varphi, \varphi\right\rangle \leq \nu_{-}\|\nabla \varphi\|_{2}^{2}+c_{\nu_{-}}\|\varphi\|_{2}^{2}, & \nu_{-}<\infty . \tag{6}
\end{array}
$$

for some $c_{\nu_{ \pm}}<\infty$, for all $\varphi \in W^{1,2}$. (Throughout the paper, given a function $f$, we denote by $f_{ \pm}$its positive/negative part.) For instance, potentials in the weak $L^{\frac{d}{2}}$ class are form-bounded, but there also exist form-bounded potentials $\notin L_{\mathrm{loc}}^{1+\varepsilon}$, for arbitrarily fixed $\varepsilon>0$.

Earlier, Kinzebulatov-Semënov [KiS3] established a priori two-sided Gaussian heat kernel bounds for (4) for $b \in \mathbf{M F}_{\delta}, \delta<\infty$ with div $b$ in the Kato class of potentials $\mathbf{K}_{\nu}^{d}$, a proper subclass of (5), (6). Example of a vector field $b$ such that div $b$ is form-bounded but is not in the Kato class is given by e.g.

$$
\begin{equation*}
b(x)=\kappa_{+}\left|x-x_{+}\right|^{-2}\left(x-x_{+}\right)-\kappa_{-}\left|x-x_{-}\right|^{-2}\left(x-x_{-}\right) \quad\left(\kappa_{ \pm}>0\right), \tag{7}
\end{equation*}
$$

where $x_{ \pm} \in \mathbb{R}^{d}$ are fixed. Here $\operatorname{div} b=\kappa_{+}(d-2)\left|x-x_{+}\right|^{-2}-\kappa_{-}(d-2)\left|x-x_{-}\right|^{-2}$ satisfies (5), (6) with $\nu_{ \pm}=\frac{4 \kappa \pm}{d-2}, c_{\nu_{ \pm}}=0$ by Hardy's inequality. This vector field indeed destroys two-sided Gaussian bound, see discussion in [KiS3]. At the level of the corresponding to (1), (7) SDE
$d X_{t}=-b\left(X_{t}\right) d t+\sqrt{2} d B_{t}$ the first term in (7) forces the diffusion process $X_{t}$ to approach $x_{+}$, while the second term pushes $X_{t}$ away from $x_{-}$, i.e. taking into account ( $\left.\operatorname{div} b\right)_{ \pm}$allows to model attraction/repulsion phenomena.

The form-boundedness of $(\operatorname{div} b)_{+}$seems to be the maximal possible assumption on div $b$ providing a "classical" regularity theory of (2). If $b$ belongs to a smaller class $\mathbf{F}_{\delta}$, then no assumption on the negative part of $\operatorname{div} b$ is needed, see Hara [H].
2. The starting point of De Giorgi's method is Caccioppoli's inequality (cf. Proposition (1). Let us outline its derivation and describe ensuing difficulties when dealing with multiplicatively formbounded $b$. First, we introduce some notations used throughout the paper. Let $B_{r}(x)$ denote the open ball in $\mathbb{R}^{d}$ of radius $r$ centered at $x$. Put

$$
B_{r}:=B_{r}(0) .
$$

Denote by $(f)_{B}$ the average of function $f$ over a ball (or some other set) $B$ :

$$
(f)_{B}:=\frac{1}{|B|}\left\langle f \mathbf{1}_{B}\right\rangle, \quad|B|:=\operatorname{Vol} B
$$

Now, to prove Caccioppoli's inequality, one multiplies equation (2) by $u \eta$, where $u \in W^{1,2}$ is a weak solution to the equation and $\eta \in C_{c}^{\infty}\left(B_{R}\right), R \leq 1$ is a $[0,1]$-valued function which is identically 1 on a concentric ball of smaller radius $r<R$ and satisfies $|\nabla \eta|^{2} / \eta \leq c(R-r)^{-2} \mathbf{1}_{B_{R}}$. Integrating and using $a \in\left(H_{\sigma, \xi}\right)$, one obtains right away

$$
\left.\left.\sigma\langle | \nabla u\right|^{2} \eta\right\rangle \leq\langle a \cdot \nabla u, u, \nabla \eta\rangle+|\langle b \cdot \nabla u, u \eta\rangle|
$$

so, applying quadratic inequality, one has

$$
\begin{equation*}
\left.\left.\left.\sigma\langle | \nabla u\right|^{2} \eta\right\rangle \leq\left.\epsilon \sigma\langle | \nabla u\right|^{2} \eta\right\rangle+\frac{\sigma}{4 \epsilon}\left\langle u^{2} \frac{|\nabla \eta|^{2}}{\eta}\right\rangle+|\langle b \cdot \nabla u, u \eta\rangle| \quad(\epsilon>0) . \tag{8}
\end{equation*}
$$

Then, in particular,

$$
\begin{equation*}
\left.\left.\sigma(1-\epsilon)\langle | \nabla u\right|^{2} \mathbf{1}_{B_{r}}\right\rangle \leq \frac{c \sigma}{4 \epsilon(R-r)^{2}}\left\langle u^{2} \mathbf{1}_{B_{R}}\right\rangle+|\langle b \cdot \nabla u, u \eta\rangle| . \tag{9}
\end{equation*}
$$

Thus, in the LHS of (9) one obtains extra information about the regularity of $u$ but on a smaller set $B_{r}$, provided that one can control $\langle b \cdot \nabla u, u \eta\rangle$ :
(a) If $b \in \mathbf{F}_{\delta}, \delta<\sigma^{2}$ (no assumptions on $\operatorname{div} b$ ), then one has, using quadratic inequality

$$
\left.\left.|\langle b \cdot \nabla u, u \eta\rangle| \leq\left.\alpha\langle | \nabla u\right|^{2} \eta\right\rangle+\left.\frac{1}{4 \alpha}\langle | b\right|^{2}, u^{2} \eta\right\rangle \quad(\alpha>0) .
$$

Now, applying $b \in \mathbf{F}_{\delta}$, minimizing in $\alpha$ and substituting the result in (8) (with $\epsilon$ chosen sufficiently small), one obtains the Caccioppoli inequality.
(b) If $b=\nabla F \in \mathbf{B M O}^{-1}$, then one obtains, using the divergence theorem,

$$
\left.\left.|\langle b \cdot \nabla u, u \eta\rangle| \leq\left.\gamma\langle | \nabla u\right|^{2} \eta\right\rangle+\frac{1}{\gamma}\langle | F-\left.(F)_{B_{R}}\right|^{2}, u^{2} \frac{|\nabla \eta|^{2}}{\eta}\right\rangle \quad(\gamma>0) .
$$

The latter yields a Caccioppoli-type inequality, where the term containing $\left|F-(F)_{B_{R}}\right|^{2}$ is handled using e.g. the John-Nirenberg inequality. We refer to [H] for details, see also [SSSZ, Zh].
(c) If $b \in \mathbf{M F}_{\delta}, \delta<\infty$ with, say, $\operatorname{div} b=0$, then one has

$$
\begin{align*}
|\langle b \cdot \nabla u, u \eta\rangle| & =\frac{1}{2}|\langle b u, u \nabla \eta\rangle| \leq \frac{1}{2}\langle | b|u, u| \nabla \eta| \rangle \\
& \leq \frac{\delta}{2}\left(\|(\nabla u) \sqrt{|\nabla \eta|}\|_{2}+\|u \nabla \sqrt{|\nabla \eta|}\|_{2}\right)\|u \sqrt{|\nabla \eta|}\|_{2}+\frac{c_{\delta}}{2}\|u \sqrt{|\nabla \eta|}\|_{2}^{2} \tag{10}
\end{align*}
$$

(see the proof of Proposition (1). The estimates

$$
\sqrt{|\nabla \eta|} \leq c(R-r)^{-\frac{1}{2}} \mathbf{1}_{B_{R}}, \quad \left\lvert\, \nabla \sqrt{|\nabla \eta| \mid} \leq c(R-r)^{-\frac{3}{2}} \mathbf{1}_{B_{R}}\right.
$$

present no problem. The difficulty is in the term $\|(\nabla u) \sqrt{|\nabla \eta|}\|_{2}$ in the RHS of (10): one has $\nabla u$ and $\nabla \eta$ at the same time. Thus, one cannot simply transition this term to the LHS of (8) as in (a). Furthermore, estimating $|\nabla \eta| \leq c(R-r)^{-1} \mathbf{1}_{B_{R}}$ in $\|(\nabla u) \sqrt{|\nabla \eta|}\|_{2}$ one obtains the norm of $\nabla u$ over a larger set than in the LHS of (9). Nevertheless, it turns out that one can iterate the resulting from (9) and (10) inequality over balls of radii between $r$ and $R$, which leads to the sought Caccioppoli's inequality, see the proof of Proposition (1. This iteration procedure ("Caccioppoli's iterations") is also used in Moser's method in the proof of Proposition 4, although in a slightly more sophisticated form.
3. Let us now recall the definitions of the class of form-bounded vector fields and the class $\mathrm{BMO}^{-1}$.

Definition 2. A vector field $b: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is said to be form-bounded if $|b| \in L_{\text {loc }}^{2}$ and there exists $\delta>0$ such that

$$
\left.\left.\langle | b\right|^{2} \varphi, \varphi\right\rangle \leq \delta\|\nabla \varphi\|_{2}^{2}+c_{\delta}\|\varphi\|_{2}^{2} \quad \forall \varphi \in W^{1,2}
$$

for some constant $c_{\delta}$ (written as $b \in \mathbf{F}_{\delta}$ ). No conditions on div $b$ are imposed.
Definition 3. A divergence-free distributional vector field $b \in\left[\mathcal{S}^{\prime}\right]^{d}$ is in the class $\mathbf{B M O}^{-1}$ if

$$
b=\nabla F \quad \text { i.e. } b_{k}=\sum_{i=1}^{d} \nabla_{i} F_{i k}, \quad 1 \leq k \leq d
$$

for matrix $F$ with entries $F_{i k}=-F_{k i} \in \mathrm{BMO}$. (Recall that $F_{i k} \in \mathrm{BMO}$ means that $F_{i k} \in L_{\mathrm{loc}}^{1}$ and

$$
\left\|F_{i k}\right\|_{\mathrm{BMO}}:=\sup _{Q} \frac{1}{|Q|} \int_{Q}\left|F-(F)_{Q}\right| d x<\infty,
$$

where the supremum taken over all cubes $Q \subset \mathbb{R}^{d}$ with sides parallel to the axes.)
The class of form-bounded vector fields $\mathbf{F}_{\delta}$ contains $L^{d}$ class, the weak $L^{d}$ class, the CampanatoMorrey class and the Chang-Wilson-Wolff class, see KiS1. It provides a posteriori Harnack inequality and Hölder continuity of solutions to (2), as long as $\delta<\sigma^{2}$, see Hara [H], and weak well-posedness of the corresponding to (11) SDE, as long as form-bound $\delta<c_{d}$ for a certain explicit constant $c_{d}<1$, see Kinzebulatov-Semënov [KiS2] and Kinzebulatov-Madou [KiM]. It also provides a posteriori upper and/or (depending on div b) lower Gaussian bound on the heat kernel of Kolmogorov operator (4), see Kinzebulatov-Semënov [KiS3]. This list of results involving form-bounded drift is far from being exhaustive. See, in particular, Zhang [Z] regarding elements
of the regularity theory of (2) under supercritical (in the sense of scaling) form-boundedness type assumption on $b$.

The quantitative role of form-bound $\delta$ in the theory of Kolmogorov operator was recognized by Kovalenko-Semënov [KS] who proved $W^{1, p}$ estimates on solutions to (1) with $b \in \mathbf{F}_{\delta}$, where the interval of admissible $p$ expands to $[2, \infty[$ as $\delta$ is taken closer and closer to zero. These estimates, with $p$ sufficiently large, allow to construct the corresponding to (1) Feller semigroup.

