REMARKS ON PARABOLIC KOLMOGOROV OPERATOR

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ABSTRACT. We obtain gradient estimates on solutions of parabolic Kolmogorov equation with singular drift in a large class. Such estimates allow to construct a Feller evolution family, which can be used to construct unique weak solutions to the corresponding stochastic differential equation.

1. Introduction and main results

We obtain gradient estimates on solutions of parabolic Kolmogorov equation

$$(\partial_t - \Delta + b(t,x) \cdot \nabla)u = 0$$

under general assumptions on a vector field $b : \mathbb{R}^{1+d} \to \mathbb{R}^d$ ($d \geq 3$). These estimates allow to construct, using an analogue of the iteration procedure in [7], a Feller evolution family that determines, for every $x \in \mathbb{R}^d$, a unique in a large class weak solution to stochastic differential equation

$$X_t = x - \int_0^t b(s, X_s)ds + \sqrt{2}B_t. \tag{1}$$

Here B_t is a d-dimensional Brownian motion.

The class of vector fields in this note is defined as follows: we write $b \in \mathbf{F}_{\delta,g}$ if

$$b \in [L^2_{\rm loc}(\mathbb{R}^{1+d})]^d$$

and there exists a constant $\delta > 0$ and a function $g = g_\delta$ of the form $g = g' + g''$ for some $0 \leq g' \in L^1(\mathbb{R})$, $0 \leq g'' \in L^\infty(\mathbb{R})$, such that for a.e. $t \in \mathbb{R}$,

$$\|b(t)f(t)\|_2^2 \leq \delta \|\nabla f(t)\|_2^2 + g(t)\|f(t)\|_2^2 \tag{2}$$

for all $f \in C_c^\infty(\mathbb{R}^{d+1})$. Here and everywhere below, $\|f(t)\|_2^2 := \int_{\mathbb{R}^d} |f(t,x)|^2 dx$, $\|\nabla f(t)\|_2^2 = \int_{\mathbb{R}^d} |\nabla_x f(t,x)|^2 dx$.

The vector fields in class $\mathbf{F}_{\delta,g}$ are called form-bounded. This class contains the well known critical Ladyzhenskaya-Prodi-Serrin class, as well as vector fields that can have stronger singularities, see examples in [3, 4].

The question of what values of constant δ are admissible is important, in particular, in light of the following example. Consider Hardy-type drift $b(x) = \sqrt{\delta}\frac{d-2}{2}|x|^{-2}x$ (which is in $\mathbf{F}_{\delta,0}$ by Hardy's inequality, but not in $\mathbf{F}_{\delta',g}$ with any $\delta' < \delta$). If $\sqrt{\delta} > \frac{2d}{d-2}$, then SDE (1) with initial

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point $x=0$ does not have a weak solution. Informally, constant δ measures the strength of the singularities of b. In the example the attraction to the origin by the singularity of b is too strong. On the other hand, by Theorem 3 below, SDE (1) with $b \in \mathbf{F}_{\delta,g}$ has a unique in appropriate class weak solution for every $x \in \mathbb{R}^d$ provided that δ satisfies the assumptions of Theorem 1. In fact, it was proved in [5] that (1) with an arbitrary $b \in \mathbf{F}_{\delta,g}$, $\delta < 4$ has at least one martingale solution for every initial point $x \in \mathbb{R}^d$.

A vector field $b \in \mathbf{F}_{\delta,g}$ can be approximated by smooth bounded vector fields b_n that preserve the form-bound δ of b; the latter is crucial for what follows.

DEFINITION 1. A sequence $\{b_n\} \subset [L^\infty(\mathbb{R}^{1+d}) \cap C^\infty(\mathbb{R}^{1+d})]^d$ of vector fields is called a regularizing sequence for $b \in \mathbf{F}_{\delta,g}$ if, for any $0<t<\infty$,

(i) $\lim_{n\to\infty} \|b_n - b\|_{L^2(Q)} = 0$, $Q=[0,t]\times K$ for every compact $K \subset \mathbb{R}^d$;

(ii) there are functions $\{g_n\}$ such that $g_n = g_n' + g_n''$, $g_n' \in L^1(\mathbb{R})$, $g_n'' \in L^\infty(\mathbb{R})$ and

$$\sup_n \int_0^t g_n(\tau)d\tau \leq c_\delta(\|g'\|_1 + t\|g''\|_\infty) \text{ for some constant } c_\delta$$

($g\prime$ and $g\prime\prime$ are from the definition of "$b \in \mathbf{F}_{\delta,g}$").

(iii) $\int_0^t \|b_n(\tau)f(\tau)\|_2^2 d\tau \leq \delta \int_0^t \|\nabla f(\tau)\|_2^2 d\tau + \int_0^t g_n(\tau)\|f(\tau)\|_2^2 d\tau \quad (n \geq 1, \quad f \in \mathcal{S}(\mathbb{R}^{1+d}))$.

($\mathcal{S}(\mathbb{R}^{1+d})$ denotes the L. Schwartz space of test functions).

The collection of all regularizing sequences for $b \in \mathbf{F}_{\delta,g}$ will be denoted by $[b]^r$.

In Section 2 we construct a regularizing sequence in $[b]^r$ for any given $b \in \mathbf{F}_{\delta,g}$.

Our first result concerns the classical solutions of Cauchy problems

$$(\partial_\tau - \Delta + b_n(\tau,x)\cdot\nabla_x)u(\tau) = 0, \quad 0 \leq s < \tau < \infty, \quad x \in \mathbb{R}^d, \quad u(s) = u_0 \in C_c^\infty(\mathbb{R}^d). \tag{3}$$

Theorem 1. *Let $b \in \mathbf{F}_{\delta,g}$. Assume that $q>d$ and $\delta>0$ satisfy the following constraints:*

$$q - 1 - \frac{q^2\delta}{4} - \frac{(q-2)^2}{4} - (q-2)\frac{q\sqrt{\delta}}{2} > 0 \quad \text{if } d=3,4,$$

$$q - 1 - \frac{q\sqrt{\delta}}{2}\left(\sqrt{\frac{q^2\delta}{4} + (q-2)^2} + q - 2\right) > 0 \quad \text{if } d \geq 5.$$

In particular, one can take

(a) *If $d=3$, then $\sqrt{\delta} = \frac{1.8}{d}$, $q = d + \frac{1}{48}$; if $d=4$, then $\sqrt{\delta} = \frac{1.4}{d}$, $q = d + 0.014$.*

(b) *If $d \geq 5$, then $\sqrt{\delta} = \frac{1}{d}$, $q = d+1$.*

(b′) *If $d \geq 5$, then $\sqrt{\delta} = (1 - \frac{a}{16+a})\frac{q-1}{q-2}\frac{1}{q}$, $a = \frac{(q-1)^2}{(q-2)^4}$, $q = d + \varepsilon$, $\forall \varepsilon \in]0,1]$.*

Let $\{b_n\} \in [b]^r$ and let $u=u_n$ be the classical solution to Cauchy problem (3)*. Then there are constants $C_i = C_i(q,d,\delta) > 0$, $i=1,2$ independent of n such that, for all $0 \leq s < t < \infty$,*

$$\sup_{s\leq\tau\leq t} \|\nabla u(\tau)\|_q^q + C_1 \int_s^t \|\nabla u(\tau)\|_{qj}^q d\tau \leq e^{C_2(\|g'\|_1 + t\|g''\|_\infty)} \|\nabla u(s)\|_q^q, \quad j := \frac{d}{d-2}.$$

Remarks. 1. Clearly, $\frac{1}{d} < f(q) := (1 - \frac{a}{16+a})\frac{q-1}{q-2}\frac{1}{q}$ for all $0 < \varepsilon \leq 1$; $\sup_{q \in [d,d+1]} f(q) = f(d)$.

