REGULARITY THEORY OF KOLMOGOROV OPERATOR REVISITED

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ABSTRACT. We consider the Kolmorogov operator $-\Delta + b \cdot \nabla$ with drift b in the class of form-bounded vector fields (containing vector fields having critical-order singularities). We characterize quantitative dependence of the Sobolev and Hölder regularity of solutions to the corresponding elliptic equation on the value of the form-bound of b.

1. Introduction and results

The goal of this paper is to refine some aspects of the regularity theory of the operator

$$-\Delta + b \cdot \nabla, \quad b \in L^1_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d), \quad d \geq 3, \tag{1}$$

required to construct a weak solution to the stochastic differential equation (SDE)

$$X_t - x = -\int_0^t b(X_s)ds + \sqrt{2}W_t, \qquad \text{where } W_t \text{ is a } d\text{-dimensional Brownian motion}, \quad x \in \mathbb{R}^d. \tag{2}$$

Recall that a weak solution to (2) is a process X_t defined on some probability space having continuous trajectories, such that (a) $\int_0^t |b(X_s)|ds < \infty$ for every $t > 0$ a.s., and (b) there exists a Brownian motion W_t on this probability space such that (X_t, W_t) satisfy (2) for every $t > 0$ a.s.

The process X_t, called a Brownian motion with drift b, plays fundamental role in the theory of diffusion processes and in the theory of elliptic and parabolic equations. The case when the drift b is singular (i.e. locally unbounded) is of special interest due to, in particular, physical applications; the problem of describing singular b such that for every $x \in \mathbb{R}^d$ there exists a (unique) weak solution to (2) is classical and has been thoroughly studied, see [BC, KrR, P, Z] and references therein. The conventional scale of singularity of b used in the literature is the scale of $L^r(\mathbb{R}^d, \mathbb{R}^d)$ spaces. The value $r = d$ is known to be optimal: regarding positive results on weak existence and uniqueness in law for SDE (2) with $|b| \in L^r$, see [P] for $r > d$, and see [Kr1, Kr2, XXZZ] for $r = d$; on the other hand, it is not difficult to find a vector field b with $|b| \in L^r$, $r < d$ such that a weak solution to (2) does not exist.

Nevertheless, the L^r scale is a rather rough measure of singularity of b, and the class $|b| \in L^d$ is far from being the maximal admissible. For instance, consider SDE (2) with Hardy drift $b(x) = \sqrt{\delta}\frac{d-2}{2}|x|^{-2}x$ (which clearly fails to be in L^d) and the initial state $x = 0$. If $\sqrt{\delta} \geq \frac{2d}{d-2}$, then this SDE does not have a weak solution (see [KiS2, Example 1]). However, if $\sqrt{\delta} < \min\{1, \frac{2}{d-2}\}$, then by [KiS2, Theorem 1] a weak solution exists.

Let us note that, generally speaking, to construct a weak solution to (2), one needs a well developed regularity theory of (1) – the operator behind SDE (2), cf. the papers cited above, see

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details below. In this paper we develop regularity theory of (1), with b in a large class of vector fields containing vector fields with entries in L^d as well as vector fields having critical-order singularities (such as the Hardy drift), that allows to construct a weak solution to SDE (2).

DEFINITION 1. A vector field $b \in L^2_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d)$ is said to be form-bounded if there exists a constant $\delta>0$ such that

$$\||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta} \quad \text{for some } \lambda=\lambda_\delta>0.$$

($\|\cdot\|_{p\to q}$ denotes the $\|\cdot\|_{L^p\to L^q}$ operator norm). The class of such vector fields is denoted by $\mathbf{F}_\delta$.

The latter condition can be re-stated as a quadratic form inequality

$$\|b\varphi\|_2^2 \leq \delta\|\nabla\varphi\|_2^2 + c_\delta\|\varphi\|_2^2, \quad \varphi \in W^{1,2},$$

with constant $c_\delta=\lambda\delta$. The constant δ is called the form-bound of b.

Note that, given a constant $k\neq 0$, if $b\in\mathbf{F}_\delta$, then $kb\in\mathbf{F}_{|k|^2\delta}$. Clearly, if $b_1\in\mathbf{F}_{\delta_1}$, $b_2\in\mathbf{F}_{\delta_2}$, then

$$b_1+b_2\in\mathbf{F}_\delta, \quad \sqrt{\delta}=\sqrt{\delta_1}+\sqrt{\delta_2}.$$

Condition $b\in\mathbf{F}_\delta$ with $\delta<1$ appears in the literature as a condition ensuring that the quadratic form corresponding to the formal operator $-\Delta+b\cdot\nabla$ determines the generator of a quasi contraction C_0 semigroup in L^2, see [Ka, Ch.VI].