The class $\mathbf{B M O}^{-1}$ contains divergence-free vector fields with entries e.g. in the CampanatoMorrey class (3), and it also contains some singular distribution vector fields [KT], which, obviously, can not be multiplicatively form-bounded. A weak solution theory of the Dirichlet problem for (11) with $b \in \mathbf{B M O}^{-1}$ was developed by Zhikov $Z \mathrm{Zh}$. This class furthermore provides a posteriori Harnack inequality and Hölder continuity of solutions to the parabolic counterpart of (21), see Friedlander-Vicol [FV], Seregin-Silvestre-Šverak-Zlatoš [SSSZ], and a posteriori two-sided Gaussian bound on Kolmogorov operator (4), see Qian-Xi QX. Earlier, a subclass of $\mathbf{B M O}^{-1}$ consisting of vector fields $b=\nabla B$ for skew-symmetric $B$ with entries in $L^{\infty}$ was considered by Osada [O].
4. We note that for $b=b_{1}+b_{2}$, where $b_{1} \in \mathbf{F}_{\delta}, b_{2} \in \mathbf{B M O}^{-1}$ one has the KLMN Theorem in $L^{2}$ via the estimate

$$
|\langle b \cdot \nabla u, v\rangle| \leq\left(\sqrt{\delta}+\|F\|_{\text {BMO }}\right)\|\nabla u\|_{2}\left(\|\nabla v\|_{2}+c\|v\|_{2}\right), \quad c=\frac{\sqrt{c_{\delta}}}{\sqrt{\delta}}
$$

(for $b_{2}$, using the compensated compactness estimate of CLMS). The KLMN Theorem provides an a posteriori solution theory for (4) in $L^{2}$. This settles for such $b$ 's the problem of a posteriori Harnack inequality. On the other hand, no analogues of the KLMN Theorem are known so far for $b \in \mathbf{M F}_{\delta}$ except in some special cases (cf. the first comment in Section [8).

Regarding the inclusion $\mathbf{F}_{\delta^{2}} \subset \mathbf{M F}_{\delta}$ mentioned above, in fact, a much stronger statement is true: $\mathbf{M F}_{\delta}$ contains the class of weakly form-bounded vector fields, that is, $|b| \in L_{\mathrm{loc}}^{1}$ and

$$
\begin{equation*}
\left\|\left||b|^{\frac{1}{2}} \varphi\left\|_{2}^{2} \leq \delta\right\|(\lambda-\Delta)^{\frac{1}{4}} \varphi \|_{2}^{2} \quad \forall \varphi \in \mathcal{W}^{\frac{1}{2}, 2}\right.\right. \tag{1}
\end{equation*}
$$

for some $\lambda>0$ (which, in turn, contains $\mathbf{F}_{\delta^{2}}$ with $c_{\delta}=\lambda \delta$, as is evident from the Heinz-Kato inequality). Here and below, $\mathcal{W}^{\alpha, p}$ denoted the Bessel potential space. Indeed,

$$
\begin{align*}
\langle | b|\varphi, \varphi\rangle & \leq \delta\left\langle(\lambda-\Delta)^{\frac{1}{2}} \varphi, \varphi\right\rangle \leq \delta\left\|(\lambda-\Delta)^{\frac{1}{2}} \varphi\right\|_{2}\|\varphi\|_{2}  \tag{11}\\
& \leq \delta\|\nabla \varphi\|_{2}\|\varphi\|_{2}+\lambda\|\varphi\|_{2}^{2} \quad \Rightarrow \quad b \in \mathbf{M F}_{\delta} . \tag{12}
\end{align*}
$$

Thus, compared to the standard form-boundedness, the multiplicative form-boundedness allows to gain twice in the a priori summability requirement on the vector field, i.e. $|b| \in L_{\mathrm{loc}}^{1}$ instead of $|b| \in L_{\text {loc }}^{2}$.

Note that $\mathbf{F}_{\delta}^{1 / 2}$ contains the Kato class $\mathbf{K}_{\delta}^{d+1}$ (e.g. by interpolation) and the Campanato-Morrey class (3) [A], hence for every $\varepsilon>0$ one can find weakly form-bounded vector fields $b$ with $|b| \notin L_{\mathrm{loc}}^{1+\varepsilon}$.

See further discussion in Section 8,

## 2. MAIN RESULTS

Our first result concerns a priori estimates for equation (2), i.e. the coefficients will be assumed to be smooth. Let $\Omega \subset \mathbb{R}^{d}$ be a bounded Lipschitz domain, $d \geq 3$.

DEfinition 4. We call a constant generic if it depends only on $d, \Omega, \sigma, \xi, \delta, c_{\delta}, \nu_{ \pm}$and $c_{\nu_{ \pm}}$.
Theorem 1. Let $a \in\left(H_{\sigma, \xi}\right)$ and $b \in \mathbf{M F}:=\cup_{\delta>0} \mathbf{M F}_{\delta}$. Assume that $a, b$ are bounded smooth. Also, assume that $\operatorname{div} b=\operatorname{div} b_{+}-\operatorname{div} b_{-}$for bounded smooth $\operatorname{div} b_{ \pm} \geq 0$ satisfying form-boundedness conditions (5), (6). Let $B_{2 R}(x) \subset \Omega$, and let $u$ be a solution to (2) in $B_{R}(x)$. Then

$$
\begin{equation*}
\|\nabla u\|_{L^{p}\left(B_{\frac{R}{2}}(x)\right)} \leq C_{0} \tag{13}
\end{equation*}
$$

for some generic $p>2$ and $C_{0}<\infty$. Moreover, if solution $u \geq 0$ in $B_{R}(x)$, then it satisfies the Harnack inequality

$$
\begin{equation*}
\sup _{B_{R / 2}(x)} u \leq C \inf _{B_{R / 2}(x)} u \tag{14}
\end{equation*}
$$

with generic constant $C$, and is Hölder continuous:

$$
\begin{equation*}
\operatorname{osc}_{B_{r}(x)} u \leq K\left(\frac{r}{R}\right)^{\gamma} \operatorname{osc}_{B_{R / 2}} u, \quad 0<r<\frac{R}{2} \tag{15}
\end{equation*}
$$

for some generic constants $K$ and $0<\gamma<1$.
Of course, a key point of Theorem 1 is that constants $C_{0}, C, K, \gamma$ do not depend on the smoothness of $a, b, \operatorname{div} b$ or the $L^{\infty}$ norms of the last two.

Theorem 1 uses the standard De Giorgi's iterations and Moser's method, with the addition of "Caccioppoli's iterations" in Propositions 1 and 4.

Remark 1. Theorem 1 extends to vector fields

$$
\begin{equation*}
b=b_{1}+b_{2}, \quad b_{1} \in \mathbf{M F}, \quad b_{2} \in \mathbf{B M O}^{-1} \tag{16}
\end{equation*}
$$

where $\operatorname{div} b_{1}$ satisfies the assumptions of Theorem 1, see Remark 2 in the end of the proof. See also comment 4 in Section 8 regarding Nash's method.

We now turn to the question of weak well-posedness of Dirichlet problem

$$
\left\{\begin{array}{l}
(-\nabla \cdot a \cdot \nabla+b \cdot \nabla) u=0 \quad \text { in } \Omega  \tag{17}\\
u-g \in W_{0}^{1,2}(\Omega)
\end{array}\right.
$$

where $a \in\left(H_{\sigma, \xi}\right)$ and $b \in \mathbf{M F}$ are assumed to be only measurable, $\operatorname{div} b \in L_{\text {loc }}^{1}$ satisfies (5), (6), and

$$
\begin{equation*}
g \in L^{\infty}, \quad\|g\|_{W^{2,2}}<\infty \tag{18}
\end{equation*}
$$

DEFINITION 5. We say that $u \in W_{\text {loc }}^{1,2}(\Omega)$ is a weak solution to equation

$$
\begin{equation*}
(-\nabla \cdot a \cdot \nabla+b \cdot \nabla) u=0 \quad \text { in } \Omega \tag{19}
\end{equation*}
$$

if
(i) $b u \in\left[L_{\mathrm{loc}}^{1}(\Omega)\right]^{d},(\operatorname{div} b) u \in L_{\mathrm{loc}}^{1}(\Omega)$ and
(ii) identity

$$
\langle a \cdot \nabla u, \nabla \varphi\rangle-\langle b u, \nabla \varphi\rangle-\langle(\operatorname{div} b) u, \varphi\rangle=0, \quad \forall \varphi \in C_{c}^{\infty}(\Omega)
$$

holds.
If $u$ is locally bounded, then $(i)$ is trivially satisfied.
We will construct a weak solution to (17) via an approximation procedure. Let us fix $C^{\infty}$ smooth bounded $b_{n}$ such that

$$
\begin{gather*}
\left.b_{n} \in \mathrm{MF}_{\delta} \text { with the same } c_{\delta} \text { (so, independent of } n\right), \quad b_{n} \rightarrow b \quad \text { in }\left[L_{\mathrm{loc}}^{1}\right]^{d},  \tag{20}\\
\operatorname{div} b_{n}=\operatorname{div} b_{n,+}-\operatorname{div} b_{n,-}, \\
0 \leq \operatorname{div} b_{n, \pm} \in C^{\infty} \cap L^{\infty} \operatorname{satisfy}(\text { (5) ), (6) }) \text { with the same } \nu_{ \pm}, c_{\nu_{ \pm}}(\text {so, independent of } n),  \tag{21}\\
\operatorname{div} b_{n, \pm} \rightarrow(\operatorname{div} b)_{ \pm} \text {in } L_{\mathrm{loc}}^{1} . \tag{22}
\end{gather*}
$$

We emphasize that div $b_{n, \pm}$ above is any pair of non-negative functions such that identity div $b_{n}=$ $\operatorname{div} b_{n,+}-\operatorname{div} b_{n,-}$ holds. We discuss a construction of such $b_{n}$ in Section 3,

We fix bounded smooth $g_{n}$ such that

$$
\begin{equation*}
g_{n} \rightarrow g \quad \text { weakly in } W_{\text {loc }}^{2,2}, \quad\left\|g_{n}\right\|_{\infty} \leq\|g\|_{\infty} \tag{23}
\end{equation*}
$$

Let us also fix $C^{\infty}$ smooth $a_{n} \in\left(H_{\sigma, \xi}\right)$,

$$
\begin{equation*}
a_{n} \rightarrow a \quad \text { in }\left[L_{\mathrm{loc}}^{1}\right]^{d \times d} . \tag{24}
\end{equation*}
$$

Theorem 2. Let $a \in\left(H_{\sigma, \xi}\right), b \in \operatorname{MF}$ with $\operatorname{div} b \in L_{\text {loc }}^{1}$ satisfying (5), (6). Let $\left(a_{n}, b_{n}, g_{n}\right)$ satisfy (20)-(24). Then solutions $u_{n}$ to the Dirichlet problems

$$
\left\{\begin{array}{l}
\left(-\nabla \cdot a_{n} \cdot \nabla+b_{n} \cdot \nabla\right) u_{n}=0 \quad \text { in } \Omega  \tag{25}\\
u_{n}=g_{n} \text { on } \partial \Omega
\end{array}\right.
$$

converge weakly in $W_{\mathrm{loc}}^{1,2}(\Omega)$ as $n \rightarrow \infty$, possibly after passing to a subsequence, to a weak solution to Dirichlet problem (17). This weak solution is bounded, satisfies the gradient estimate (13) and, if $g \geq 0$, satisfies the Harnack inequality (14) and is Hölder continuous, cf. (15).

The proof of convergence/existence part of Theorem 2 requires some care, since the moment one puts problem (17) for $a=a_{n}, b=b_{n}, g=g_{n}$ in the form

$$
\left\{\begin{array}{l}
(-\nabla \cdot a \cdot \nabla+b \cdot \nabla) v=-f \\
v \in W_{0}^{1,2}(\Omega)
\end{array}\right.
$$

(so $u=v+g$ ), then the right-hand side $f \equiv f_{n}:=-\nabla \cdot a \cdot \nabla g+b \cdot \nabla g$ is, in general, not bounded in $W^{-1,2}$ uniformly in $n$ (if it is uniformly bounded in $W^{-1,2}$, then the proof is straightforward).

One now arrives at the question: does a weak solution to (17) constructed in Theorem 2 (an "approximation solution") depend on the choice of the approximation procedure ( $a_{n}, b_{n}, g_{n}$ )?

In the next theorem, which is essentially a consequence of Theorem $\square$ and Gehring's Lemma, a "generic constant" can also depend on $\|g\|_{W^{2,2}(\Omega)}$.