2. In the assumptions of Theorem 1 we actually obtain a stronger regularity estimate:

$$\sup_{s\leq\tau\leq t} \|\nabla u(\tau)\|_q^q + c_0 \int_s^t \||\nabla u(\tau)|^{\frac{q-2}{2}}\partial_\tau u(\tau)\|_2^2 d\tau + c_1 \sum_{i=1}^d \int_s^t \langle |\nabla_i \nabla u(\tau)|^2, |\nabla u_n(\tau)|^{q-2}\rangle d\tau$$
$$\leq e^{c_2(\|g'\|_1 + t\|g''\|)} \|\nabla u(s)\|_q^q.$$

The gradient estimates of the type established in Theorem 1 play an important role in study of parabolic and stochastic equations. For example, similar estimates are used in [1] to study stochastic transport and continuity equations (although under more restrictive assumptions on b than the class $\mathbf{F}_{\delta,g}$, see [6] in this regard).

Let $C_\infty(\mathbb{R}^d)$ denote the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity, endowed with the sup-norm.

Theorem 2. *Let $\{b_n\} \in [b]^r \subset \mathbf{F}_{\delta,g}$ with δ and $q > d$ satisfying the assumptions of Theorem 1. Let u_n be the classical solution to Cauchy problem*

$$(\partial_t - \Delta + b_n \cdot \nabla) u_n(t) = 0, \quad 0 \leq s < t < \infty, \quad u_n(s) = f \in C_c^1(\mathbb{R}^d). \tag{4}$$

For each $n = 1, 2, \dots$ and $0 \leq s \leq t < \infty$ define operators $U_n^{t,s} \in \mathcal{B}(C_\infty)$ by

$$U_n^{t,s} f := u_n(t), \qquad U^{s,s} = 1.$$

Then the limit

$$U^{t,s} := s\text{-}C_\infty(\mathbb{R}^d)\text{-}\lim_n U_n^{t,s} \quad \textit{(uniformly in } 0 \leq s < t \leq 1)$$

exists and determines a Feller evolution family on $\mathcal{M} = \{(t,s) \in \mathbb{R}_+^2 \mid 0 < t - s\} \times C_\infty(\mathbb{R}^d)$.

Remarks. 1. The limit $u(t) = U^{t,s} f$ does not depend on the choice of concrete regularization $\{b_n\} \in [b]^r$ (in this sense, the "approximation solution" u to Cauchy problem $\partial_t - \Delta + b \cdot \nabla = 0$, $u(s) = f$ is unique). Moreover, one can show that $u = U^{t,s} f$, $f \in C_\infty \cap L^2$ is a weak solution of $\partial_t - \Delta + b \cdot \nabla = 0$ in the usual sense, and that it satisfies the gradient estimates in Theorem 1 if $f \in C_\infty \cap W^{1,q}$.

2. Theorem 1 can be extended to non-homogeneous parabolic equation with form-bounded right-hand side, moreover, the corresponding gradient estimates can be localized, which, together with Theorem 2, allows to prove the following result (see [3] for details).

Theorem 3. *Let $b \in \mathbf{F}_{\delta,g}$ with $q > d$ close to d and δ satisfying conditions of Theorem 1. Then there exist probability measures $\mathbb{P}_x$, $x \in \mathbb{R}^d$ on $(C([0,T], \mathbb{R}^d), \sigma(\omega_r \mid 0 \leq r \leq t))$, where ω_t is the coordinate process, satisfying*

$$\mathbb{E}_x[f(\omega_r)] = P^{0,r} f(x), \quad 0 \leq r \leq T, \quad f \in C_\infty(\mathbb{R}^d),$$

where $P^{t,r}(b) := U^{T-t,T-r}(\tilde{b})$, $\tilde{b}(t,x) = b(T-t,x)$, such that $\mathbb{P}_x$ is a weak solution to SDE

$$X_t = x - \int_0^t b(s, X_s) ds + \sqrt{2} B_t. \tag{5}$$

Moreover $\mathbb{P}_x$ is unique in a large class of weak solutions (see [3]*).*

The assertions of Theorem 1 and Theorem 2 for $q > d$ close to d, but under more restrictive assumption $\sqrt{\delta} < \frac{1}{d}$, are contained in [2]. In this paper we improve these results and to some extent simplify the corresponding proofs. In particular, in the proof of gradient estimates we do not try to exclude the time derivative $\partial_\tau u$ as was done in [2], but use it, thus needing less restrictive assumptions on δ.

2. Construction of a regularizing sequence for a $b \in \mathbf{F}_{\delta,g}$

Set $E^1_\varepsilon f(\tau,x) = e^{\varepsilon\Delta_\tau} f(\tau,x)$, $E^d_\varepsilon f(\tau,x) = e^{\varepsilon\Delta_x} f(\tau,x)$, $E^{1+d}_\varepsilon = E^1_\varepsilon E^d_\varepsilon$,

$$b_n(\tau,x) := E^{1+d}_{\varepsilon_n}(\mathbf{1}_{Q_n} b)(\tau,x), \qquad Q_n = [0,n] \times B_d(0,n).$$

Select $\{\varepsilon_n\} \downarrow 0$ from the requirement $\lim_n \int_0^t \|E^{1+d}_{\varepsilon_n}(\mathbf{1}_{Q_n} b)(\tau) - (\mathbf{1}_{Q_n} b)(\tau)\|_2^2 d\tau = 0$.

Note that $|E\phi| \leq \sqrt{E|\phi|^2}$, $|E(\phi\psi)| \leq \sqrt{E|\phi|^2}\sqrt{E|\psi|^2}$. We have (for a.e. $t > 0$)

$$\begin{aligned}|E^d_{\varepsilon_n} \mathbf{1}_{B_d(0,n)} b(\tau,x)|^2 &\leq \langle e^{\varepsilon_n \Delta_d}(x,\cdot) \mathbf{1}_{B_d(0,n)} |b(\tau,\cdot)|^2\rangle \leq \delta \|\nabla_x \sqrt{E^d_{\varepsilon_n}(x,\cdot)}\|_2^2 + g(\tau) \\ &= \varepsilon_n^{-1}\delta \langle \frac{|x-\cdot|^2}{4\varepsilon_n} e^{\varepsilon_n\Delta_d}(x,\cdot)\rangle + g(\tau) \leq C(d)\varepsilon_n^{-1}\delta + g(\tau).\end{aligned}$$

Thus $|E^{d+1}_{\varepsilon_n}(\mathbf{1}_{Q_n} b)(\tau,x)| \leq \sqrt{C(d)\delta}\varepsilon_n^{-\frac{1}{2}} + \sqrt{E^1_{\varepsilon_n} g(\tau)}$ an so $|b_n| \in L^\infty(\mathbb{R}^{1+d})$. It is clear that b_n are smooth.