Let us list some sub-classes of $\mathbf{F}_\delta$ defined in elementary terms.

1. A vector field $b=b_1+b_2\in L^d(\mathbb{R}^d,\mathbb{R}^d)+L^\infty(\mathbb{R}^d,\mathbb{R}^d)$ is in $\mathbf{F}_\delta$ with δ that can be chosen arbitrarily small.

Indeed, representing $b=\mathsf{f}+v$, where $\|\mathsf{f}\|_d<\varepsilon$, $v\in L^\infty(\mathbb{R}^d,\mathbb{R}^d)$, one can estimate, using the Hölder inequality and the Sobolev Embedding Theorem,

$$\begin{aligned}\||b|(\lambda-\Delta)^{-\frac{1}{2}}g\|_2 &\leq \|\mathsf{f}\|_d\|(\lambda-\Delta)^{-\frac{1}{2}}g\|_{\frac{2d}{d-2}} + \|v\|_\infty\lambda^{-\frac{1}{2}}\|g\|_2 \qquad (g\in L^2)\\ &\leq \big(c_S\|\mathsf{f}\|_d+\|v\|_\infty\lambda^{-\frac{1}{2}}\big)\|g\|_2\\ &\leq (c_S+1)\varepsilon\|g\|_2 \quad \text{for } \lambda=\varepsilon^{-2}\|v\|_\infty^{-2}.\end{aligned}$$

2. The class $\mathbf{F}_\delta$ also contains vector fields having critical-order singularities, such as $b(x):=\pm\sqrt{\delta}\frac{d-2}{2}|x|^{-2}x$, as follows from the Hardy inequality $\||x|^{-1}\varphi\|_2^2\leq\frac{4}{(d-2)^2}\|\nabla\varphi\|_2^2$, $\varphi\in W^{1,2}$. (And, of course, $b\notin\mathbf{F}_{\delta_2}$ if $\delta_2<\delta$.) The last example shows that $\mathbf{F}_\delta$ contains $L^d(\mathbb{R}^d,\mathbb{R}^d)+L^\infty(\mathbb{R}^d,\mathbb{R}^d)$ as a proper sub-class.

More generally, $\mathbf{F}_\delta$ contains, as a proper sub-class, vector fields b with $|b|$ in $L^{d,\infty}$ (the weak L^d class). Recall that a measurable function $h:\mathbb{R}^d\to\mathbb{R}$ is in $L^{d,\infty}$ if $\|h\|_{d,\infty}:=\sup_{s>0}s|\{x\in\mathbb{R}^d : |h(x)|>s\}|^{1/d}<\infty$. If $|b|$ in $L^{d,\infty}$, then

$$\begin{aligned}b\in\mathbf{F}_{\delta_1} \quad \text{with } \sqrt{\delta_1} &= \||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}\\ &\leq \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\||x|^{-1}(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}\\ &\leq \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}2^{-1}\frac{\Gamma\big(\frac{d-2}{4}\big)}{\Gamma\big(\frac{d+2}{4}\big)} = \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\frac{2}{d-2}.\end{aligned}$$

where $\Omega_d=\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)$ is the volume of the unit ball in $\mathbb{R}^d$, see [KPS, Prop. 2.5, 2.6, Cor. 2.9].

3. The class $\mathbf{F}_\delta$ contains the vector fields b with $|b|^2$ in the Campanato-Morrey class ($s>1$)

$$\left\{v\in L^s_{\rm loc}:\left(\frac{1}{|Q|}\int_Q|v(x)|^sdx\right)^{\frac{1}{s}}\leq c_s l(Q)^{-2}\text{ for all cubes }Q\right\};$$

the latter is a proper sub-class of $\mathbf{F}_\delta$, see [CWW].