Theorem 3. In the assumptions of Theorem 圆, let also

$$
g \in W_{\mathrm{loc}}^{2,2+\epsilon_{1}} \cap W_{\mathrm{loc}}^{1, \frac{1+\epsilon}{\epsilon}} \text { for some } 0<\epsilon_{1}, \epsilon<1, \quad g_{n} \rightarrow g \quad \text { in } W_{\mathrm{loc}}^{2,2+\epsilon_{1}} \cap W_{\mathrm{loc}}^{1, \frac{1+\epsilon}{\epsilon}},
$$

and, in the assumption (5) on $(\operatorname{div} b)_{+}$, let $c_{\nu_{+}}=0$.
There exists a generic $p \in[1+\epsilon, 2[$ such that if, in addition to (20),

$$
|b| \in L_{\mathrm{loc}}^{p}, \quad b_{n} \rightarrow b \quad \text { in }\left[L_{\mathrm{loc}}^{p}\right]^{d},
$$

then the approximation solution to Dirichlet problem (17) constructed in Theorem 图 does not depend on the choice of $\left(a_{n}, b_{n}, g_{n}\right)$, and is in this sense unique.

Zhikov [Zh] investigated approximation uniqueness for Dirichlet problem

$$
\left\{\begin{array}{l}
(-\Delta+b \cdot \nabla) v=-f, \quad f \in W^{-1,2} \\
v \in W_{0}^{1,2}(\Omega)
\end{array}\right.
$$

with divergence-free $b$, singling out two classes of the approximation uniqueness results:

1) $b=\nabla B$ with $B$ is of bounded mean oscillation on $\Omega$ or $\lim _{q \rightarrow \infty} q^{-1}\|B\|_{L^{q}(\Omega)}=0$ (the question what properties of $b$ ensure the existence of such $B$, satisfying some integrability assumptions, is non-trivial),
2) $b \in\left[L^{2}(\Omega)\right]^{d}$ or $\lim _{\varepsilon \downarrow 0} \varepsilon\|b\|_{L^{2-\varepsilon}(\Omega)}=0$.

Theorem 3 thus shows that one can step away from $p=2$ in the condition $|b| \in L^{p}(\Omega)$ by a fixed constant, provided that $b$ is multiplicatively form-bounded. One can now justifiably pose the question if one can remove the additional to $b \in \mathbf{M F}$ assumption " $|b| \in L_{\mathrm{loc}}^{p}$ for a certain $1<p<2$ " completely.

## 3. Smooth approximation of coefficients

Let measurable $a \in\left(H_{\sigma, \xi}\right), b \in \mathbf{M F}$, assume that $\operatorname{div} b \in L_{\text {loc }}^{1}$ satisfies (5), (6), and boundary data $g$ satisfies (18). We discuss the question of constructing $a_{n}, b_{n}, g_{n}$ satisfying the assumptions (20)-(24) before Theorem 2,

It is not difficult to construct a bounded smooth approximation of matrix $a$ and $g$. The question of how to approximate $b$ by bounded smooth vector fields is more subtle since we need to control both the multiplicative form-bound and the form-bound of the (positive/negative part of the) divergence of the approximating vector fields $b_{n}$. We can put e.g.

$$
b_{n}:=\gamma_{\varepsilon_{n}} * b \quad \text { for } \varepsilon_{n} \downarrow 0,
$$

where $\gamma_{\varepsilon}(y):=\varepsilon^{-d} \gamma(y / \varepsilon)$ is the Friedrichs mollifier, $\gamma(y):=c e^{-\frac{1}{|y|^{2}-1}}$ for $|y|<1$ and is zero otherwise, with $c$ adjusted to $\langle\gamma\rangle=1$, cf. KiS1, KiS3]. Indeed, the following is true for every $n \geq 1$ :

1) $b_{n} \in L^{\infty} \cap C^{\infty}$. The second inclusion follows from $b \in\left[L_{\text {loc }}^{1}\right]^{d}$ and the standard properties of Friedrichs mollifiers. To see the first inclusion, fix $x \in \mathbb{R}^{d}$ and estimate

$$
\begin{aligned}
\left|E_{\varepsilon} b(x)\right| & =\left|\left\langle b(\cdot) \sqrt{\gamma_{\varepsilon}(x-\cdot)}, \sqrt{\gamma_{\varepsilon}(x-\cdot)}\right\rangle\right| \\
& \left.\leq\left.\delta\langle | \nabla \sqrt{\gamma_{\varepsilon}(x-\cdot)}\right|^{2}\right\rangle^{\frac{1}{2}}+c_{\delta} \\
& \left.\left(\text { we use }\left.\langle | \nabla \sqrt{\gamma_{\varepsilon}(x-\cdot)}\right|^{2}\right\rangle=C^{2} \varepsilon^{-2}\right) \\
& \leq C \varepsilon^{-1}+c_{\delta}
\end{aligned}
$$

for appropriate $C>0$ (clearly, independent of $x$ ).
2) $b_{n} \in \mathbf{M F}_{\delta}$ with the same $c_{\delta}$ (thus, independent of $n$ ). Let $\varphi \in C_{c}^{\infty}$. First, let us note that for $\varphi_{m}:=\varphi+\frac{e^{-|x|^{2}}}{m}$ we have

$$
\begin{aligned}
\left\|\nabla \sqrt{\gamma_{\varepsilon} *\left|\varphi_{m}\right|^{2}}\right\|_{2} & =\left\|\frac{\gamma_{\varepsilon} *\left(\left|\varphi_{m}\right||\nabla| \varphi_{m} \mid\right)}{\sqrt{\gamma_{\varepsilon} *\left|\varphi_{m}\right|^{2}}}\right\|_{2} \\
& \leq\left\|\sqrt{\left.\gamma_{\varepsilon} *|\nabla| \varphi_{m}\right|^{2}}\right\|_{2}=\left\|\gamma_{\varepsilon} *|\nabla| \varphi_{m}\right\|^{2}\left\|_{1}^{\frac{1}{2}} \leq\right\| \nabla\left|\varphi_{m}\right|\left\|_{2} \leq\right\| \nabla \varphi_{m} \|_{2}
\end{aligned}
$$

(we need term $\frac{e^{-|x|^{2}}}{m}$ to make sure that $\gamma_{\varepsilon} *\left|\varphi_{m}\right|^{2}>0$ everywhere). Now, taking $m \rightarrow \infty$, we obtain

$$
\left\|\nabla \sqrt{\gamma_{\varepsilon} *|\varphi|^{2}}\right\|_{2} \leq\|\nabla \varphi\|_{2}
$$

By $b \in \mathbf{M F}_{\delta}$, we have for all $\varphi \in C_{c}^{\infty}$,

$$
\begin{aligned}
\langle | \gamma_{\varepsilon} * b|\varphi, \varphi\rangle & \left.\leq\left.\langle | b\left|, \gamma_{\varepsilon} *\right| \varphi\right|^{2}\right\rangle \\
& \leq \delta\left\|\nabla \sqrt{\gamma_{\varepsilon} *|\varphi|^{2}}\right\|_{2}\left\|\sqrt{\gamma_{\varepsilon} *|\varphi|^{2}}\right\|_{2}+c_{\delta}\left\|\sqrt{\gamma_{\varepsilon} *|\varphi|^{2}}\right\|_{2} \\
& \leq \delta\|\nabla \varphi\|_{2}\|\varphi\|_{2}+c_{\delta}\|\varphi\|_{2}^{2},
\end{aligned}
$$

as needed.
3) We put $\operatorname{div} b_{n, \pm}:=E_{\varepsilon_{n}}(\operatorname{div} b)_{ \pm} \geq 0$, i.e. we mollify $(\operatorname{div} b)_{+}:=\operatorname{div} b \vee 0$ and $(\operatorname{div} b)_{-}:=$ $-(\operatorname{div} b \wedge 0)$. Then, clearly,

$$
\operatorname{div} b_{n}=\operatorname{div} b_{n,+}-\operatorname{div} b_{n,-} .
$$

The proof of the boundedness, smoothness and form-boundedness (5), (6) of $\operatorname{div} b_{n,+}$, $\operatorname{div} b_{n,-}$ follow the argument in 1), 2).

The convergence $b_{n} \rightarrow b$ in $\left[L_{\mathrm{loc}}^{1}\right]^{d},\left(\operatorname{div} b_{n}\right)_{ \pm} \rightarrow \operatorname{div} b_{ \pm}$in $L_{\text {loc }}^{1}$ follows from the properties of Friedrichs mollifiers.

Finally, the extra local summability/Sobolev regularity assumptions on $b, g$ in Theorem 3 transfer to $b_{n}, g_{n}$ without any problems, by the properties of Friedrichs mollifiers.

## 4. Caccioppoli's inequality

Proposition 1. Let $a, b$ satisfy the assumptions of Theorem 1. Let $u$ be a solution to equation (21) in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{d}$. Set $v:=(u-c)_{+}, c \in \mathbb{R}$. Then, for all $x \in \Omega$ and all $0<r<R$, where $R$ is bounded from above by some $R_{0} \leq 1$ such that $B_{R_{0}}(x) \subset \subset \Omega$, we have

$$
\begin{equation*}
\left.\left.\langle | \nabla v\right|^{2} \mathbf{1}_{B_{r}(x)}\right\rangle \leq K(R-r)^{-2}\left\langle v^{2} \mathbf{1}_{B_{R}(x)}\right\rangle \tag{26}
\end{equation*}
$$

for a generic constant $K$.
Proof. Step 1 (a pre-Caccioppoli's inequality). Without loss of generality, $x=0$. We fix $[0,1]-$ valued smooth cut-off functions $\left\{\eta=\eta_{r_{1}, r_{2}}\right\}_{0<r_{1}<r_{2}<R}$ on $\mathbb{R}^{d}$ satisfying

$$
\eta= \begin{cases}1 & \text { in } B_{r_{1}}, \\ 0 & \text { in } \mathbb{R}^{d}-\bar{B}_{r_{2}},\end{cases}
$$

and

$$
\begin{align*}
\frac{|\nabla \eta|^{2}}{\eta} & \leq \frac{c}{\left(r_{2}-r_{1}\right)^{2}} \mathbf{1}_{B_{r_{2}}}  \tag{27}\\
\sqrt{|\nabla \eta|} & \leq \frac{c}{\sqrt{r_{2}-r_{1}}} \mathbf{1}_{B_{r_{2}}}  \tag{28}\\
\mid \nabla \sqrt{|\nabla \eta|} & \leq \frac{c}{\left(r_{2}-r_{1}\right)^{\frac{3}{2}}} \mathbf{1}_{B_{r_{2}}} \tag{29}
\end{align*}
$$

for some constant $c$. For instance, one can take, for $r_{1} \leq|y| \leq r_{2}$,

$$
\eta(y):=1-\int_{1}^{1+\frac{|y|-r_{1}}{r_{2}-r_{1}}} \varphi(s) d s, \quad \text { where } \varphi(s):=C e^{-\frac{1}{\frac{1}{4}-\left(s-\frac{3}{2}\right)^{2}}}, \quad \operatorname{sprt} \varphi=[1,2],
$$

with constant $C$ adjusted to $\int_{1}^{2} \varphi(s) d s=1$.
We multiply equation (21) by $v \eta$ and integrate to obtain

$$
\langle a \cdot \nabla v,(\nabla v) \eta\rangle+\langle a \cdot \nabla v, v \nabla \eta\rangle+\langle b \cdot \nabla v, v \eta\rangle=0,
$$

so

$$
\left.\left.\sigma\langle | \nabla v\right|^{2} \eta\right\rangle+\langle a \cdot \nabla v, v \nabla \eta\rangle+\langle b \cdot \nabla v, v \eta\rangle \leq 0,
$$

Applying quadratic inequality in the second term, we obtain

$$
\begin{align*}
\left.\left.(\sigma-\epsilon)\langle | \nabla v\right|^{2} \eta\right\rangle & \leq \frac{1}{4 \sigma^{2} \epsilon}\left\langle v^{2} \frac{|\nabla \eta|^{2}}{\eta}\right\rangle-\langle b \cdot \nabla v, v \eta\rangle \quad(\epsilon>0) \\
& \leq \frac{1}{4 \sigma^{2} \epsilon}\left\langle v^{2} \frac{|\nabla \eta|^{2}}{\eta}\right\rangle+\frac{1}{2}\langle b v, v \nabla \eta\rangle+\frac{1}{2}\left\langle\operatorname{div} b, v^{2} \eta\right\rangle \\
& =: I_{1}+I_{2}+I_{3} . \tag{30}
\end{align*}
$$

By (27),

$$
I_{1} \leq \frac{c}{4 \sigma^{2} \epsilon\left(r_{2}-r_{1}\right)^{2}}\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2}^{2}
$$