Next, for $f \in \mathcal{S}(\mathbb{R}^{1+d})$, $\int_0^t \|b_n f\|_2^2 = \int_0^t \|b_n f_t\|_2^2$, where $f_t(\tau,x) := \mathbf{1}_{[0,t]} f(\tau,x)$, and

$$\begin{aligned}\int_0^t \|b_n f\|_2^2 &\leq \int_0^t \langle E^1_{\varepsilon_n}(\mathbf{1}_{[0,n]} b^2), E^d_{\varepsilon_n}|f_t|^2\rangle \leq \int_{\mathbb{R}^1} \langle \mathbf{1}_{[0,n]} b^2, E^{1+d}_{\varepsilon_n}|f_t|^2\rangle \\ &\leq \delta \int_0^n \|\nabla \sqrt{E^{1+d}_{\varepsilon_n}|f_t|^2}\|_2^2 + \int_0^n g E^1_{\varepsilon_n}\langle E^d_{\varepsilon_n}|f_t|^2\rangle \\ &\leq \delta \int_0^n E^1_{\varepsilon_n}\langle E^d_{\varepsilon_n}|\nabla|f_t||^2\rangle + \int_0^n g E^1_{\varepsilon_n}\langle E^d_{\varepsilon_n}|f_t|^2\rangle \\ &\leq \delta \int_0^n E^1_{\varepsilon_n}\|\nabla|f_t|\|_2^2 + \int_0^n g E^1_{\varepsilon_n}\|f_t\|_2^2.\end{aligned}$$

$$\begin{aligned}\int_0^n E^1_{\varepsilon_n}\|\nabla|f_t|\|_2^2 &= \int_{\mathbb{R}^1} \mathbf{1}_{[0,n]} E^1_{\varepsilon_n}(\mathbf{1}_{[0,t]}\|\nabla|f|\|_2^2) \\ &= \int_{\mathbb{R}^1} (E^1_{\varepsilon_n}\mathbf{1}_{[0,n]})\mathbf{1}_{[0,t]}\|\nabla|f|\|_2^2 \leq \int_0^t \|\nabla f\|_2^2.\end{aligned}$$

$$\int_0^n g E^1_{\varepsilon_n}\|f_t\|_2^2 \leq \int_0^t (E^1_{\varepsilon_n} g)\|f\|_2^2.$$

Therefore

$$\int_0^t \|b_n f\|_2^2 \leq \delta \int_0^t \|\nabla f\|_2^2 + \int_0^t g_n \|f\|_2^2, \qquad g_n(\tau) := E^1_{\varepsilon_n} g(\tau).$$

It is seen now that $\{b_n\}$ is regularizing sequence of b.

Remark 1. Let E^1_ε and E^d_ε denote K. Friedrichs mollifiers in one and in d variables, respectively. We could define

$$b_n := E^{1+d}_{\varepsilon_n}(\mathbf{1}_{[0,n]}b).$$

Then, arguing as above, one easily concludes that $\{b_n\}$ is regularizing sequence for b.

3. Proof of Theorem 1

Proof. Denote $w = \nabla_x u(\tau, x)$, $\phi := -\nabla \cdot (w|w|^{q-2}) \equiv -\sum_{i=1}^d \nabla_i (w_i |w|^{q-2})$. Since b_n is smooth and bounded, we can multiply the equation by $\bar{\phi}$ and integrate by parts to obtain

$$q^{-1}\partial_\tau \|w\|_q^q + I_q + (q-2)J_q = X_q, \tag{6}$$

where

$$I_q := \sum_{i=1}^d \langle |\nabla w_i|^2, |w|^{q-2}\rangle, \quad J_q := \langle |\nabla |w||^2, |w|^{q-2}\rangle, \quad X_q := \operatorname{Re}\langle b_n \cdot w, \nabla \cdot (w|w|^{q-2})\rangle.$$

1. Case $d = 3$, $d = 4$. Clearly, $X_q = \operatorname{Re}\langle b_n \cdot w, |w|^{q-2}\Delta u\rangle + (q-2)\operatorname{Re}\langle b_n \cdot w, |w|^{q-3} w \cdot \nabla |w|\rangle$,

$$\begin{aligned}
\operatorname{Re}\langle b_n \cdot w, |w|^{q-2}\Delta u\rangle &= \operatorname{Re}\langle b_n \cdot w, |w|^{q-2}(\partial_\tau u + b_n \cdot w)\rangle \\
&= B_q + \operatorname{Re}\langle b_n \cdot w, |w|^{q-2}\partial_\tau u\rangle \\
&= B_q + \operatorname{Re}\langle (-\partial_\tau u u + \Delta u), |w|^{q-2}\partial_\tau u\rangle \\
&= B_q - \langle |\partial_\tau u|^2, |w|^{q-2}\rangle - q^{-1}\partial_\tau \|w\|_q^q - (q-2)\operatorname{Re}\langle |w|^{q-3} w \cdot \nabla |w|, \partial_\tau u\rangle \\
\operatorname{Re}\langle b_n \cdot w, |w|^{q-2}\Delta u\rangle &\le B_q - \langle |\partial_\tau u|^2, |w|^{q-2}\rangle - q^{-1}\partial_\tau \|w\|_q^q + (q-2)J_q^{\frac12}\langle |\partial_\tau u|^2, |w|^{q-2}\rangle^{\frac12} \\
&\le -q^{-1}\partial_\tau \|w\|_q^q + B_q + \frac{(q-2)^2}{4}J_q \\
&\le -q^{-1}\partial_\tau \|w\|_q^q + \left[\frac{q^2\delta}{4} + \frac{(q-2)^2}{4}\right]J_q + g_n(\tau)\|w\|_q^q.
\end{aligned}$$

$$\begin{aligned}
|\langle b_n \cdot w, |w|^{q-3} w \cdot \nabla |w|\rangle| &\le B_q^{\frac12} J_q^{\frac12} \le \left[\frac{1}{4\varepsilon}B_q + \varepsilon J_q\right] \\
&\le \left[\frac{1}{4\varepsilon}\frac{q^2\delta}{4} + \varepsilon\right]J_q + \frac{g_n(\tau)}{4\varepsilon}\|w\|_q^q \\
&= \frac{q\sqrt{\delta}}{2}J_q + \frac{g_n(\tau)}{q\sqrt{\delta}}\|w\|_q^q \qquad (\varepsilon = \frac{q\sqrt{\delta}}{4}).
\end{aligned}$$

Thus

$$X_q \le -\frac{1}{q}\partial_\tau \|w\|_q^q + \left[\frac{q^2\delta}{4} + \frac{(q-2)^2}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]J_q + (\frac{q-2}{q\sqrt{\delta}} + 1)g_n(\tau)\|w\|_q^q,$$

and hence

$$\frac{2}{q}\partial_\tau \|w\|_q^q + \left[q - 1 - \frac{q^2\delta}{4} - \frac{(q-2)^2}{4} - (q-2)\frac{q\sqrt{\delta}}{2}\right]J_q \le \left(\frac{q-2}{q\sqrt{\delta}} + 1\right)g_n(\tau)\|w\|_q^q.$$

Set $\mu_\tau := \frac{q}{2}\big(\frac{q-2}{q\sqrt{\delta}}+1\big)\int_s^\tau g_n(r)dr$, so

$$\frac{2}{q}\partial_\tau\big(e^{-\mu_\tau}\|w(\tau)\|_q^q\big)+\Big[q-1-\frac{q^2\delta}{4}-\frac{(q-2)^2}{4}-(q-2)\frac{q\sqrt{\delta}}{2}\Big]e^{-\mu_\tau}J_q(\tau)\leq 0. \qquad (\star)$$

It is readily seen that

$$q-1-\frac{q^2\delta}{4}-\frac{(q-2)^2}{4}-(q-2)\frac{q\sqrt{\delta}}{2}>0.$$

holds in the assumption (a) for $d=3,4$.