4. Let us note that, for every $\varepsilon>0$, there exists a $b\in\mathbf{F}_\delta$ such that $b\notin L^{2+\varepsilon}_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d)$, e.g.

$$|b|^2(x)=C\frac{\mathbf{1}_{B(0,1+\alpha)}-\mathbf{1}_{B(0,1-\alpha)}}{\big||x|-1\big|^{-1}(-\ln\big||x|-1\big|)^\beta},\quad\beta>1,\quad 0<\alpha<1.$$

In contrast to the other classes of singular vector fields mentioned above, the class $\mathbf{F}_\delta$ is defined, loosely speaking, in terms of the operators that constitute (1).

Let $b\in\mathbf{F}_\delta$. By [KS, Theorem 1 and Lemma 5], if $\delta<\min\{1,\big(\frac{2}{d-2}\big)^2\}$, then for every $p\in\big[2,2/\sqrt{\delta}\big[$ there exists a realization $\Lambda_p(b)$ of the formal operator $-\Delta+b\cdot\nabla$ on L^p as the (minus) generator of a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup $e^{-t\Lambda_p(b)}$ such that $u:=(\mu+\Lambda_p(b))^{-1}f$, $f\in L^p$ ($\Leftrightarrow$ solution to the elliptic equation $(\mu+\Lambda_p(b))u=f$) satisfies for all $\mu>\mu_1\equiv\mu_1(d,p,\delta)>0$

$$\|\nabla u\|_p\leq K_1(\mu-\mu_1)^{-\frac{1}{2}}\|f\|_p.$$

Concerning the regularity of higher-order derivatives of u, the authors in [KS] establish the next bound on the following non-linear characteristics of u:

$$\|\nabla|\nabla u|^{\frac{p}{2}}\|_2^{\frac{2}{p}}\leq K_2(\mu-\mu_1)^{\frac{1}{p}-\frac{1}{2}}\|f\|_p$$

Here constants $K_i=K_i(d,p,\delta)<\infty$ ($i=1,2$). Then, by the Sobolev Embedding Theorem, $\|\nabla u\|_{pj}\leq C\|f\|_p$, $j=\frac{d}{d-2}$, and so there exists $p>\max\{2,d-2\}$ such that $u\in C^{0,\gamma}$ with the Hölder continuity exponent $\gamma=1-\frac{d-2}{p}$.

The results in [KS] capture quantitative dependence of Sobolev and Hölder regularity of u on the value of form-bound δ. The latter serves as a measure of the "size" of singularity of b. Note that, from this point of view, the class $b\in L^d(\mathbb{R}^d,\mathbb{R}^d)$ does not contain vector fields having critical-order singularities, for by Example 1 such b is in $\mathbf{F}_\delta$ with arbitrarily small δ.

In our main result (Theorem 1) we establish regularity of higher-order derivatives of solution to the elliptic equation u under the *same* assumption on the form-bound δ as in [KS] (and thus without losing the size of singularity of b). This will allow us, in particular, to considerably simplify, in comparison with [KS], the construction of the corresponding Feller semigroup (and thus of the corresponding diffusion process), see Remark 1 below.

The method. Our starting object is an $\mathcal{B}(L^p)$-valued function $\Theta_p(\mu,b)$, $\mu>\mu_0$, a "candidate for the resolvent of $-\Delta+b\cdot\nabla$ in L^p". We prove that, for smooth approximations b_n of b, $\Theta_p(\mu,b_n)$ indeed coincides with the resolvent $(\mu-\Delta+b_n\cdot\nabla)^{-1}\in\mathcal{B}(L^p)$ (which exists by the classical theory) for $\mu>0$ sufficiently large. Armed with this fact, we show that, for a general $b\in\mathbf{F}_\delta$ with δ smaller than a certain explicit constant, the operator-valued function $\Theta(\mu,b)$ is the resolvent of a closed densely defined operator $-\Lambda_p(b)$ generating a C_0 semigroup on L^p. This operator is the sought operator realization of $-\Delta + b\cdot\nabla$ in L^p. The regularity properties of the resolvent $(\mu+\Lambda_p(b)^{-1}$, and thus of the solution u to the equation $(\mu+\Lambda_p(b))u = f$ (cf. Corollary 1 below), then follow immediately from the definition of $\Theta_p(\mu,b)$. Concerning the relationship between $\Lambda_p(b)$ and the formal operator $-\Delta+b\cdot\nabla$, we can show, arguing as in the proof of Theorem 1.3 in [Ki], that

$$\langle \Lambda_p(b)u,v\rangle = \langle u,-\Delta v\rangle + \langle b\cdot\nabla u, v\rangle,\quad u\in D(\Lambda_p(b)),\quad v\in C_c^\infty.$$