Regarding $I_{2}$, we have by $b \in \mathbf{M F}_{\delta}$,

$$
\begin{aligned}
2 I_{2} \leq\langle | b|v, v| \nabla \eta| \rangle & \leq \delta\|\nabla(v \sqrt{|\nabla \eta|})\|_{2}\|v \sqrt{|\nabla \eta|}\|_{2}+c_{\delta}\|v \sqrt{|\nabla \eta|}\|_{2}^{2} \\
& \leq \delta\left(\|(\nabla v) \sqrt{|\nabla \eta|}\|_{2}+\|v \nabla \sqrt{|\nabla \eta|}\|_{2}\right)\|v \sqrt{|\nabla \eta|}\|_{2}+c_{\delta}\|v \sqrt{|\nabla \eta|}\|_{2}^{2}
\end{aligned}
$$

Hence, using (28), (29), we obtain

$$
\begin{aligned}
2 I_{2} & \leq \delta c\left(\frac{1}{\sqrt{r_{2}-r_{1}}}\left\|(\nabla v) \mathbf{1}_{B_{r_{2}}}\right\|_{2}+\frac{1}{\left(r_{2}-r_{1}\right)^{\frac{3}{2}}}\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2}\right) \frac{1}{\sqrt{r_{2}-r_{1}}}\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2} \\
& +\frac{c_{\delta} c}{r_{2}-r_{1}}\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2}^{2} .
\end{aligned}
$$

Thus, since $r_{2}-r_{1}<1$,

$$
\begin{aligned}
I_{2} & \leq \frac{C_{1}}{r_{2}-r_{1}}\left\|(\nabla v) \mathbf{1}_{B_{r_{2}}}\right\|_{2}\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2} \\
& +C_{1}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2}^{2}
\end{aligned}
$$

for appropriate constant $C_{1}$. Finally, recalling that $\operatorname{div} b=\operatorname{div} b_{+}-\operatorname{div} b_{-}$for bounded smooth $\operatorname{div} b_{ \pm} \geq 0$ that satisfy (5), (6), we have by (5)

$$
\begin{aligned}
I_{3} & \leq \frac{1}{2}\left\langle\operatorname{div} b_{+}, v^{2} \eta\right\rangle \\
& \left.\leq \frac{\nu_{+}}{2}\left(\left.\langle | \nabla v\right|^{2} \eta\right\rangle+4^{-1}\left\langle v^{2} \frac{|\nabla \eta|^{2}}{\eta}\right\rangle\right)+\frac{c_{\nu_{+}}}{2}\left\langle v^{2} \eta\right\rangle \\
& \left.\leq\left.\frac{\nu_{+}}{2}\langle | \nabla v\right|^{2} \eta\right\rangle+\frac{c_{1}}{\left(r_{2}-r_{1}\right)^{2}}\left\langle v^{2} \mathbf{1}_{B_{r_{2}}}\right\rangle, \quad c_{1}:=4^{-1} c \nu_{+}+\frac{c_{\nu_{+}}}{2} .
\end{aligned}
$$

Substituting the above estimates on $I_{1}, I_{2}$ and $I_{3}$ in (30), selecting $\epsilon$ sufficiently small and using our assumption $\nu_{+}<2 \sigma$, we obtain

$$
\begin{aligned}
\left.\left.\langle | \nabla v\right|^{2} \eta\right\rangle & \leq \frac{C_{1}}{r_{2}-r_{1}}\left\|(\nabla v) \mathbf{1}_{B_{r_{2}}}\right\|_{2}\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2} \\
& +C_{2}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left\|v \mathbf{1}_{B_{r_{2}}}\right\|_{2}^{2}
\end{aligned}
$$

Hence

$$
\begin{align*}
\left.\left.\langle | \nabla v\right|^{2} \mathbf{1}_{B_{r_{1}}}\right\rangle & \leq \frac{C_{1}}{r_{2}-r_{1}}\left\|(\nabla v) \mathbf{1}_{B_{r_{2}}}\right\|_{2}\left\|v \mathbf{1}_{B_{R}}\right\|_{2} \\
& +C_{2}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2} \tag{31}
\end{align*}
$$

We can divide (31) by $\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2}$ :

$$
\begin{equation*}
\frac{\left\|(\nabla v) \mathbf{1}_{B_{r_{1}}}\right\|_{2}^{2}}{\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2}} \leq \frac{C_{1}}{r_{2}-r_{1}} \frac{\left\|(\nabla v) \mathbf{1}_{B_{r_{2}}}\right\|_{2}}{\left\|v \mathbf{1}_{B_{R}}\right\|_{2}}+C_{2}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right) . \tag{32}
\end{equation*}
$$

This is a pre-Cacciopolli inequality that we will now iterate.

Step 2 (Caccioppoli's iterations). Fix $r$ as in the formulation of the theorem (so $0<r<R$ ) and put in (32)

$$
r_{1}:=R-\frac{R-r}{2^{n-1}}, \quad r_{2}:=R-\frac{R-r}{2^{n}}, \quad n=1,2, \ldots
$$

so $r_{2}-r_{1}=\frac{R-r}{2^{n}}$. Then, denoting the LHS of (32) by

$$
a_{n}^{2}:=\frac{\left\|(\nabla v) \mathbf{1}_{B_{R-\frac{R-r}{2-r}}}\right\|_{2}^{2}}{\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2}}
$$

the inequality (32) can be written as

$$
a_{n}^{2} \leq C(R-r)^{-1} 2^{n} a_{n+1}+C^{2}(R-r)^{-2} 2^{2 n}+C^{2}
$$

for appropriate $C$ independent of $n$. We multiply the latter by $(R-r)^{2}$ and divide by $C^{2} 2^{2 n}$ ( $\geq 1$ ). Then, setting

$$
y_{n}:=\frac{(R-r) a_{n}}{C^{2} 2^{n}}
$$

we obtain

$$
\begin{equation*}
y_{n}^{2} \leq 1+(R-r)^{2}+y_{n+1}, \quad n=1,2, \ldots \tag{33}
\end{equation*}
$$

Now we can iterate (33), estimating all $y_{n}$ via nested square roots $1+(R-r)^{2}+\sqrt{1+(R-r)^{2}+\sqrt{\cdots}}$. Or we can simply note that $\beta:=\sup _{n \geq 1} y_{n}$ satisfies

$$
\beta^{2} \leq 1+(R-r)^{2}+\beta
$$

(Note that $\beta<\infty$ since all $a_{n}$ 's are bounded by a (non-generic) constant $\left\|(\nabla v) \mathbf{1}_{B_{R}}\right\|_{2} /\left\|v \mathbf{1}_{B_{R}}\right\|_{2}<$ $\infty$.) Hence

$$
\beta \leq \frac{1+\sqrt{1+4\left(1+(R-r)^{2}\right)}}{2}
$$

which implies $\beta^{2} \leq 3+2(R-r)^{2}$ and thus $y_{n}^{2} \leq 3+2(R-r)^{2}, n=1,2, \ldots$ So, taking $n=1$, we arrive at

$$
\left\|(\nabla v) \mathbf{1}_{B_{r}}\right\|_{2}^{2} /\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2} \leq K(R-r)^{-2}
$$

for a generic constant $K$ (we used $R \leq 1$ to get rid of the constant term in the RHS). This is the claimed inequality.

## 5. Proof of Theorem $\mathbb{1}$

5.1. The sup bound. The main result of this section is Proposition 3. It will follow from the next result.

Proposition 2. Fix $1<\theta<\frac{d}{d-2}$. There exists a generic constant $K$ such that for all $0<R \leq$ $R_{0} \leq 1$, where $B_{R_{0}}(x) \subset \subset \Omega$,

$$
\begin{equation*}
\sup _{B_{\frac{R}{2}}(x)} u_{+} \leq K\left(\frac{1}{\left|B_{R}(x)\right|}\left\langle u_{+}^{2 \theta} \mathbf{1}_{B_{R}(x)}\right\rangle\right)^{\frac{1}{2 \theta}} . \tag{34}
\end{equation*}
$$

We prove Proposition 2, armed with Proposition 1, using De Giorgi's method. Here we can follow Hara $[\mathrm{H}$. For reader's convenience, we included the details.

Proof. Without loss of generality, $x=0$. Proposition 1 yields

$$
\|v\|_{W^{1,2}\left(B_{r}\right)} \leq \tilde{K}(R-r)^{-1}\|v\|_{L^{2}\left(B_{R}\right)}, \quad v:=(u-c)_{+}, \quad c \in \mathbb{R}
$$

(we used $R-r \leq 1$ ). By the Sobolev Embedding Theorem,

$$
\|v\|_{L^{\frac{2 d}{d-2}}\left(B_{r}\right)} \leq C(R-r)^{-1}\|v\|_{L^{2}\left(B_{R}\right)} .
$$

So, by Hölder's inequality,

$$
\begin{equation*}
\|v\|_{L^{\frac{2 d}{d-2}}\left(B_{r}\right)} \leq C(R-r)^{-1}\left|B_{R}\right|^{\frac{\theta-1}{2 \theta}}\|v\|_{L^{2 \theta}\left(B_{R}\right)} . \tag{35}
\end{equation*}
$$

Set

$$
R_{m}:=R\left(\frac{1}{2}+\frac{1}{2^{m+1}}\right), \quad B_{m} \equiv B_{R_{m}}, \quad m \geq 0
$$

so $B_{R}=B_{0} \supset B_{1} \supset \cdots \supset B_{R / 2}$. Then, by (35),

$$
\|v\|_{L^{\frac{2 d}{d-2}\left(B_{m+1}\right)}}^{2} \leq \hat{C} 2^{2 m}\left|B_{m}\right|^{\frac{2}{d}+1-\frac{1}{\theta}}\|v\|_{L^{2 \theta}\left(B_{m}\right)}^{2} .
$$

On the other hand,

$$
\frac{1}{\left|B_{R}\right|}\left\langle v^{2 \theta} \mathbf{1}_{B_{m+1}}\right\rangle \leq\left(\frac{1}{\left|B_{R}\right|}\left\langle v^{\frac{2 d}{d-2}} \mathbf{1}_{B_{m+1}}\right\rangle\right)^{\theta \frac{d-2}{d}}\left(\frac{\left|B_{m+1} \cap\{v>0\}\right|}{\left|B_{R}\right|}\right)^{1-\theta \frac{d-2}{d}}
$$

Applying the previous inequality in the first multiple in the RHS, we obtain

$$
\frac{1}{\left|B_{R}\right|}\left\langle v^{2 \theta} \mathbf{1}_{B_{m+1}}\right\rangle \leq \tilde{C} \frac{2^{2 \theta m}}{\left|B_{R}\right|}\left\langle v^{2 \theta} \mathbf{1}_{B_{m}}\right\rangle\left(\frac{\left|B_{m+1} \cap\{v>0\}\right|}{\left|B_{R}\right|}\right)^{1-\theta \frac{d-2}{d}} .
$$

Now, put $v_{m}:=\left(u-c_{m}\right)_{+}$where

$$
c_{m}:=c\left(1-2^{-m}\right) \rightarrow c
$$

Then

$$
\frac{1}{c^{2 \theta}\left|B_{R}\right|}\left\langle v_{m+1}^{2 \theta} \mathbf{1}_{B_{m+1}}\right\rangle \leq \tilde{C} \frac{2^{2 \theta m}}{c^{2 \theta}\left|B_{R}\right|}\left\langle v_{m+1}^{2 \theta} \mathbf{1}_{B_{m}}\right\rangle\left(\frac{\left|B_{m+1} \cap\left\{u>c_{m+1}\right\}\right|}{\left|B_{R}\right|}\right)^{1-\theta \frac{d-2}{d}} .
$$

Hence, using

$$
\frac{\left|B_{m+1} \cap\left\{u>c_{m+1}\right\}\right|}{\left|B_{R}\right|} \leq \frac{\left(c_{m+1}-c_{m}\right)^{-2 \theta}}{\left|B_{R}\right|}\left\langle v_{m}^{2 \theta} \mathbf{1}_{B_{m+1}}\right\rangle,
$$

we obtain

$$
\frac{1}{c^{2 \theta}\left|B_{R}\right|}\left\langle v_{m+1}^{2 \theta} \mathbf{1}_{B_{m+1}}\right\rangle \leq C 2^{2 \theta m\left(2-\theta \frac{d-2}{d}\right)}\left(\frac{1}{c^{2 \theta}\left|B_{R}\right|}\left\langle v_{m}^{2 \theta} \mathbf{1}_{B_{m}}\right\rangle\right)^{2-\theta \frac{d-2}{d}}
$$