Finally, using the uniform Sobolev inequality and the bound $\int_0^t g_n \leq c_\delta(\|g'\|_1+t\|g''\|_\infty)$, we obtain from $(\star)$

$$\sup_{s\leq r\leq t}\|w(r)\|_q^q+c_1\int_s^t\|w(\tau)\|_{qj}^q d\tau\leq e^{C_2(\|g'\|_1+t\|g''\|_\infty)}\|\nabla u(s)\|_q^q,$$

Here we have used that $U_n^{s_1,s}u(s)=e^{(s_1-s)\Delta}u(s)-\int_s^{s_1}U_n^{s_1,\tau}b_n\cdot\nabla e^{(\tau-s)\Delta}u(s)d\tau$ and, for $s_1-\tau\leq 1$,

$$\|\nabla U_n^{s_1,\tau}\|_{q\to q}\leq\frac{c_n}{\sqrt{s_1-\tau}},\quad \|\nabla\int_s^{s_i}U_n^{s_1,\tau}b_n\cdot\nabla e^{(\tau-s)\Delta}u(s)d\tau\|_q\leq 2c_n\|b_n\|_\infty\sqrt{s_1-s}\|\nabla u(s)\|_q,$$

so that $\lim_{s_1\downarrow s}\|\nabla U_n^{s_1,s}u(s)\|_q=\lim_{s_1\downarrow s}\|\nabla e^{(s_1-s)\Delta}u(s)\|_q=\|\nabla u(s)\|_q$.

2. Case $d\geq 5$. Now we estimate the term $X_q' := \mathrm{Re}\langle b_n\cdot w,|w|^{q-2}\Delta u\rangle$ as follows.

$$\begin{aligned}X_q'&=\mathrm{Re}\langle-\partial_\tau u+\Delta u,|w|^{q-2}\Delta u\rangle\\&=\langle|\Delta u|^2,|w|^{q-2}\rangle-\mathrm{Re}\langle\partial_\tau u,|w|^{q-2}\Delta u\rangle,\end{aligned}$$

$$\begin{aligned}X_q'&=\mathrm{Re}\langle b_n\cdot w,|w|^{q-2}(\partial_\tau u+b_n\cdot w)\rangle\\&=B_q+\mathrm{Re}\langle b_n\cdot w,|w|^{q-2}\partial_\tau u\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\langle|\Delta u|^2,|w|^{q-2}\rangle&=B_q+\mathrm{Re}\langle\partial_\tau u,|w|^{q-2}(b_n\cdot w+\Delta u)\rangle\\&=B_q+\mathrm{Re}\langle\partial_\tau u,|w|^{q-2}(-\partial_\tau u+2\Delta u)\rangle\\&=B_q-\langle|\partial_\tau u|^2,|w|^{q-2}\rangle+2\mathrm{Re}\langle\partial_\tau u,|w|^{q-2}\Delta u\rangle\\&=B_q-\langle|\partial_\tau u|^2,|w|^{q-2}\rangle-\frac{2}{q}\partial_\tau\|w\|_q^q-2(q-2)\mathrm{Re}\langle\partial_\tau u,|w|^{q-3}w\cdot\nabla|w|\rangle\\&\leq B_q-\langle|\partial_\tau u|^2,|w|^{q-2}\rangle-\frac{2}{q}\partial_\tau\|w\|_q^q+(q-2)^2J_q+\langle|\partial_\tau u|^2,|w|^{q-2}\rangle\\&=B_q-\frac{2}{q}\partial_\tau\|w\|_q^q+(q-2)^2J_q;\end{aligned}$$

$$
\begin{aligned}
X'_q &\leq \langle |\Delta u|^2, |w|^{q-2}\rangle^{\frac{1}{2}} B_q^{\frac{1}{2}} \leq \epsilon \langle |\Delta u|^2, |w|^{q-2}\rangle + \frac{1}{4\epsilon} B_q \\
&\leq -\frac{2\epsilon}{q}\partial_\tau \|w\|_q^q + \Big(\epsilon + \frac{1}{4\epsilon}\Big) B_q + (q-2)^2 \epsilon J_q \\
&\leq -\frac{2\epsilon}{q}\partial_\tau \|w\|_q^q + \Big(\frac{q^2\delta}{4}\epsilon + \frac{1}{4\epsilon}\frac{q^2\delta}{4} + (q-2)^2\epsilon\Big) J_q + \Big(\epsilon + \frac{1}{4\epsilon}\Big) g_n(\tau)\|w\|_q^q \\
&\quad \text{(here we put } \epsilon = \frac{q\sqrt{\delta}}{4}\Big(\frac{q^2\delta}{4} + (q-2)^2\Big)^{-\frac{1}{2}}\text{)} \\
&= -\frac{2\epsilon}{q}\partial_\tau \|w\|_q^q + \frac{q\sqrt{\delta}}{2}\sqrt{\frac{q^2\delta}{4} + (q-2)^2}\; J_q + \Big(\epsilon + \frac{1}{4\epsilon}\Big) g_n(\tau)\|w\|_q^q.
\end{aligned}
$$

Note that $X_q = X'_q + X''_q$, $X''_q = (q-2)\mathrm{Re}\langle b_n \cdot w, |w|^{q-3} w \cdot \nabla |w|\rangle$. Estimating X''_q as in Step **1**, $X''_q \leq (q-2)\Big(\frac{q\sqrt{\delta}}{2} J_q + \frac{g_n}{q\sqrt{\delta}}\|w\|_q^q\Big)$, we have

$$
X_q \leq -\frac{2\epsilon}{q}\partial_\tau \|w\|_q^q + \frac{q\sqrt{\delta}}{2}\Big(\sqrt{\frac{q^2\delta}{4} + (q-2)^2} + q - 2\Big)\; J_q + \Big(\epsilon + \frac{1}{4\epsilon} + \frac{q-2}{q\sqrt{\delta}}\Big) g_n(\tau)\|w\|_q^q.
$$

Finally,

$$
\begin{aligned}
&\frac{1+2\epsilon}{q}\partial_\tau \|w\|_q^q + \Big[q - 1 - \frac{q\sqrt{\delta}}{2}\Big(\sqrt{\frac{q^2\delta}{4} + (q-2)^2} + q - 2\Big)\Big] J_q \\
&\qquad \leq \Big(\epsilon + \frac{1}{4\epsilon} + \frac{q-2}{q\sqrt{\delta}}\Big) g_n(\tau)\|w\|_q^q.
\end{aligned}
$$

We are left to show that

$$
q - 1 - \frac{q\sqrt{\delta}}{2}\Big(\sqrt{\frac{q^2\delta}{4} + (q-2)^2} + q - 2\Big) > 0. \tag{$\star'$}
$$

assuming that $d \geq 5$, $\sqrt{\delta} \leq (1 - \frac{a}{16+a})\frac{q-1}{q-2}\frac{1}{q}$, $a = \frac{a}{16+a}$, $a = \frac{(q-1)^2}{(q-2)^4}$, $q = d + \varepsilon$, $\forall \varepsilon \in]0,1]$.