Here and below,

$$\langle h\rangle := \int_{\mathbb{R}^d} h(x)d\mathcal{L}^d,\quad \langle h,g\rangle := \langle h\bar{g}\rangle.$$

Notations. Let $\mathcal{W}^{\alpha,p}$, $\alpha>0$ denote the Bessel potential space endowed with norm $\|f\|_{p,\alpha} := \|g\|_p$, $f=(1-\Delta)^{-\frac{\alpha}{2}}g$, $g\in L^p$. Let $\mathcal{W}^{-\alpha,p'}$, $p'=\frac{p}{p-1}$, be the anti-dual of $\mathcal{W}^{\alpha,p}$.

Let $\mathcal{B}(X,Y)$ be the space of bounded linear operators between Banach spaces $X\to Y$. Set $\mathcal{B}(X) := \mathcal{B}(X,X)$.

Denote by $\upharpoonright$ the restriction of an operator to a subspace.

For $p\geq 2$, put

$$c_{\delta,p} := \left(\frac{p}{2}\delta + \frac{p-2}{2}\sqrt{\delta}\right)^{\frac{1}{p}}\left(p-1-(p-1)\frac{p-2}{2}\sqrt{\delta} - \frac{p(p-2)}{4}\delta\right)^{-\frac{1}{p}}$$

and

$$b^{\frac{2}{p}} := |b|^{\frac{2}{p}-1}b,\qquad \mathcal{E} := \bigcup_{\varepsilon>0} e^{-\varepsilon|b|}L^p.$$

Theorem 1 (Main result)**.** *Let $d\geq 3$. Assume that $b\in\mathbf{F}_\delta$, $\delta<1$. Then for every $p\in\left[2,\frac{2}{\sqrt{\delta}}\right[$ the formal operator $-\Delta+b\cdot\nabla$ has a realization $\Lambda_p(b)$ in L^p as the generator of a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup $e^{-t\Lambda_p(b)}$ such that:*

(i) The resolvent of $-\Lambda_p(b)$ admits the representation

$$\big(\mu+\Lambda_p(b)\big)^{-1} = \Theta_p(\mu,b)$$

for all $\mu>\mu_0\equiv\mu_0(d,p,\delta)>0$, where

$$\Theta_p(\mu,b) := (\mu-\Delta)^{-1} - Q_p(1+T_p)^{-1}G_p$$

for operators $Q_p, G_p, T_p\in\mathcal{B}(L^p)$ defined as follows:

$$G_p := b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-1},$$

and Q_p, T_p are the extensions by continuity of densely defined operators

$$Q_p\upharpoonright\mathcal{E} := (\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}},\quad T_p\upharpoonright\mathcal{E} := b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}.$$

We have

$$\|G_p\|_{p\to p}\leq C_1\mu^{-\frac{1}{2}+\frac{1}{p}},\quad \|Q_p\|_{p\to p}\leq C_2\mu^{-\frac{1}{2}-\frac{1}{p}},\quad \|T_p\|_{p\to p}\leq c_{\delta,p}<1,$$

(ii) For each $2\leq r<p<q<\infty$ and $\mu>\mu_0$, define operators

$$G_p(r) := b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{r}}\in\mathcal{B}(L^p),\qquad Q_p(q) := (\mu-\Delta)^{-\frac{1}{2}+\frac{1}{q}}|b|^{1-\frac{2}{p}}\qquad \text{on } \mathcal{E}.$$

Then $Q_p(q) \in \mathcal{B}(L^p)$[1], *and the resolvent admits the representation*

$$\big(\mu + \Lambda_p(b)\big)^{-1} = (\mu - \Delta)^{-1} - (\mu - \Delta)^{-\frac{1}{2} - \frac{1}{q}} Q_p(q)(1 + T_p)^{-1} G_p(r)(\mu - \Delta)^{-\frac{1}{2} + \frac{1}{r}}$$

for all $\mu > \mu_0$.

(*iii*)

$$e^{-t\Lambda_p(b_n)} \to e^{-t\Lambda_p(b)} \quad \text{in } L^p \quad \text{locally uniformly in } t \geq 0,$$

where $b_n := e^{\epsilon_n \Delta}(\mathbf{1}_n b)$, $\mathbf{1}_n$ *is the indicator of* $\{x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n\}$, $\epsilon_n \downarrow 0$, $n \geq 1$, *and* $\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_p(b_n)) = \mathcal{W}^{2,p}$.