Denote $x_{m}:=\frac{1}{c^{2 \theta}\left|B_{R}\right|}\left\langle v_{m}^{2 \theta} \mathbf{1}_{B_{m}}\right\rangle$ and fix $c$ by

$$
c^{2 \theta}:=C^{\frac{1}{\alpha}} \gamma^{\frac{1}{\alpha^{2}}} \frac{1}{\left|B_{R}\right|}\left\langle v^{2 \theta} \mathbf{1}_{B_{R}}\right\rangle \quad \text { where } \alpha:=1-\theta \frac{d-2}{d}, \quad \gamma:=2^{2 \theta\left(2-\theta \frac{d-2}{d}\right)}
$$

Thus, we have

$$
x_{m+1} \leq C \gamma^{m} x_{m}^{1+\alpha}
$$

where, clearly, $x_{0}=C^{-\frac{1}{\alpha}} \gamma^{-\frac{1}{\alpha^{2}}}$. Hence, by a standard result [G, Sect.7.2], $x_{m} \rightarrow 0$ as $m \rightarrow \infty$. It follows that

$$
\sup _{B_{R / 2}} u_{+} \leq c
$$

which yields the claimed inequality.
Proposition 3. For every $0<p<\infty$ there exists a generic constant $K$ such that, for all $0<R \leq R_{0} \leq 1$, where $B_{R_{0}}(x) \subset \subset \Omega$,

$$
\sup _{B_{\frac{R}{2}}(x)} u \leq K\left(\frac{1}{\left|B_{R}(x)\right|}\left\langle u_{+}^{p} \mathbf{1}_{B_{R}}(x)\right\rangle\right)^{\frac{1}{p}}
$$

Proof. We follow [H]. If $p \geq 2 \theta$ for some $\theta<\frac{d}{d-2}$, then the result follows by Proposition 2 and Hölder's inequality. If $0<p<2 \theta$, then the proof goes as follows. Proposition 2 yields: for all $0<r<R \leq R_{0}$ (without loss of generality, $x=0$ ),

$$
\sup _{B_{r}} u_{+} \leq K\left(\frac{1}{(R-r)^{d}}\left\langle u_{+}^{2 \theta} \mathbf{1}_{B_{R}}\right\rangle\right)^{\frac{1}{2 \theta}}
$$

Hence

$$
\sup _{B_{r}} u_{+} \leq K\left(\frac{1}{(R-r)^{d}}\left\langle u_{+}^{p} \mathbf{1}_{B_{R}}\right\rangle\right)^{\frac{1}{2 \theta}}\left(\sup _{B_{R}} u_{+}\right)^{1-\frac{q}{2 \theta}} .
$$

Now, applying Young's inequality, we arrive at

$$
\sup _{B_{r}} u_{+} \leq \frac{1}{2} \sup _{B_{R}} u_{+}+\tilde{K} \frac{1}{(R-r)^{d}}\left\langle u_{+}^{p} \mathbf{1}_{B_{R}}\right\rangle, \quad 0<r<R \leq 1
$$

The result now follows upon applying [G, Lemma 6.1].
5.2. The inf bound. The main result of this section is Proposition [5,

Proposition 4. There exists generic constants $C$ and $q>0$ such that, if $u \geq c_{0}>0$ is a solution to (2) in $B_{2 R}(x) \subset \subset \Omega$, then

$$
\left(\frac{1}{\left|B_{R}(x)\right|}\left\langle u^{q} \mathbf{1}_{B_{R}(x)}\right\rangle\right)\left(\frac{1}{\left|B_{R}(x)\right|}\left\langle u^{-q} \mathbf{1}_{B_{R}}(x)\right\rangle\right) \leq C^{2} .
$$

The proof of Proposition 4 consists of an iteration-type procedure similar to the one in the proof of Cacciopolli's inequality (Proposition [1), and Moser's method, cf. [H].

Proof. Step 1. We work over a ball $B_{r}(y) \subset B_{2 R}(x), r \leq 1$. Without loss of generality, $y=0$. Let $\zeta_{m}$ be $[0,1]$-valued smooth cut-off functions:

$$
\zeta_{m}(x)=\left\{\begin{array}{ll}
1 & |x| \leq r_{m}, \\
0 & |x| \geq r_{m+1},
\end{array} \quad r_{m}:=r\left(1-\frac{1}{2^{m}}\right), \quad m \geq 1,\right.
$$

satisfying

$$
\left|\nabla \zeta_{m}\right| \leq C \frac{2^{m}}{r}, \quad|\nabla| \nabla \zeta_{m}| | \leq C \frac{4^{m}}{r^{2}}
$$

We multiply equation (2) by $u^{-1} \zeta_{m}^{2}$, obtaining, after integrating by parts, using $\sigma I \leq a \leq \xi I$ and applying quadratic inequality,

$$
\sigma\left\langle(\nabla w)^{2} \zeta_{m}^{2}\right\rangle \leq \varepsilon \xi\left\langle(\nabla w)^{2} \zeta_{m}^{2}\right\rangle+\frac{\xi}{\varepsilon}\left\langle\left(\nabla \zeta_{m}\right)^{2}\right\rangle+\left\langle b \cdot \nabla w, \zeta_{m}^{2}\right\rangle .
$$

where $w:=\log u$. Hence, provided that $\varepsilon$ is fixed by $C_{1}:=\sigma-\varepsilon \xi>0$,

$$
\begin{equation*}
C_{1}\left\langle(\nabla w)^{2} \zeta_{m}^{2}\right\rangle \leq C_{2} 4^{m} r^{d-2}+\left\langle b \cdot \nabla w, \zeta_{m}^{2}\right\rangle \tag{36}
\end{equation*}
$$

We need to estimate the last term. Integrating by parts, we have

$$
\left\langle b \cdot \nabla w, \zeta_{m}^{2}\right\rangle=-2\left\langle b(w-c), \zeta_{m} \nabla \zeta_{m}\right\rangle-\left\langle\operatorname{div} b, w \zeta_{m}^{2}\right\rangle
$$

for any constant $c$ (we will chose its value later). Thus,

$$
\begin{equation*}
\left.\left\langle b \cdot \nabla w, \zeta_{m}^{2}\right\rangle \leq\left. 2\langle | b| | \nabla \zeta_{m}\right|^{2}\right\rangle^{\frac{1}{2}}\langle | b\left|(w-c)^{2} \zeta_{m}^{2}\right\rangle^{\frac{1}{2}}+\left\langle\operatorname{div} b_{-},(w-c) \zeta_{m}^{2}\right\rangle, \tag{37}
\end{equation*}
$$

where, recall $\operatorname{div} b=\operatorname{div} b_{+}-\operatorname{div} b_{-}$for bounded smooth $\operatorname{div} b_{ \pm} \geq 0$ that satisfy (5), (6).

1) We estimate the first multiple in the RHS of (377): by $b \in \operatorname{MF}$ (taking into account $r \leq 1$ ):

$$
\begin{equation*}
\langle | b\left|\left|\nabla \zeta_{m}\right|^{2}\right\rangle^{\frac{1}{2}} \leq C_{3} 2^{\frac{3}{2} m} r^{\frac{d-3}{2}} . \tag{38}
\end{equation*}
$$

2) The second multiple is estimated as

$$
\langle | b\left|(w-c)^{2} \zeta_{m}^{2}\right\rangle^{\frac{1}{2}} \leq \sqrt{\delta}\left(\left\|\zeta_{m} \nabla w\right\|_{2}+\left\|(w-c) \nabla \zeta_{m}\right\|_{2}\right)^{\frac{1}{2}}\left\|(w-c) \zeta_{m}\right\|_{2}^{\frac{1}{2}}+\sqrt{c_{\delta}}\left\|(w-c) \zeta_{m}\right\|_{2}
$$

Therefore, setting $B_{m}:=B_{r_{m}}$, we have

$$
\begin{aligned}
\langle | b\left|(w-c)^{2} \zeta_{m}^{2}\right\rangle^{\frac{1}{2}} & \leq \sqrt{\delta}\left(\|\nabla w\|_{L^{2}\left(B_{m+1}\right)}+2^{m} r^{-1}\|w-c\|_{L^{2}\left(B_{m+1}\right)}\right)^{\frac{1}{2}}\|w-c\|_{L^{2}\left(B_{m+1}\right)}^{\frac{1}{2}} \\
& +\sqrt{c_{\delta}}\|w-c\|_{L^{2}\left(B_{m+1}\right)} .
\end{aligned}
$$

Select $c$ to be the average $(w)_{B_{m+1}}:=\left|B_{m+1}\right|^{-1}\left\langle w \mathbf{1}_{B_{m+1}}\right\rangle$ of $w$ over $B_{m+1}$. Then, using the Poincaré inequality $\|w-c\|_{L^{2}\left(B_{m+1}\right)} \leq C_{0} r\|\nabla w\|_{L^{2}\left(B_{m+1}\right)}$, we obtain

$$
\begin{equation*}
\langle | b\left|(w-c)^{2} \zeta_{m}^{2}\right\rangle^{\frac{1}{2}} \leq C_{4} 2^{\frac{1}{2} m} r^{\frac{1}{2}}\|\nabla w\|_{L^{2}\left(B_{m+1}\right)} . \tag{39}
\end{equation*}
$$

3) Finally, we estimate the last term in the RHS of (37):

$$
\begin{equation*}
\left\langle\operatorname{div} b_{-},(w-c) \zeta_{m}^{2}\right\rangle \leq\left\langle\operatorname{div} b_{-},(w-c)^{2} \zeta_{m}^{2}\right\rangle^{\frac{1}{2}}\left\langle\operatorname{div} b_{-}, \zeta_{m}^{2}\right\rangle^{\frac{1}{2}}, \tag{40}
\end{equation*}
$$

where, by the form-boundedness assumption (6),

$$
\begin{align*}
& \left\langle\operatorname{div} b_{-},(w-c)^{2} \zeta_{m}^{2}\right\rangle \\
& \quad \leq \nu_{-}\left(1+\varepsilon_{1}\right)\left\|(\nabla w) \zeta_{m}\right\|_{2}^{2}+\nu_{-}\left(1+\varepsilon_{1}^{-1}\right)\left\|(w-c) \nabla \zeta_{m}\right\|_{2}^{2}+c_{\nu_{-}}\left\|(w-c) \zeta_{m}\right\|_{2}^{2}  \tag{41}\\
& \quad \leq \nu_{-}\left(1+\varepsilon_{1}\right)\left\|(\nabla w) \zeta_{m}\right\|_{2}^{2}+\nu_{-}\left(1+\varepsilon_{1}^{-1}\right) 4^{m} r^{-2}\|w-c\|_{L^{2}\left(B_{m+1}\right)}^{2}+c_{\nu_{-}}\|w-c\|_{L^{2}\left(B_{m+1}\right)}^{2}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle\operatorname{div} b_{-}, \zeta_{m}^{2}\right\rangle & \leq \nu_{-}\left\|\nabla \zeta_{m}\right\|_{2}^{2}+c_{\nu_{-}}\left\|\zeta_{m}\right\|_{2}^{2} \\
& \leq C_{5} 4^{m} r^{d-2} . \tag{42}
\end{align*}
$$

Hence, applying (41), (42) in (40), using the quadratic inequality and the Poincaré inequality as above, we obtain

$$
\begin{equation*}
\left\langle(\operatorname{div} b)_{-},(w-c) \zeta_{m}^{2}\right\rangle \leq C_{6} \varepsilon_{2}\left\|(\nabla w) \zeta_{m}\right\|_{2}+\frac{C_{7}}{4 \varepsilon_{2}} 4^{m} r^{d-2}+C_{7} 4^{m} r^{\frac{d}{2}-1}\|\nabla w\|_{L^{2}\left(B_{m+1}\right)} \tag{43}
\end{equation*}
$$