Set $\sqrt{\delta} = (1-\mu)\frac{q-1}{q-2}\frac{1}{q}$, $0 < \mu < 1$. Then ($\star'$) will follow from

$$
q - 1 - (1-\mu)\frac{q-1}{2} > (1-\mu)\frac{q-1}{2}\sqrt{1 + \frac{q^2}{4(q-2)^2}(1-\mu)^2\frac{(q-1)^2}{(q-1)^2}\frac{1}{q^2}}.
$$

The latter is equivalent to

$$
16\mu > (1-\mu)^4 \frac{(q-1)^2}{(q-2)^4}
$$

which clearly follows from $16\mu \geq (1-\mu)\frac{(q-1)^2}{(q-2)^4}$. In turn the latter is equivalent to

$$
\mu \geq \frac{a}{16+a}, \quad a = \frac{(q-1)^2}{(q-2)^4}.
$$

Finally, with $\mu = \frac{a}{16+a}$ it is seen that $\frac{1}{d} < (1-\mu)\frac{q-1}{q-2}\frac{1}{q}$ for $q = d + \varepsilon$ and all $0 < \varepsilon \leq 1$.

3. Let $d \geq 3$, $\sqrt{\delta} = \frac{1}{d}$, $q = d+1$. It is seen that ($\star'$) is equivalent to $d > 1$. Note that ($\star'$) fails if $q > d+1$ and $\sqrt{\delta} = \frac{1}{d}$. □

Remarks. ($\star'$) still holds for $\mu = 1 + \frac{8}{a} - \sqrt{(1+\frac{8}{a})^2 - 1}$ $(< \frac{a}{16+a})$.

4. Proof of Theorem 2

Claim 1. *Let u_n be the classical solution of* (3). *Then, for every $r \in]\frac{2}{2-\sqrt{\delta}}, \infty[$, $\{u_n\}$ is a Cauchy sequence in $L^\infty([s,t], L^r(\mathbb{R}^d))$.*

Proof. Below we allow $\delta < 4$, so we do not use the gradient bounds of Theorem 1. Without loss of generality we will suppose that $f = \text{Re} f$, and so u_n is real, and that r is a rational number (so u_n^{r-1} is well defined even if u_n is sign changing).

(**a**). Let $k > 2$. Define

$$\eta(t) := \begin{cases} 0, & \text{if } t < k, \\ \left(\frac{t}{k} - 1\right)^k, & \text{if } k \leq t \leq 2k, \\ 1, & \text{if } 2k < t, \end{cases} \qquad \text{and } \zeta(x) = \eta(\tfrac{|x-o|}{R}), \quad R > 0.$$

Note that $|\nabla \zeta| \leq R^{-1} \mathbf{1}_{\nabla\zeta} \zeta^{1-\frac{1}{k}}$. Here $\mathbf{1}_{\nabla\zeta}$ denotes the indicator of the support of $|\nabla \zeta|$.

Set $v := \zeta u_n(\tau)$. Clearly,

$$\langle \zeta(\partial_\tau - \Delta + b_n \cdot \nabla) u_n(\tau), v^{r-1} \rangle = 0,$$

$$\langle (\partial_\tau - \Delta + b_n \cdot \nabla) v, v^{r-1} \rangle = \langle [-\Delta, \zeta]_- u_n + u_n b_n \cdot \nabla \zeta, v^{r-1} \rangle, \tag{$\star$}$$

where

$$\begin{aligned} \langle [-\Delta, \zeta]_- u_n, v^{r-1} \rangle &= \frac{2}{r'} \langle \nabla v^{\frac{r}{2}}, u_n v^{\frac{r}{2}-1} \nabla \zeta \rangle - \langle \nabla \zeta, v^{r-1} \cdot \nabla u_n \rangle \\ &= \frac{2}{r'} \langle \nabla v^{\frac{r}{2}}, v^{\frac{r}{2}} \frac{\nabla \zeta}{\zeta} \rangle - \frac{2}{r} \langle \frac{\nabla \zeta}{\zeta}, v^{\frac{r}{2}} \nabla v^{\frac{r}{2}} \rangle + \langle \frac{|\nabla \zeta|^2}{\zeta^2}, v^r \rangle \\ &= \frac{2(r-2)}{r} \langle \nabla v^{\frac{r}{2}}, v^{\frac{r}{2}} \frac{\nabla \zeta}{\zeta} \rangle + \langle \frac{|\nabla \zeta|^2}{\zeta^2}, v^r \rangle. \end{aligned}$$

By the quadratic estimates

$$\begin{aligned} \langle u_n b_n \cdot \nabla \zeta, v^{r-1} \rangle &= \langle b_n \cdot \frac{\nabla \zeta}{\zeta}, v^r \rangle \\ &\leq \frac{\mu\sqrt{\delta}}{r} \|\nabla v^{\frac{r}{2}}\|_2^2 + \frac{r\sqrt{\delta}}{4\mu} \langle \frac{|\nabla \zeta|^2}{\zeta^2}, |v|^r \rangle + \frac{\mu g_n(\tau)}{r\sqrt{\delta}} \|v\|_r^r \quad (\mu > 0), \end{aligned}$$

$$\frac{2(r-2)}{r} \langle \nabla v^{\frac{r}{2}}, v^{\frac{r}{2}} \frac{\nabla \zeta}{\zeta} \rangle \leq \frac{\mu\sqrt{\delta}}{r} \|\nabla v^{\frac{r}{2}}\|_2^2 + \frac{(r-2)^2}{r\mu\sqrt{\delta}} \langle \frac{|\nabla \zeta|^2}{\zeta^2}, |v|^r \rangle,$$

we get from ($\star$)

$$\partial_\tau \|v\|_r^r + 2\Big(\frac{2}{r'} - (1+\mu)\sqrt{\delta}\Big) \|\nabla v^{\frac{r}{2}}\|_2^2 \leq \Big(\frac{(r-2)^2}{\mu\sqrt{\delta}} + \frac{r^2\sqrt{\delta}}{4\mu} + r\Big) \langle \frac{|\nabla \zeta|^2}{\zeta^2}, |v|^r \rangle + \frac{r+\mu}{\sqrt{\delta}} g_n(\tau) \|v\|_r^r.$$

Recalling that $\frac{2}{r'} > \sqrt{\delta}$, we can find $\mu > 0$ such that $\frac{2}{r'} - (1+\mu)\sqrt{\delta} \geq 0$. Thus

$$\partial_\tau \|v\|_r^r \leq \Big(\frac{4(r-2)^2 + r^2\delta}{4\mu\sqrt{\delta}} + r\Big)\langle \frac{|\nabla\zeta|^2}{\zeta^2}, |v|^r\rangle + \frac{r+\mu}{\sqrt{\delta}} g_n(\tau)\|v\|_r^r \qquad (\star\star)$$

Next, $\langle \frac{|\nabla\zeta|^2}{\zeta^2}, |v|^r\rangle \leq R^{-2}\|\mathbf{1}_{\nabla\zeta}\zeta^{-2\theta}|v|^r\|_1$, $\theta := k^{-1}$. Since $\|u_n\|_\infty \leq \|f\|_\infty$, $\|\mathbf{1}_{\nabla\zeta}\|_{\frac{r}{2\theta}} \leq c(d,\theta)R^{\frac{2\theta d}{r}}$, and

$$\|\mathbf{1}_{\nabla\zeta}\zeta^{-2\theta}|v|^r\|_1 \leq \|\mathbf{1}_{\nabla\zeta}u_n^{2\theta}\|_{\frac{r}{2\theta}}\|v\|_r^{r-2\theta} \leq \|\mathbf{1}_{\nabla\zeta}\|_{\frac{r}{2\theta}}\|u_n\|_\infty^{2\theta}\|v\|_r^{r-2\theta},$$

we obtain, using the Young inequality, the crucial estimate (*on which the whole proof rests*)

$$\langle \frac{|\nabla\zeta|^2}{\zeta^2}, |v|^r\rangle \leq \frac{2\theta}{r}[c(d)]^{\frac{r}{2\theta}} R^{d-\frac{r}{\theta}}\|f\|_\infty^r + \frac{r-2\theta}{r}\|v\|_r^r.$$