Theorem 1(*i*),(*ii*) immediately yields

Corollary 1. *In the assumptions of Theorem 1, for every* $2 \leq r < p < q < \infty$ *and* $\mu > \mu_0$,

$$\big(\mu + \Lambda_p(b)\big)^{-1} \in \mathcal{B}\big(\mathcal{W}^{-1+\frac{2}{r},p},\ \mathcal{W}^{1+\frac{2}{q},p}\big). \tag{$\star$}$$

In particular,

$$D\big(\Lambda_p(b)\big) \subset \mathcal{W}^{1+\frac{2}{q},p}, \quad q > p.$$

The previous corollary and the Sobolev Embedding Theorem give

Corollary 2. *For* $d \geq 4$, *if* $\delta < \big(\frac{2}{d-2}\big)^2$ *then there exists* $p > d-2$ *such that* $u := (\mu + \Lambda_p(b))^{-1} f$, $f \in L^p$ *satisfies* $u \in C^{0,\gamma}$, $\gamma < 1 - \frac{d-2}{p}$. (*For* $d = 3$ *the corresponding inclusion can be improved, see remarks below.*)

Denote

$$C_\infty := \{f \in C(\mathbb{R}^d) : \lim_{x \to \infty} f(x) = 0\} \quad \text{(endowed with the sup-norm)}.$$

The resolvent representation of Theorem 1(*ii*) yields rather easily (following the arguments in the proof of Theorem 1.5 in [Ki], see also the proof of Theorem 5.6 in [KiS])

Theorem 2. *Let* $d \geq 3$. *Assume that* $b \in \mathbf{F}_\delta$, $\delta < \min\{1, \big(\frac{2}{d-2}\big)^2\}$. *Fix* $p > \max\{2, d-2\}$. *Then*

$$(\mu + \Lambda_{C_\infty}(b))^{-1} := \big(\Theta_p(\mu, b) \upharpoonright L^p \cap C_\infty\big)^{\text{clos}}_{C_\infty \to C_\infty}, \quad \mu > \mu_0,$$

determines the resolvent of the generator of a positivity preserving contraction C_0 *semigroup on* C_∞ *("Feller semigroup"), such that*

$$e^{-t\Lambda_{C_\infty}(b)} = s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(b_n)} \quad \text{locally uniformly in } t \geq 0,$$

where b_n*'s were defined in Theorem 1(iii),* $\Lambda_{C_\infty}(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_{C_\infty}(b_n)) = (1-\Delta)^{-1} C_\infty$.

Recall that, by a standard result [BG, Ch. I.9], given a Feller semigroup T^t on C_∞, there exist probability measures $\{\mathbb{P}_x\}_{x \in \mathbb{R}^d}$ on the space of right-continuous functions $X : [0, \infty[\to \bar{\mathbb{R}}^d$ having left limits ($\bar{\mathbb{R}}^d$ is the one-point compactification of $\mathbb{R}^d$) such that

$$\mathbb{E}_{\mathbb{P}_x}[f(X_t)] = (T^t f)(x), \quad f \in C_\infty, \quad x \in \mathbb{R}^d. \tag{3}$$

[1]the extension of $Q_p(q)$ by continuity will be denoted again by $Q_p(q)$

Let $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ be the probability measures determined by $T^t := e^{-t\Lambda_{C_\infty}(b)}$. Theorems 1 and 2 provide an information about the Feller semigroup $e^{-t\Lambda_{C_\infty}(b)}$ sufficient to repeat the argument in [KiS2], which then yields that for every $x \in \mathbb{R}^d$ the probability measure $\mathbb{P}_x$ is concentrated on finite continuous trajectories and determines a (unique in appropriate sense) weak solution to SDE (2).

Remarks. 1. In Theorem 2, we obtain the assertion of [KS, Theorem 2] in a simpler way, i.e. without appealing to rather sophisticated $L^p \to L^\infty$ Moser-type iteration procedure of the cited paper.