We now apply (41), (42) and (43) (with $\varepsilon_{2}$ chosen sufficiently small) in (40), arriving at

$$
\left\langle(\nabla w)^{2} \zeta_{m}^{2}\right\rangle \leq C^{2} 4^{m} r^{d-2}+C 4^{m} r^{\frac{d}{2}-1}\|\nabla w\|_{L^{2}\left(B_{m+1}\right)}
$$

for appropriate constant $C$. Therefore,

$$
\begin{equation*}
\|\nabla w\|_{L^{2}\left(B_{m}\right)}^{2} \leq C^{2} 4^{m} r^{d-2}+C 4^{m} r^{\frac{d}{2}-1}\|\nabla w\|_{L^{2}\left(B_{m+1}\right)} \tag{44}
\end{equation*}
$$

for all $m=1,2, \ldots$
Step 2. We are going to iterate inequality (44). Put

$$
x_{m}:=\frac{\|\nabla w\|_{L^{2}\left(B_{m}\right)}}{C 2^{m} r^{\frac{d}{2}-1}}
$$

so (44) becomes

$$
x_{m}^{2} \leq 1+2^{m+1} x_{m+1} .
$$

We may assume without loss of generality that all $x_{m} \geq 1$ (if $x_{m_{0}} \leq 1$ for some $m_{0}$, then we are already done). Thus,

$$
x_{m}^{2} \leq 2^{m+2} x_{m+1} .
$$

On the other hand, all $x_{m}(m=1,2, \ldots)$ are bounded by a non-generic but independent of $m$ constant $\frac{\|\nabla w\|_{L^{2}\left(B_{r}\right)}}{2 C r^{\frac{d}{2}-1}}$. Hence we can iterate the previous inequality:

$$
x_{m}^{2} \leq 2^{m+2} x_{m+1} \leq 2^{m+2} 2^{\frac{m+3}{2}} x_{m+2}^{\frac{1}{2}} \leq 2^{m+2} 2^{\frac{m+3}{2}} \ldots 2^{\frac{m+n}{2^{n}}} x_{m+1+n}^{\frac{1}{2^{n}}} \leq 2^{\sum_{n=1}^{\infty} \frac{m+1+n}{2^{n}}}=: c(m)
$$

In particular, $x_{1}^{2} \leq c(1)$, which yields

$$
\begin{equation*}
\|\nabla w\|_{L^{2}\left(B_{r / 2}\right)} \leq K r^{\frac{d}{2}-1} \tag{45}
\end{equation*}
$$

for a generic constant $K$.
Step 3. Inequality (45) is the point of departure for Moser's method. Namely, applying Poincaré's and Hölder's inequalities, we obtain

$$
\begin{equation*}
\frac{1}{r^{d}}\langle | w-(w)_{r / 2}\left|\mathbf{1}_{B_{r / 2}}\right\rangle \leq \tilde{K}, \tag{46}
\end{equation*}
$$

where $(w)_{r / 2}$ is the average of $w$ over $B_{r / 2}$.
The centre $y$ of the ball in (46) was chosen arbitrarily, and the constant $\tilde{K}$ does not depend on this choice. Thus, by (46), $w \in \operatorname{BMO}\left(B_{2 R}(x)\right)$ (in what follows, for brevity, $x=0$ ). Now, by the John-Nirenberg inequality:

$$
\left\langle e^{\left.q\left|w-(w)_{R}\right|_{B_{R}}\right\rangle \leq C R^{d}, ~}\right.
$$

for some generic $q>0$. So, we have

$$
\begin{aligned}
\left\langle u^{q} \mathbf{1}_{B_{R}}\right\rangle\left\langle u^{-q} \mathbf{1}_{B_{R}}\right\rangle & =\left\langle e^{q w} \mathbf{1}_{B_{R}}\right\rangle\left\langle e^{-q w} \mathbf{1}_{B_{R}}\right\rangle \\
& =\left\langle e^{q\left(w-(w)_{R}\right)} \mathbf{1}_{B_{R}}\right\rangle\left\langle e^{-q\left(w-(w)_{R}\right)} \mathbf{1}_{B_{R}}\right\rangle \leq C^{2} R^{2 d},
\end{aligned}
$$

as needed.
Proposition 5 (Moser). There exists generic constants $C_{0}$ and $q>0$ such that, if $u \geq c_{0}>0$ is a solution to (2) in $B_{2 R}(x) \subset \Omega$, then

$$
\left(\frac{1}{\left|B_{R}(x)\right|}\left\langle u^{q} \mathbf{1}_{B_{R}(x)}\right\rangle\right)^{\frac{1}{q}} \leq C_{0} \inf _{B_{R / 2}(x)} u .
$$

Proof. Let $x=0$. Multiplying equation (2) by $u^{-p}(p>1)$, we obtain that $u^{-p+1}$ is a sub-solution of (2):

$$
-\nabla \cdot a \cdot \nabla u^{-p+1}+b \cdot \nabla u^{-p+1} \leq 0 .
$$

We can repeat the proofs of Proposition 1 and of Proposition 2 for positive sub-solutions of (2) essentially word by word. Fix $p$ by $p-1=\frac{q}{2 \theta}$ for any $1<\theta<\frac{d}{d-2}$. Then, by Proposition 2,

$$
\sup _{B_{R / 2}} u^{-1} \leq K^{\frac{2 \theta}{q}}\left(\frac{1}{\left|B_{R}\right|}\left\langle u^{-q} \boldsymbol{1}_{B_{R}}\right\rangle\right)^{\frac{1}{q}} .
$$

Hence

$$
\begin{aligned}
& \inf _{B_{R / 2}} u \geq K^{-\frac{2 \theta}{q}}\left(\frac{1}{\left|B_{R}\right|}\left\langle u^{-q} \mathbf{1}_{B_{R}}\right\rangle\right)^{-\frac{1}{q}} \\
& \text { (we are applying Proposition (4) } \\
& \geq C^{-\frac{2}{q}} K^{-\frac{2 \theta}{q}}\left(\frac{1}{\left|B_{R}\right|}\left\langle u^{q} \mathbf{1}_{B_{R}}\right\rangle\right)^{\frac{1}{q}},
\end{aligned}
$$

as needed.

We are in position to complete the proof of Theorem 1 Propositions 3 and 5 yield Harnack's inequality for $u \geq c_{0}>0$ in $B_{2 R}(x)$. A simple limiting argument allows to extend it to $u \geq 0$ in $B_{2 R}(x)$. The Hölder continuity of $u$ now follows using a standard argument. The gradient estimate is a standard consequence of Cacciopolli's inequality (Proposition (1), see e.g. [H.

Remark 2. The iteration procedure of Propositions $\mathbb{1}$ and $\mathbb{4}$ extends to vector fields

$$
b=b_{1}+b_{2}, \quad b_{1} \in \mathbf{M F}, \quad b_{2} \in \mathbf{B M O}^{-1}
$$

where $\operatorname{div} b_{1}$ satisfies the assumptions of Theorem 1 , $b_{2}=\nabla F$ for skew-symmetric $F$ with entries in BMO. Namely, repeating the proof of Proposition $⿴$ for such $b$ (cf. the proof of (53) below), one obtains an extra term $\left\langle b_{2} \cdot \nabla v, v \eta\right\rangle$, which one estimates as in [H, Lemma 8]:

$$
\left|\left\langle b_{2} \cdot \nabla v, v \eta\right\rangle\right| \leq \varepsilon\langle | \nabla u|\eta\rangle+\frac{1}{4 \varepsilon}\left\langle\left(F-(F)_{B_{R}}\right)^{2} v^{2} \frac{|\nabla \eta|^{2}}{\eta}\right\rangle, \quad \varepsilon>0,
$$

and so the pre-Caccioppoli's inequality now takes form

$$
\frac{\left\|(\nabla v) \mathbf{1}_{B_{r_{1}}}\right\|_{2}^{2}}{\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2}} \leq \frac{C_{1}}{r_{2}-r_{1}} \frac{\left\|(\nabla v) \mathbf{1}_{B_{r_{2}}}\right\|_{2}}{\left\|v \mathbf{1}_{B_{R}}\right\|_{2}}+C_{2}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left(1+\frac{\left\|\left(F-(F)_{B_{R}}\right) v \mathbf{1}_{B_{R}}\right\|_{2}^{2}}{\left\|v \mathbf{1}_{B_{R}}\right\|_{2}^{2}}\right) .
$$

One can now iterate this inequality in the same was as it was done in the proof of Proposition arriving, instead of (26), at a Caccioppoli-type inequality as in [H] (cf. (b) in the introduction):

$$
\left.\left.\langle | \nabla v\right|^{2} \mathbf{1}_{B_{r}}\right\rangle \leq K(R-r)^{-2}\left\langle\left[1+\left(F-(F)_{B_{R}}\right)^{2}\right] v^{2} \mathbf{1}_{B_{R}}\right\rangle, \quad B_{R}=B_{R}(x) .
$$

Having the last inequality at hand, one then runs De Giorgi's method as in H . The proof of Proposition 4 is modified similarly; then we refer again to $[\mathrm{H}$. This allows to extend Theorem 1 to vector fields $b=b_{1}+b_{2}$ as above.

## 6. Proof of Theorem 2

Given $\Omega \subset \mathbb{R}^{d}$, put $\|f\|_{p, \Omega}:=\left(\int_{\Omega}|f| d x\right)^{\frac{1}{p}}$.
It is convenient to put Dirichlet problem (17) in an equivalent form (at the level of weak solutions)

$$
\left\{\begin{array}{l}
(-\nabla \cdot a \cdot \nabla+b \cdot \nabla) v=-f  \tag{47}\\
v \in W_{0}^{1,2}(\Omega)
\end{array}\right.
$$

where $f:=-\nabla \cdot a \cdot \nabla g+b \cdot \nabla$. Then the sought $u$ is given by $u=v+g$.
Considering (47) for $a=a_{n}, b=b_{n}, g=g_{n}, f=f_{n}$ and, accordingly, $v=v_{n}$, we multiply the equation by $v_{n}$ and integrate to obtain

$$
\sigma\left\|\nabla v_{n}\right\|_{2, \Omega}^{2} \leq \frac{1}{2}\left\langle\operatorname{div} b_{n}, v_{n}^{2}\right\rangle-\left\langle f_{n}, v_{n}\right\rangle,
$$

so, in particular,

$$
c \sigma \epsilon\left\|v_{n}\right\|_{2, \Omega}^{2}+\sigma(1-\epsilon)\left\|\nabla v_{n}\right\|_{2, \Omega}^{2} \leq \frac{1}{2}\left\langle\operatorname{div} b_{n,+}, v_{n}^{2}\right\rangle-\left\langle f_{n}, v_{n}\right\rangle
$$

for a small $\epsilon>0$. In the RHS of the last inequality, using the fact that $\operatorname{div} b_{n,+}$ satisfies (5), we estimate

$$
\frac{1}{2}\left\langle\operatorname{div} b_{n,+}, v_{n}^{2}\right\rangle \leq \frac{\nu_{+}}{2}\left\|\nabla v_{n}\right\|_{2}^{2}+\frac{c_{\nu_{+}}}{2}\left\|v_{n}\right\|_{2}^{2}
$$

and, applying quadratic inequality twice, we have

$$
\left|\left\langle f_{n}, v_{n}\right\rangle\right| \leq \sigma\left\|\nabla g_{n}\right\|_{2, \Omega}\left\|\nabla v_{n}\right\|_{2, \Omega}+\alpha\langle | b_{n}\left|, v_{n}^{2}\right\rangle+\frac{1}{4 \alpha}\langle | b_{n}\left|,\left|\nabla g_{n}\right|^{2}\right\rangle \quad(\alpha>0)
$$

(we are using $b_{n} \in \mathbf{M F}_{\delta}$ for some $\delta<\infty, c_{\delta}$ independent of $n$ )

$$
\begin{aligned}
& \leq \sigma \beta\left\|\nabla v_{n}\right\|_{2, \Omega}^{2}+\frac{\sigma}{4 \beta}\left\|\nabla g_{n}\right\|_{2, \Omega}^{2} \quad(\beta>0) \\
& +\alpha\left(\delta\left\|\nabla v_{n}\right\|_{2, \Omega}\left\|v_{n}\right\|_{2, \Omega}+c_{\delta}\left\|v_{n}\right\|_{2, \Omega}^{2}\right) \\
& +\frac{1}{4 \alpha}\left(\delta\left\|\nabla \mid \nabla g_{n}\right\|\left\|_{2, \Omega}\right\| \nabla g_{n}\left\|_{2, \Omega}+c_{\delta}\right\| g_{n} \|_{2, \Omega}^{2}\right)
\end{aligned}
$$