Fix θ by $0 < \theta < \frac{r}{d+2r}$. Now from $(\star\star)$ we obtain the inequality

$$\partial_\tau\|v\|_r^r \leq M(r,d,\delta)R^{-\gamma}\|f\|_\infty^r + N(r,d,\delta)\|v\|_r^r, \quad \gamma = \frac{r}{\theta} - d > 0, \qquad (\star\star\star)$$

from which we conclude that, for a given $\hat{\varepsilon} > 0$ there is R such that $\sup_{\tau\in[s,t],n}\|\zeta u_n(\tau)\|_r \leq \frac{\hat{\varepsilon}}{2}$, and so

$$\sup_{\tau\in[s,t],\, n,m\geq 1}\|(1_{B^c(o,2kR)})(u_n(\tau) - u_m(\tau))\|_r < \hat{\varepsilon}.$$

(**b**). Let $k > 2$. Define

$$\eta(t) := \begin{cases} 1, & \text{if } t < 2k, \\ \big(1 - \frac{1}{k}(t-2k)\big)^k, & \text{if } 2k \leq t \leq 3k, \\ 0, & \text{if } 3k < t, \end{cases} \quad \text{and } \zeta(x) := \eta(\tfrac{|x-o|}{R}), \quad R > 0.$$

Set $h := u_m - u_n$. Clearly, for r rational and $v = \zeta h(\tau)$,

$$\langle(\partial_\tau h - \Delta h + b_m\cdot\nabla h), \zeta v^{r-1}\rangle = F,$$

$$\partial_\tau\|v\|_r^r + 4(r')^{-1}\|\nabla v^{\frac{r}{2}}\|_2^2 + 2\langle b_m v^{\frac{r}{2}}, \nabla v^{\frac{r}{2}}\rangle \leq rF, \quad r' = \frac{r}{r-1},$$

where

$$F = \langle[-\Delta,\zeta]_- h, v^{r-1}\rangle + \langle(b_n - b_m)\cdot\nabla u_n, \zeta v^{r-1}\rangle + \langle b_m\cdot\nabla\zeta, hv^{r-1}\rangle,$$

$$\langle[-\Delta,\zeta]_- h, v^{r-1}\rangle = \frac{2(r-2)}{r}\langle\nabla v^{\frac{r}{2}}, v^{\frac{r}{2}}\frac{\nabla\zeta}{\zeta}\rangle + \langle\frac{|\nabla\zeta|^2}{\zeta^2}, v^r\rangle,$$

$$\langle\nabla v^{\frac{r}{2}}, v^{\frac{r}{2}}\frac{\nabla\zeta}{\zeta}\rangle \leq \|\nabla v^{\frac{r}{2}}\|_2\langle\frac{|\nabla\zeta|^2}{\zeta^2}, |v|^r\rangle^{\frac{1}{2}},$$

$$\langle b_m\cdot\nabla\zeta, hv^{r-1}\rangle = \langle b_m v^{\frac{r}{2}}\cdot\frac{\nabla\zeta}{\zeta}, v^{\frac{r}{2}}\rangle \leq \|b_m v^{\frac{r}{2}}\|_2\langle\frac{|\nabla\zeta|^2}{\zeta^2}, |v|^r\rangle^{\frac{1}{2}},$$

$$\|b_m v^{\frac{r}{2}}\|_2^2 \leq \delta\|\nabla v^{\frac{r}{2}}\|_2^2 + g_n\|v\|_r^r.$$

Using these estimates and fixing $\epsilon > 0$ by $2r'^{-1} - (1+\epsilon)\sqrt{\delta} \geq 0$, we have

$$\partial_\tau\|v\|_r^r + 2\big(2r'^{-1} - (1+\epsilon)\sqrt{\delta}\big)|\nabla v^{\frac{r}{2}}\|_2^2 \leq \big(\frac{(r-2)^2}{\epsilon r} + \frac{r}{4\epsilon} + r\big)\langle\frac{|\nabla\zeta|^2}{\zeta^2}|v|^r\rangle + (\epsilon+2)g_n\|v\|_r^r + F_1,$$

$$F_1 = \langle\zeta|b_n - b_m|^2\rangle^{\frac{1}{2}}\langle\zeta|\nabla u_n|^2, |v|^{2(r-1)}\rangle^{\frac{1}{2}}.$$

Again using the estimate $\langle\frac{|\nabla\zeta|^2}{\zeta^2}|v|^r\rangle \leq MR^{-\gamma}\|f\|_\infty^r + N\|v\|_r^r$, $\gamma>0$, and setting $\mu_\tau = NC\tau + (\epsilon+2)\int_s^\tau g_n(s)ds$, where $C=C(r,\delta)=\frac{(r-2)^2}{\epsilon r}+\frac{r}{4\epsilon}+r$, we obtain that

$$e^{-\mu_t}\|v(t)\|_r^r \leq \|v(s)\|_r^r + MCR^{-\gamma}\|f\|_\infty^r\int_s^t e^{-\mu_\tau}d\tau + \int_s^t e^{-\mu_\tau}F_1(\tau)d\tau,$$

$$\|v(t)\|_r^r \leq MCR^{-\gamma}\|f\|_\infty^r e^{\mu_t}t + e^{\mu_t}\int_s^t F_1(\tau)d\tau,$$

$$\int_s^t F_1(\tau)d\tau \leq \Big(\int_0^t\langle\zeta|b_n-b_m|^2\rangle d\tau\Big)^{\frac12}\Big(\int_s^t\langle\zeta|\nabla u_n|^2\rangle d\tau\Big)^{\frac12}\|f\|_\infty^{r-1}.$$

We estimate $\int_s^t\langle\zeta|\nabla u_n|^2\rangle d\tau$ as follows. Note that $\langle\partial_\tau u_n - \Delta u_n + b_n\cdot\nabla u_n, \zeta u_n\rangle = 0$, and so

$$\frac12\partial_\tau\langle\zeta u_n^2\rangle + \langle\zeta|\nabla u_n|^2\rangle + \langle\nabla u_n, u_n\nabla\zeta\rangle + \langle b_n\cdot\nabla u_n,\zeta u_n\rangle = 0,$$

$$\partial_\tau\langle\zeta u_n^2\rangle + \langle\zeta|\nabla u_n|^2\rangle \leq 2(\langle\frac{|\nabla\zeta|^2}{\zeta}\rangle + \langle\zeta|b_{t,b}|^2\rangle)\|f\|_\infty^2,$$

$$\begin{aligned}\int_s^t\langle\zeta|\nabla u_n|^2\rangle d\tau &\leq \|f\|_2^2 + \Big(2t\langle\frac{(\nabla\zeta)^2}{\zeta}\rangle + \int_0^t\langle\zeta|b_n|^2\rangle d\tau\Big)\|f\|_\infty^2\\ &\leq \|f\|_2^2 + tL(R)\|f\|_\infty^2.\end{aligned}$$

Thus, we arrived at

$$\begin{aligned}\|v(t)\|_r^r &\leq MCR^{-\gamma}\|f\|_\infty^r e^{\mu_t}t\\ &+(\|f\|_2+\sqrt{tL(R)}\|f\|_\infty)\|f\|_\infty^{r-1}e^{\mu_t}\sqrt{t}\int_0^t\langle\zeta|b_n-b_m|^2\rangle d\tau.\end{aligned}$$

By the definition of b_n, $\lim_{n,m}\int_0^t\langle\zeta|b_n-b_m|^2\rangle d\tau = 0$, and hence for given $\hat\epsilon>0$ and $R<\infty$ there is a number $P<\infty$ such that

$$\sup_{\tau\in[s,t],n,m\geq P}\|1_{B(o,2kR)}(u_n(\tau)-u_m(\tau))\|_r < \hat\varepsilon.$$

□

The proof of Theorem 2 follows from the next claim.