2. There is an analogue of Theorem 1 for vector fields b in the class of weakly form-bounded vector fields that contains $\mathbf{F}_\delta$ as a proper sub-class, see [Ki, Theorem 1.3], see also [KiS, Theorem 4.3]. However, there one obtains a different regularity result,

$$\big(\mu + \Lambda_p(b)\big)^{-1} \in \mathcal{B}(\mathcal{W}^{-1+\frac{1}{r},p}, \mathcal{W}^{1+\frac{1}{q},p}),$$

with *strictly smaller* values of δ (and so these two results should be viewed as essentially incomparable). Moreover, the proof of the cited result appeals to abstract L^p inequalities for symmetric Markov generators, while the proof of Theorem 1 is elementary.

3. If $b \in \mathbf{F}_\delta$, $\delta < 1$, then one can show that $D(\Lambda_2(b)) \subset W^{2,2}$. In particular, if $d = 3$, $(\mu+\Lambda_2(b))^{-1}$ maps L^2 to $W^{1,6}$, and so $D(\Lambda_2(b)) \subset C^{0,\gamma}$ with $\gamma = \frac{1}{2}$.

4. For a general $b \in \mathbf{F}_\delta$, for p large the $W^{2,p}$ estimates on solution u to the corresponding elliptic equation do not exist, see detailed discussion in [KiS, sect. 4].

5. In Theorems 1 and 2 we obtain the same condition $\delta < \min\{1, (\frac{2}{d-2})^2\}$ as in [KS, Theorem 1]. One can thus ask whether this condition is sharp. Incidentally, the constant $(\frac{2}{d-2})^2$ coincides with the constant in Hardy's inequality.

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2. Proof of Theorem 1

Proposition 1. *(j) Set* $G_p = b^{\frac{2}{p}} \cdot \nabla(\mu-\Delta)^{-1}$, $Q_p = (\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}$, $T_p = b^{\frac{2}{p}} \cdot \nabla(\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}$. Q_p, T_p *are densely defined (on* $\mathcal{E}$*) operators. Then there exists* $\mu_0 = \mu_0(d,p,\delta) > 0$ *such that*

$$\|G_p\|_{p\to p} \le C_1\mu^{-\frac{1}{2}+\frac{1}{p}}, \quad \|Q_p\|_{p\to p} \le C_2\mu^{-\frac{1}{2}-\frac{1}{p}}, \quad \|T_p\|_{p\to p} \le c_{\delta,p} < 1, \qquad \mu > \mu_0,$$

where $c_{\delta,p} := \left(\frac{p}{2}\delta + \frac{p-2}{2}\sqrt{\delta}\right)^{\frac{1}{p}}\left(p-1-(p-1)\frac{p-2}{2}\sqrt{\delta} - \frac{p(p-2)}{4}\delta\right)^{-\frac{1}{p}}$.

(jj) Set $G_p(r) = b^{\frac{2}{p}} \cdot \nabla(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{r}}$, $Q_p(q) = (\mu-\Delta)^{-\frac{1}{2}+\frac{1}{q}}|b|^{1-\frac{2}{p}}$, *where* $2 \le r < p < q < \infty$. $Q_p(q)$ *is a densely defined (on* $\mathcal{E}$*) operator. Then for* $\mu > \mu_0$

$$\|G_p(r)\|_{p\to p} \le K_{1,r}, \qquad \|Q_p(q)\|_{p\to p} \le K_{2,q}.$$

The extension of $Q_p(q)$ *by continuity we denote again by* $Q_p(q)$.

Proof. It suffices to consider the case $p > 2$.

(j) (**a**) Set $u := (\mu - \Delta)^{-1}|b|^{1-\frac{2}{p}}f$, $0 \leq f \in L^p$. Then

$$\begin{aligned}\|T_p f\|_p^p &= \|b^{\frac{2}{p}}\nabla u\|_p^p = \langle |b|^2|\nabla u|^p\rangle \\ &= \||b|(\lambda-\Delta)^{-\frac{1}{2}}(\lambda-\Delta)^{\frac{1}{2}}|\nabla u|^{\frac{p}{2}}\|_2^2 \qquad (\lambda = \lambda_\delta) \\ &\leq \||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}^2\|(\lambda-\Delta)^{\frac{1}{2}}|\nabla u|^{\frac{p}{2}}\|_2^2 \\ &= \delta\|(\lambda-\Delta)^{\frac{1}{2}}|\nabla u|^{\frac{p}{2}}\|_2^2 = \delta\big(\lambda\|\nabla u\|_p^p + \|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\big).\end{aligned}$$

It remains to prove the principal inequality

$$\delta\big(\lambda\|\nabla u\|_p^p + \|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\big) \leq c_{\delta,p}^p\|f\|_p^p, \tag{$*$}$$

and conclude that $\|T_p\|_{p\to p} \leq c_{\delta,p}$.