Now, selecting in the previous three inequalities $\alpha, \beta$ sufficiently small and using $\nu_{+}<2 \sigma$, we obtain

$$
\begin{aligned}
\left\|v_{n}\right\|_{W^{1,2}(\Omega)}^{2} & \leq C\|g\|_{W^{2,2}(\Omega)}+C_{1}\left\|v_{n}\right\|_{2, \Omega} \\
& \leq C\|g\|_{W^{2,2}(\Omega)}+C_{1}|\Omega|^{\frac{1}{2}}\left\|v_{n}\right\|_{\infty, \Omega}
\end{aligned}
$$

for some $C, C_{1}<\infty$. Hence, taking into account that, by the maximum principle, $\left\|v_{n}\right\|_{L^{\infty}(\Omega)} \leq$ $2\left\|g_{n}\right\|_{\infty}\left(\leq 2\|g\|_{\infty}<\infty\right)$, we have

$$
\begin{equation*}
\left\|v_{n}\right\|_{W^{1,2}(\Omega)}^{2} \leq C\|g\|_{W^{2,2}(\Omega)}+2 C_{1}|\Omega|^{\frac{1}{2}}\|g\|_{\infty} \tag{48}
\end{equation*}
$$

This allows to conclude that there exists a subsequence $\left\{v_{n}\right\}$ and a function $v \in W_{0}^{1,2}(\Omega)$ such that

$$
\begin{equation*}
v_{n} \rightarrow v \quad \text { weakly in } W_{0}^{1,2}(\Omega), \text { strongly in } L_{\mathrm{loc}}^{2}(\Omega) \tag{49}
\end{equation*}
$$

Let us show that thus constructed $v$ is a weak solution to (47):

1) For a given $\varphi \in C_{c}^{\infty}(\Omega)$, we can write

$$
\begin{aligned}
\left\langle a_{n} \cdot \nabla v_{n}, \nabla \varphi\right\rangle+\left\langle b_{n} \cdot \nabla v_{n}, \varphi\right\rangle & =\left\langle a_{n} \cdot \nabla v_{n}, \nabla \varphi\right\rangle-\left\langle b_{n} v_{n}, \nabla \varphi\right\rangle-\left\langle\left(\operatorname{div} b_{n}\right) v_{n}, \varphi\right\rangle \\
& =\langle a \cdot \nabla v, \nabla \varphi\rangle-\langle b v, \nabla \varphi\rangle-\langle\operatorname{div} \mathrm{b} v, \varphi\rangle \\
& +\left\langle\left(a_{n}-a\right) \cdot \nabla v_{n}, \nabla \varphi\right\rangle+\left\langle a \cdot\left(\nabla v_{n}-\nabla v\right), \nabla \varphi\right\rangle \\
& -\left\langle\left(b_{n}-b\right) v_{n}, \nabla \varphi\right\rangle-\left\langle b\left(v_{n}-v\right), \nabla \varphi\right\rangle \\
& -\left\langle\left(\operatorname{div} b_{n}-\operatorname{div} b\right) v_{n}, \nabla \varphi\right\rangle-\left\langle(\operatorname{div} b)\left(v_{n}-v\right), \varphi\right\rangle .
\end{aligned}
$$

The -6 th term in the RHS, i.e. $\left\langle\left(a_{n}-a\right) \cdot \nabla v_{n}, \nabla \varphi\right\rangle$, tends to 0 : we use $\left\|\nabla v_{n}\right\|_{L^{p}(\operatorname{sprt} \varphi)}<\infty$ for some $p>2$ (Theorem (1) and convergence $a_{n} \rightarrow a$ in $L^{p^{\prime}}\left(\operatorname{sprt} \varphi\right.$ ) (use (24) and $a_{n} \in\left(H_{\sigma, \xi}\right)$ ).

The -5 th term $\left\langle a \cdot\left(\nabla v_{n}-\nabla v\right), \nabla \varphi\right\rangle$, tends to 0 since $v_{n} \rightarrow v$ weakly in $W_{0}^{1,2}(\Omega)$.
The -4 th term $\left\langle\left(b_{n}-b\right) v_{n}, \nabla \varphi\right\rangle$ tends to 0 due to (20) and since $v_{n}$ are uniformly bounded on $\Omega\left(\left\|v_{n}\right\|_{L^{\infty}(\Omega)} \leq 2\left\|g_{n}\right\|_{\infty}\right.$ by the maximum principle, where, by our choice of $g_{n},\left\|g_{n}\right\|_{\infty} \leq\|g\|_{\infty}<$ $\infty)$.

The -3 rd term $\left\langle b\left(v_{n}-v\right), \nabla \varphi\right\rangle$ goes to 0 by $|b| \in L_{\text {loc }}^{1}$ and the Dominated Convergence Theorem, using the uniform boundedness of $v_{n}$ on $\Omega$ and a.e. convergence $v_{n} \rightarrow v$, which follows from (49) (possibly after passing to a subsequence using a diagonal argument).

The -2 nd term $\left\langle\left(\operatorname{div} b_{n}-\operatorname{div} b\right) v_{n}, \nabla \varphi\right\rangle$ goes to 0 by convergence (22) and since $v_{n}$ are uniformly bounded on $\Omega$.

The -1 st term $\left\langle(\operatorname{div} b)\left(v_{n}-v\right), \varphi\right\rangle$ goes to 0 by $\operatorname{div} b \in L_{\text {loc }}^{1}$ and, again, the Dominated Convergence Theorem, using uniform boundedness of $v_{n}$ and a.e. convergence $v_{n} \rightarrow v$.
2) Next, in view of our assumptions on $g$ and $g_{n}$,

$$
\begin{aligned}
\left\langle f_{n}, \varphi\right\rangle & =\left\langle a_{n} \cdot \nabla g_{n}, \nabla \varphi\right\rangle+\left\langle b_{n} g_{n}, \nabla \varphi\right\rangle \\
& \rightarrow\langle f, \varphi\rangle
\end{aligned}
$$

using the same argument as in 1), taking into account that, by (23) and the Rellich-Kondrashov Theorem, we may assume that $g_{n} \rightarrow g$ a.e. on $\Omega$ (of course, possibly after passing to a subsequence of $\left\{g_{n}\right\}$ )

Combining 1) and 2), we obtain that $v$ is a weak solution to (47). Moreover, since $v_{n}$ are (uniformly in $n$ ) bounded on $\Omega$, so is $v$.

Now, we have $u_{n}=v_{n}+g_{n}$, so, in view of our conditions on $g_{n}$ and $g$,

$$
u_{n} \rightarrow u:=v+g \quad \text { weakly in } W_{0}^{1,2}(\Omega),
$$

and so $u_{n} \rightarrow u$ strongly in $L_{\text {loc }}^{2}(\Omega)$, possibly after passing to a subsequence. Further, since $u_{n}$ are bounded on $\Omega$ by the maximum principle, so is $u$. The last statement of the theorem now follows from Theorem 1

## 7. Proof of Theorem 3

1. To establish uniqueness of the approximation solution, it suffices to show that solutions $\left\{v_{n}\right\}$ to

$$
\left\{\begin{array}{l}
\left(-\nabla \cdot a_{n} \cdot \nabla+b_{n} \cdot \nabla\right) v_{n}=-f_{n}  \tag{50}\\
v_{n}=0 \text { on } \partial \Omega,
\end{array}\right.
$$

where $f_{n}:=-\nabla \cdot a_{n} \cdot \nabla g_{n}+b_{n} \cdot \nabla g_{n}$, constitute a Cauchy sequence in $L^{2}(\Omega)$. (Then, clearly, solutions $u_{n}=v_{n}+g_{n}$ to (25) constitute a Cauchy sequence in $L^{2}(\Omega)$.) In fact, subtracting the equations for $v_{n}, v_{m}$ and setting $h:=v_{n}-v_{m}$, we obtain

$$
-\nabla \cdot a_{n} \cdot \nabla h+b_{n} \cdot \nabla h-\nabla \cdot\left(a_{n}-a_{m}\right) \cdot \nabla v_{m}+\left(b_{n}-b_{m}\right) \cdot \nabla v_{m}=-f_{n}+f_{m},
$$

Then, multiplying the previous identity by $h$ and integrating, we obtain

$$
\begin{aligned}
\sigma\|\nabla h\|_{2}^{2}-\frac{1}{2}\left\langle\operatorname{div} b_{n}, h^{2}\right\rangle & \leq\left|\left\langle\left(a_{n}-a_{m}\right) \cdot \nabla v_{m}, \nabla h\right\rangle\right|+\left|\left\langle\left(b_{n}-b_{m}\right) \cdot \nabla v_{m}, h\right\rangle\right| \\
& +\left|\left\langle f_{n}-f_{m}, h\right\rangle\right| .
\end{aligned}
$$

Hence, using $\frac{1}{2}\left\langle\left(\operatorname{div} b_{n,+}\right), h^{2}\right\rangle \leq \frac{1}{2} \nu_{+}\|\nabla h\|_{2}^{2}$ (by the assumption of the theorem), we have

$$
\begin{align*}
\left(\sigma-\nu_{+}\right)\|\nabla h\|_{2} & \leq\left|\left\langle\left(a_{n}-a_{m}\right) \cdot \nabla v_{m}, \nabla h\right\rangle\right|+\left|\left\langle\left(b_{n}-b_{m}\right) \cdot \nabla v_{m}, h\right\rangle\right| \\
& +\left|\left\langle f_{n}-f_{m}, h\right\rangle\right| . \tag{51}
\end{align*}
$$

Recall that, by our assumption, $\sigma-\frac{\nu_{+}}{2}>0$. Thus, our goal is to show that all terms in the RHS of (51) tend to 0 as $n, m \rightarrow \infty$; this would imply that $\left\{v_{m}\right\}$ is indeed a Cauchy sequence in $L^{2}(\Omega)$.

1) Let us get rid of the last term in the RHS of (51):

$$
\begin{aligned}
\left\langle f_{n}-f_{m}, h\right\rangle=\left\langle\left(a_{n}-a_{m}\right)\right\rangle & =\left\langle\left(a_{n}-a_{m}\right) \cdot \nabla g_{n}, \nabla h\right\rangle \\
& +\left\langle a_{m} \cdot \nabla\left(g_{n}-g_{m}\right), \nabla h\right\rangle \\
& +\left\langle\left(b_{n}-b_{m}\right) \cdot \nabla g_{n}, h\right\rangle \\
& +\left\langle b_{m} \cdot \nabla\left(g_{n}-g_{m}\right), h\right\rangle .
\end{aligned}
$$

All four terms in the RHS tends to 0 as $n, m \rightarrow \infty$. This follows, upon applying Hölder's inequality, from the uniform boundedness of $|\nabla h|$ in $L^{2}(\Omega)$ (cf. (48) in the proof of Theorem (2) and convergence $\nabla g_{n}-\nabla g_{m} \rightarrow 0$ in $\left[L_{\mathrm{loc}}^{\frac{1+\epsilon}{\epsilon}}\right]^{d}, b_{n}-b_{m} \rightarrow 0$ in $\left[L_{\mathrm{loc}}^{p}\right]^{d}(p \geq 1+\epsilon)$ as $n, m \rightarrow \infty$.
2) We now treat the first two terms in the RHS of (51).

$$
\begin{aligned}
\left|\left\langle\left(b_{n}-b_{m}\right) \cdot \nabla v_{m}, h\right\rangle\right| & \leq\left\|b_{n}-b_{m}\right\|_{p, \Omega}\left\|\nabla v_{m}\right\|_{p^{\prime}, \Omega}\|h\|_{\infty, \Omega} \\
& \leq\left\|b_{n}-b_{m}\right\|_{p, \Omega}\left\|\nabla v_{m}\right\|_{p^{\prime}, \Omega} 2\|g\|_{\infty} .
\end{aligned}
$$

If we can prove a uniform in $n$ bound

$$
\begin{equation*}
\left\|\nabla v_{m}\right\|_{p^{\prime}, \Omega} \leq C \quad \text { for some } p^{\prime}>2 \tag{52}
\end{equation*}
$$

it would imply that $\left\langle\left(b_{n}-b_{m}\right) \cdot \nabla v_{m}, h\right\rangle \rightarrow 0$ as $n, m \rightarrow \infty$, since $b_{n}-b_{m} \rightarrow 0$ in $\left[L_{\text {loc }}^{p}\right]^{d}$ by the assumption of the theorem, with $p\left(=\frac{p^{\prime}}{p^{\prime}-1}\right)<2$.