Claim 2. *$\{u_n\}$ is a Cauchy sequence in $L_{\infty,\infty}$.*

Here by $L_{p,r}=L_{p,r}([s,t]\times\mathbb{R}^d)$ we denote the Banach space of real functions on $[s,t]\times\mathbb{R}^d$ having finite norm

$$\|v\|_{p,r} := (\int_s^t\|v(\tau)\|_r^p d\tau)^{\frac1p}, \quad \|v\|_{\infty,\infty} := \sup_{\tau\in[s,t]}\|v(\tau)\|_\infty.$$

Proof. **1**. Again, first we allow $\delta < 4$. Note that $h(\tau) = u_m(\tau) - u_n(\tau)$ satisfies the identity

$$(\frac{d}{d\tau} - \Delta + b_m \cdot \nabla)h = (b_n - b_m) \cdot \nabla u_n, \quad h(s) = 0.$$

Multiplying the identity by $h|h|^{r-2}$, $r > \frac{2}{2-\sqrt{\delta}}$ and integrating by parts, we obtain

$$\frac{1}{r}\partial_\tau \|v\|_2^2 + \frac{4}{rr'}\|\nabla v\|_2^2 + \frac{2}{r}\mathrm{Re}\langle b_m \cdot \nabla v, v\rangle = \mathrm{Re}\langle (b_n - b_m) \cdot \nabla u_n, v|v|^{1-\frac{2}{r}}\rangle,$$

where $v = h|h|^{\frac{r-2}{2}}$. Now, using the quadratic estimates and the definition of class $\mathbf{F}_{\delta,g}$, we have

$$\begin{aligned}|\langle b_m \cdot \nabla v, v\rangle| &\leq \varepsilon \|b_m v\|_2^2 + (4\varepsilon)^{-1}\|\nabla v\|_2^2 \\ &\leq (\varepsilon\delta + (4\varepsilon)^{-1})\|\nabla v\|_2^2 + \varepsilon g_n(\tau)\|v\|_2^2 \\ &= \sqrt{\delta}\|\nabla v\|_2^2 + (2\sqrt{\delta})^{-1} g_n(\tau)\|v\|_2^2 \quad (\varepsilon = (2\sqrt{\delta})^{-1},\ n > m)\end{aligned}$$

and

$$\begin{aligned}|\langle (b_n - b_m) \cdot \nabla u_n, v|v|^{1-\frac{2}{r}}\rangle| &\leq \langle (|b_n| + |b_m|)|v|, |v|^{1-\frac{2}{r}}|\nabla u_n|\rangle \\ &\leq \eta\delta\|\nabla v\|_2^2 + \eta^{-1}\||v|^{1-\frac{2}{r}}\nabla u_n\|_2^2 + \eta g_n(\tau)\|v\|_2^2 \quad (\eta > 0),\end{aligned}$$

and hence obtain the inequality

$$\begin{aligned}&\frac{1}{r}\partial_\tau\|v\|_2^2 + \left(\frac{4}{rr'} - \frac{2}{r}\sqrt{\delta} - \eta\delta\right)\|\nabla v\|_2^2 \\ &\qquad \leq \eta^{-1}\||v|^{1-\frac{2}{r}}\nabla u_n\|_2^2 + ((r\sqrt{\delta})^{-1} + \eta)g_n(\tau)\|v\|_2^2.\end{aligned}$$

Since $r > \frac{2}{2-\sqrt{\delta}} \Leftrightarrow \frac{2}{r'} - \sqrt{\delta} > 0$, we can choose $k > 2$ so large that

$$\frac{4}{rr'} - \frac{2}{r}\sqrt{\delta} = \frac{2}{r}(\frac{2}{r'} - \sqrt{\delta}) = 2r^{-k+1}.$$

Fix η by

$$\eta\delta = \frac{4}{rr'} - \frac{2}{r}\sqrt{\delta} - r^{-k+1} \ \ (= r^{-k+1}).$$

Thus

$$\begin{aligned}&\partial_\tau\|v\|_2^2 + r^{-k}\|\nabla v\|_2^2 \\ &\qquad \leq \delta r^{k-1}\||v|^{1-\frac{2}{r}}\nabla u_n\|_2^2 + (\delta^{-\frac{1}{2}} + \delta^{-1}r^{-k+2})g_n(\tau)\|v\|_2^2.\end{aligned}$$

So, multiplying this inequality by $e^{-\mu_\tau}$, $\mu_\tau := (\delta^{-\frac{1}{2}} + \delta^{-1})\int_s^\tau g_n(s)ds$, integrating over $[s,t]$, and then using the inequality

$$\mu_\tau \leq \bar{\mu}_t := (\delta^{-\frac{1}{2}} + \delta^{-1})c_\delta(\|g'\|_1 + t\|g''\|_\infty)$$

we obtain

$$\sup_{s\leq\tau\leq t}\|v(\tau)\|_2^2 + r^{-k}\int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq r^k e^{\bar{\mu}_t}\int_s^t \||v|^{1-\frac{2}{r}}(\tau)\nabla u_n(\tau)\|_2^2 d\tau.$$

From the last inequality we obtain, using uniform Sobolev inequality $c_d^{-1}\|v\|_{2j}^2 \leq \|\nabla v\|_2^2$ and Hölder's inequality:

$$c_d r^k \sup_{s\leq\tau\leq t}\|v(\tau)\|_2^2+\int_s^t \|\nabla v\|_{2j}^2 d\tau \leq c_d r^{2k} e^{\bar{\mu}_t} \int_s^t \||v|^{1-\frac{2}{r}}\nabla u_n\|_2^2 d\tau$$

$$\leq c_d r^{2k} e^{\bar{\mu}_t} \int_s^t \|\nabla u_n\|_{2x}^2 \|v^{1-\frac{2}{r}}\|_{2x'}^2 d\tau, \quad x>1, \quad x' := \frac{x}{x-1}.$$

2. Now let d, δ and $q>d$ satisfy the assumptions of Theorem 1. Thus

$$\sup_{s\leq\tau\leq t}\|\nabla u(\tau)\|_q^2 \leq e^{2C_2 q^{-1}(\|g'\|_1+t\|g''\|)}\|\nabla u(s)\|_q^2.$$

Selecting $x := \frac{q}{2}$ and putting $C_3 = 2\delta^{-1}c_\delta + 2C_2 q^{-1}$, we obtain

$$c_d r^k \|h\|_{\infty,r}^r + \|h\|_{r,rj}^r \leq c_d r^{2k} e^{C_3(\|g'\|_1+t\|g''\|)}\|\nabla u(s)\|_q^2 \int_s^t \|h\|_{x'(r-2)}^{r-2} d\tau.$$

Set $D := c_d e^{C_3(\|g'\|_1+t\|g''\|)}\|\nabla u(s)\|_q^2$. Then the last inequalities take form

$$c_d r^k \|h\|_{\infty,r} + \|h\|_{r,rj} \leq D^{\frac{1}{r}}(r^{\frac{1}{r}})^{2k}\|h\|_{r-2,x'(r-2)}^{1-\frac{2}{r}}. \tag{$\star$}$$

Let us use first Hölder and then Young inequalities:

$$\|h\|_{\frac{r}{1-\beta},\frac{rd}{d-2+2\beta}}^r \leq \|h\|_{\infty,r}^{\beta r}\|h\|_{r,rj}^{(1-\beta)r} \leq \beta\|h\|_{\infty,r}^r + (1-\beta)\|h\|_{r,rj}^r, \quad 0<\beta<1.$$

Therefore, we obtain from ($\star$) the inequalities

$$\|h\|_{\frac{r}{1-\beta},\frac{rd}{d-2+2\beta}} \leq D^{\frac{1}{r}}(r^{\frac{1}{r}})^{2k}\|h\|_{r-2,x'(r-2)}^{1-\frac{2}{r}}.$$

Let $d\geq 5$, $\sqrt{\delta}=d^{-1}$ and $q=d+1$. Define $\beta=\frac{2}{d^2+d+2}$, $j_1=\frac{d}{d-2+2\beta}$ and $\mathfrak{t}=\frac{j_1}{x'}$. Then $\mathfrak{t}=\frac{1}{1-\beta}$. In other cases we select $\beta \in]0,q-d]$ such that $\mathfrak{t}=\frac{1}{1-\beta}$. Thus,

$$\|h\|_{\mathfrak{t}r,j_1 r} \leq D^{\frac{1}{r}}(r^{\frac{1}{r}})^{2k}\|h\|_{r-2,x'(r-2)}^{1-\frac{2}{r}}.$$

Fix $r_0 > \frac{2}{2-\sqrt{\delta}}$. Successively setting $x'(r_1-2)=r_0$, $x'(r_2-2)=j_1 r_1$, $x'(r_3-2)=j_1 r_2, \dots$, so that

$$r_n = (\mathfrak{t}-1)^{-1}\Big(\mathfrak{t}^n(\frac{r_0}{x'}+2) - \mathfrak{t}^{n-1}\frac{r_0}{x'} - 2\Big),$$

we obtain from the last inequality that

$$\|h\|_{\mathfrak{t}r_n,j_1 r_n} \leq D^{\alpha_n}\Gamma_n\|h\|_{\frac{r_0}{x'},r_0}^{\gamma_n},$$

where

$$\begin{aligned}\alpha_n =&\frac{1}{r_1}\Big(1-\frac{2}{r_2}\Big)\Big(1-\frac{2}{r_3}\Big)\dots\Big(1-\frac{2}{r_n}\Big)+\frac{1}{r_2}\Big(1-\frac{2}{r_3}\Big)\Big(1-\frac{2}{r_4}\Big)\dots\Big(1-\frac{2}{r_n}\Big)\\ &+\dots+\frac{1}{r_{n-1}}\Big(1-\frac{2}{r_n}\Big)+\frac{1}{r_n};\\ \gamma_n =&\Big(1-\frac{2}{r_1}\Big)\Big(1-\frac{2}{r_2}\Big)\dots\Big(1-\frac{2}{r_n}\Big);\\ \Gamma_n =&\Big[r_n^{r_n^{-1}}r_{n-1}^{r_{n-1}^{-1}(1-2r_n^{-1})}r_{n-2}^{r_{n-2}^{-1}(1-2r_{n-1}^{-1})(1-2r_n^{-1})}\dots r_1^{r_1^{-1}(1-2r_2^{-1})\dots(1-2r_n^{-1})}\Big]^{2k}.\end{aligned}$$

Since $\alpha_n=(\mathfrak{t}^n-1)r_n^{-1}(\mathfrak{t}-1)^{-1}$ and $\gamma_n=r_0\mathfrak{t}^{n-1}(x'r_n)^{-1}$,

$$\alpha_n\leq\alpha\equiv\Big(\frac{r_0}{x'}+2-\frac{r_0}{j_1}\Big)^{-1}=\frac{j_1}{r_0}\Big(\mathfrak{t}-1+2\frac{j_1}{r_0}\Big)^{-1},$$

and

$$\inf_n\gamma_n>\gamma=\frac{r_0}{x'}(\frac{r_0}{x'}+\frac{2\mathfrak{t}}{\mathfrak{t}-1})^{-1}>0,\qquad\sup_n\gamma_n<1.$$

Also, since

$$\Gamma_n^{\frac{1}{2k}}=r_n^{r_n^{-1}}r_{n-1}^{\mathfrak{t}r_n^{-1}}r_{n-2}^{\mathfrak{t}^2r_n^{-1}}\dots r_1^{\mathfrak{t}^{n-1}r_n^{-1}}$$

and $b\mathfrak{t}^n\leq r_n\leq a\mathfrak{t}^n$, where $a=r_1(\mathfrak{t}-1)^{-1},\ b=r_1\mathfrak{t}^{-1}$, we have

$$\begin{aligned}\Gamma_n^{\frac{1}{2k}}&\leq(a\mathfrak{t}^n)^{(b\mathfrak{t}^n)^{-1}}(a\mathfrak{t}^{n-1})^{(b\mathfrak{t}^{n-1})^{-1}}\dots(a\mathfrak{t})^{(b\mathfrak{t})^{-1}}\\ &=\Big[a^{(1-\mathfrak{t}^{-n})(\mathfrak{t}-1)^{-1}}\mathfrak{t}^{\sum_{i=1}^n i\mathfrak{t}^{-i}}\Big]^{\frac{1}{b}}\leq\Big[a^{(\mathfrak{t}-1)^{-1}}\mathfrak{t}^{\mathfrak{t}(\mathfrak{t}-1)^{-2}}\Big]^{\frac{1}{b}}.\end{aligned}$$

Finally, note that $\|h\|_{r_0,r_0}\to0$ as $n,m\uparrow\infty$, and so $\|h\|^{\gamma_n}_{\frac{r_0}{x'},r_0}\leq(t-s)^{\frac{\gamma_n}{r_0(x-1)}}\|h\|^{\gamma}_{r_0,r_0}$ for all large n,m.

Define $\nu(\tau)=\tau^{\frac{\gamma}{r_0(x-1)}}$ if $0<\tau\leq1$ and $\tau^{\frac{1}{r_0(x-1)}}$ if $\tau>1$.

Therefore, we conclude that there are constants $B<\infty$ and $\gamma>0$ such that the following inequality is valid

$$\|h\|_{\infty,\infty}\leq B(t-s)\|h\|^{\gamma}_{r_0,r_0},\quad B(t-s)=B\nu(t-s)e^{\alpha C_3\|g''\|_\infty t}.$$

It remains to note that $\|h\|_{L^{r_0}([s,t]\times\mathbb{R}^d)}\to0$ uniformly in $s\in[0,t]$ according to Claim 1. □

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