The estimate (52) is also what is needed to prove $\left\langle\left(a_{n}-a_{m}\right) \cdot \nabla v_{m}, \nabla h\right\rangle \rightarrow 0$, since $a_{n}-a_{m} \rightarrow 0$ in $L^{q}(\Omega)$ for any $q<\infty$.

Thus, the proof of Theorem 3 will be completed once we prove (52).
2. Proof of (52). Write for brevity $v=v_{m}, a=a_{m}, b=b_{m}$ (the constants below are independent of $m$ ). We extend $v$ to $\mathbb{R}^{d}$ by zero. It suffices to establish

$$
\begin{equation*}
\left.\left.\left.(R / 4)^{-d}\langle | \nabla v\right|^{2} \mathbf{1}_{B_{\frac{R}{4}}(x)}\right\rangle \leq C\left[\left(\left.R^{-d}\langle | \nabla v\right|^{\frac{2}{\theta}} \mathbf{1}_{B_{R}}(x)\right\rangle\right)^{\theta}+R^{-d}\left\langle k^{2}\right\rangle\right] \tag{53}
\end{equation*}
$$

for generic constants $\theta>1, C$, for all $R \leq R_{0}, x \in \Omega$, for some function $k \in L_{\text {loc }}^{2+\epsilon}, \epsilon>0$. Then Gehring's Lemma will yield (52).

Let us prove (53). If $B_{\frac{R}{2}}(x) \subset \Omega$, then we put $w:=v-(v)_{B_{R}(x)}$, otherwise $w:=v$. Without loss of generality, $x=0$. As in the proof of Proposition 亿 we fix [0, 1]-valued smooth cut-off functions $\left\{\eta=\eta_{r_{1}, r_{2}}\right\}_{0<r_{1}<r_{2}<R}$ on $\mathbb{R}^{d}$ such that

$$
\eta= \begin{cases}1 & \text { in } B_{r_{1}}, \\ 0 & \text { in } \mathbb{R}^{d}-\bar{B}_{r_{2}},\end{cases}
$$

satisfying (27)-(29). We multiply equation $-\nabla \cdot a \cdot \nabla w+b \cdot \nabla w=-f$ by $w \eta$, integrate over $\mathbb{R}^{d}$, and argue as in the proof of (31) to obtain

$$
\begin{align*}
\left.\left.\sigma\langle | \nabla w\right|^{2} \eta\right\rangle & \leq \frac{C_{1}}{r_{2}-r_{1}}\left\|(\nabla w) \mathbf{1}_{B_{r_{2}}}\right\|_{2}\left\|w \mathbf{1}_{B_{r_{2}}}\right\|_{2} \\
& +C_{2}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left\|w \mathbf{1}_{B_{\frac{R}{2}}}\right\|_{2}^{2} \\
& +|\langle f, w \eta\rangle| . \tag{54}
\end{align*}
$$

In comparison with (31), we now have an extra term $|\langle f, w \eta\rangle|$. We deal with it as follows:

$$
\begin{aligned}
|\langle f, w \eta\rangle| & =|\langle a \cdot \nabla g,(\nabla w) \eta+w \nabla \eta\rangle+\langle b \cdot \nabla g, w \eta\rangle| \\
& \left.\left.\left.\leq\left.\alpha\langle | \nabla w\right|^{2} \eta\right\rangle+\alpha\left\langle w^{2}\right| \nabla \eta| \rangle+\left.\frac{\xi^{2}}{4 \alpha}\langle | \nabla g\right|^{2} \eta\right\rangle+\gamma\langle | b\left|w^{2} \eta\right\rangle+\left.\frac{1}{4 \gamma}\langle | b| | \nabla g\right|^{2} \eta\right\rangle \quad(\alpha, \gamma>0)
\end{aligned}
$$

Now, applying $b \in$ MF and substituting the result in (54), we obtain

$$
\begin{aligned}
\left.\left.(\sigma-\alpha-\gamma)\langle | \nabla w\right|^{2} \eta\right\rangle & \leq \frac{C_{1}}{r_{2}-r_{1}}\left\|(\nabla w) \mathbf{1}_{B_{r_{2}}}\right\|_{2}\left\|w \mathbf{1}_{B_{r_{2}}}\right\|_{2} \\
& +C_{2}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left\|w \mathbf{1}_{B_{R}}\right\|_{2}^{2}+C_{3}(\alpha, \gamma)\left\langle w^{2}\right| \nabla \eta| \rangle+C_{4}(\alpha, \gamma)\|g\|_{W^{2,2}\left(B_{R}\right)} .
\end{aligned}
$$

Hence, fixing $\alpha$ and $\gamma$ sufficiently small so that $\sigma-\alpha-\gamma>0$, we obtain

$$
\begin{aligned}
\left.\left.\langle | \nabla w\right|^{2} \mathbf{1}_{r_{1}}\right\rangle & \leq \frac{C_{1}^{\prime}}{r_{2}-r_{1}}\left\|(\nabla w) \mathbf{1}_{B_{r_{2}}}\right\|\left\|_{2}\right\| w \mathbf{1}_{B_{r_{2}}} \|_{2} \\
& +C_{2}^{\prime}\left(1+\frac{1}{\left(r_{2}-r_{1}\right)^{2}}\right)\left\|w \mathbf{1}_{B_{R}}\right\|_{2}^{2}+C_{4}^{\prime}(\alpha, \gamma)\|g\|_{W^{2,2}\left(B_{R}\right)}
\end{aligned}
$$

We now iterate this inequality in the same way as in the proof of Proposition [1, selecting

$$
r_{1}:=R-\frac{R}{2^{n-1}}, \quad r_{2}:=R-\frac{R}{2^{n}}, \quad n=1,2, \ldots
$$

arriving, upon taking $n \rightarrow \infty$, to

$$
\left\|(\nabla w) \mathbf{1}_{B_{\frac{R}{4}}}\right\|_{2}^{2} \leq C\left[\frac{1}{R^{2}}\left\|w \mathbf{1}_{B_{R}}\right\|_{2}^{2}+\|g\|_{W^{2,2}\left(B_{R}\right)}\right]
$$

By the Sobolev-Poincaré inequality (or by the Sobolev inequality, if $B_{\frac{R}{2}}(x) \not \subset \Omega$ ), we have

$$
\left\|(\nabla w) \mathbf{1}_{B_{\frac{R}{4}}}\right\|_{2}^{2} \leq C\left[\frac{1}{R^{2}}\left\|(\nabla w) \mathbf{1}_{B_{R}}\right\|_{\frac{2 d}{d+2}}^{2}+\|g\|_{W^{2,2}\left(B_{R}\right)}\right],
$$

so

$$
\left.\left.\left.R^{-d}\langle | \nabla w\right|^{2} \mathbf{1}_{B_{\frac{R}{4}}}\right\rangle \leq C\left[\left(\left.R^{-d}\langle | \nabla w\right|^{\frac{2}{\theta}} \mathbf{1}_{B_{R}}\right\rangle\right)^{\theta}+R^{-d}\|g\|_{W^{2,2}\left(B_{R}\right)}\right], \quad \theta=\frac{d+2}{d} .
$$

Now Gehring's Lemma yields (52)) and thus ends the proof.

## 8. Further discussion

1. In Kinzebulatov-Semënov KiS4, the authors show that applying the Lions variation approach for $\partial_{t}-\Delta+b \cdot \nabla$ in the Bessel space $\mathcal{W}^{1 / 2,2}$ rather than $L^{2}$ allows to enlarge the class of admissible vector fields from the classical form-bounded vector fields $\mathbf{F}_{\delta}$ to the weakly formbounded vector fields $\mathbf{F}_{\delta}^{1 / 2} \subset \mathbf{M F}_{\delta}$. (In fact, the class $\mathbf{F}_{\delta}$ is dictated by the Lions approach ran in $L^{2}$.) Hence one obtains existence and uniqueness of weak solution to Cauchy problem for $\partial_{t}-\Delta+b \cdot \nabla, b \in \mathbf{F}_{\delta}^{1 / 2}$ in $\mathcal{W}^{1 / 2,2}$. This result does not impose any assumptions on div $b$, but requires $\delta<1$. Since having a divergence-free $b$ one expects to have no constraints of the value of $\delta$ except that it is finite, this result does not settle the question of a posteriori theory for (4), even for $a=I$ and weakly-form bounded $b$.
2. Requiring $b \in \mathbf{F}_{\delta}^{1 / 2}$ with $\delta$ sufficiently small (without any assumptions on div $b$ ) yields $\mathcal{W}^{1+\frac{1}{p}-, p}$-regularity theory of $\partial_{t}-\Delta+b \cdot \nabla$, with the interval of admissible $p$ expanding to $] 1, \infty[$ as $\delta \downarrow 0$, see [Ki, KiS1].
3. In absence of any assumptions on $\operatorname{div} b$, De Giorgi's method yields the Harnack inequality for (2) when $b \in \mathbf{F}_{\delta}, \delta<\sigma^{2}$. In view of the previous comment, one can ask if De Giorgi's method also works for $a=I$ and $b \in \mathbf{F}_{\delta}^{1 / 2}$ with weak form-bound $\delta<1$. One obstacle when working directly with $\mathbf{F}_{\delta}^{1 / 2}$ is the need to handle non-local operators. Interestingly, a larger class $\mathbf{M F}_{\boldsymbol{\delta}}$ allows one to stay in the local setting at expense of imposing additional assumptions on div $b$. (One practical outcome of this is that when one approximates $b$ by bounded smooth vector fields $b_{n}$, e.g. in the proofs of Theorems 2 and 3, it is easier to control simultaneously the multiplicative form-bound of $b_{n}$ and the form-bounds of $\operatorname{div} b_{n, \pm}$, than to control the weak form-bound of $b_{n}$ and the form-bound of $\operatorname{div} b_{n, \pm}$.)
4. Both classes $\mathbf{M F}_{\delta}$ and $\mathbf{B M O}^{-1}$ are contained in a larger class: $b \in\left[\mathcal{S}^{\prime}\right]^{d}$ such that

$$
\begin{equation*}
|\langle b \varphi, \varphi\rangle| \leq \delta\|\nabla \varphi\|_{2}\|\varphi\|_{2}+c_{\delta}\|\varphi\|_{2}^{2} \quad \forall \varphi \in C_{c}^{\infty} . \tag{55}
\end{equation*}
$$

This class was considered in [KiS4] where it was proved that (55), together with the hypothesis "div $b$ in the Kato class of potentials with sufficiently small Kato norm", provides a priori Gaussian upper bound on the heat kernel of (4); an a priori Gaussian lower bound in [KiS4] is proved under somewhat stronger assumption (16).
5. There is an analogy between the approximation uniqueness for Dirichlet problem for (21), discussed in Theorem 3, and the uniqueness of "good solution" to Dirichlet problem for nondivergence form elliptic equations studied by Krylov, Safonov and Nadirashvili among others, see discussion in [Sa]. The analogy is not just formal: being able to treat a large class of drifts allows one to put non-divergence form equations in divergence form (this was exploited e.g. in [KiS2] in the study of SDEs with diffusion coefficients critical discontinuities and form-bounded drifts.)
6. The iteration procedure used in the proof of Caccioppoli's inequality in Proposition $\mathbb{1}$ also works for the corresponding parabolic equation $\partial_{t}-\nabla \cdot a \cdot \nabla+b \cdot \nabla=0$ where the class $\mathbf{M F}_{\delta}$ is now defined as the class of time-inhomogeneous vector fields $b \in\left[L_{\text {loc }}^{1}\left(\mathbb{R}_{+} \times \mathbb{R}^{d}\right)\right]^{d}$ such that for a.e. $t \in \mathbb{R}_{+}$,

$$
\langle | b(t)|\varphi, \varphi\rangle \leq \delta\|\nabla \varphi\|_{2}\|\varphi\|_{2}+c_{\delta}\|\varphi\|_{2}^{2}, \quad \forall \varphi \in W^{1,2} .
$$

(furthermore, constant $c_{\delta}$ can be replaced by a function of time). We are interested, in particular, in applications to weak well-posedness of SDEs, which require regularity estimates on solution to Cauchy problem in $\mathbb{R}^{d}$

$$
\left(\partial_{t}-\Delta+b \cdot \nabla\right) u=|\mathrm{f}| g, \quad u(0)=0
$$

where $\mathrm{f} \in \mathrm{MF}_{\mu}, g \in C_{c}^{2}$, cf. KiM, KiS5] for details. The proof of such estimates for multiplicatively form-bounded $b$, f presents its own set of difficulties, which we plan to address elsewhere.

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