First, we prove an a priori variant of $(*)$, i.e. for $u := (\mu - \Delta)^{-1}|b|^{1-\frac{2}{p}}f$ with $b = b_n$. Since our assumptions on δ involve only strict inequalities, we may assume, upon selecting appropriate $\varepsilon_n \downarrow 0$, that $b_n \in \mathbf{F}_\delta$ with the same $\lambda = \lambda_\delta$ for all n.

Set

$$w := \nabla u, \quad I_q := \sum_{r=1}^d \langle (\nabla_r w)^2|w|^{p-2}\rangle, \quad J_q := \langle (\nabla|w|)^2|w|^{p-2}\rangle.$$

We multiply $(\mu - \Delta)u = |b|^{1-\frac{2}{p}}f$ by $\phi := -\nabla\cdot(w|w|^{p-2})$ and integrate by parts to obtain

$$\mu\|w\|_p^p + I_p + (p-2)J_p = \langle |b|^{1-\frac{2}{p}}f, -\nabla\cdot(w|w|^{p-2})\rangle, \tag{4}$$

where

$$\begin{aligned}&\langle |b|^{1-\frac{2}{p}}f, -\nabla\cdot(w|w|^{p-2})\rangle = \langle |b|^{1-\frac{2}{p}}f, (-\Delta u)|w|^{p-2} - (p-2)|w|^{p-3}w\cdot\nabla|w|\rangle \\ &(\text{use the equation } -\Delta u = -\mu u + |b|^{1-\frac{2}{p}}f) \\ &= \langle |b|^{1-\frac{2}{p}}f, \big(-\mu u + |b|^{1-\frac{2}{p}}f\big)|w|^{p-2}\rangle - (p-2)\langle |b|^{1-\frac{2}{p}}f, |w|^{p-3}w\cdot\nabla|w|\rangle.\end{aligned}$$

Remark 1. We have used the idea of [KS] of working with the test function $\phi = -\nabla\cdot(w|w|^{p-2})$. It allows to, essentially, differentiate the equation while avoiding differentiating its coefficients/right-hand side.

It is interesting to note that, similarly to [KS], above we had to use the same equation twice. One could use it only once, but this would lead to more restrictive assumptions on δ.

We have

1) $\langle |b|^{1-\frac{2}{p}}f, (-\mu u)|w|^{p-2}\rangle \leq 0$,

2) $|\langle |b|^{1-\frac{2}{p}}f, |w|^{p-3}w\cdot\nabla|w|\rangle| \leq \alpha J_p + \frac{1}{4\alpha}N_p$ $(\alpha > 0)$, where $N_p := \langle |b|^{1-\frac{2}{p}}f, |b|^{1-\frac{2}{p}}f|w|^{p-2}\rangle$,

so, the RHS of (4) $\leq (p-2)\alpha J_p + \big(1 + \frac{p-2}{4\alpha}\big)N_p$, where, in turn,

$$\begin{aligned}N_p &\leq \langle |b|^2|w|^p\rangle^{\frac{p-2}{p}}\langle f^p\rangle^{\frac{2}{p}} \\ &\leq \frac{p-2}{p}\langle |b|^2|w|^p\rangle + \frac{2}{p}\|f\|_p^p \qquad (\text{use } b \in \mathbf{F}_\delta \Leftrightarrow \|b\varphi\|_2^2 \leq \delta\|\nabla\varphi\|_2^2 + \lambda\delta\|\varphi\|_2^2,\ \varphi \in W^{1,2}) \\ &\leq \frac{p-2}{p}\left(\frac{p^2}{4}\delta J_q + \lambda\delta\|w\|_p^p\right) + \frac{2}{p}\|f\|_p^p.\end{aligned}$$

Thus, applying $I_q \geq J_q$ in the LHS of (4), we obtain

$$(\mu - c_0)\|w\|_p^p + \left[p-1-(p-2)\left(\alpha + \frac{1}{4\alpha}\frac{p(p-2)}{4}\delta\right) - \frac{p(p-2)}{4}\delta\right]\frac{4}{p^2}\|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2 \leq \left(1 + \frac{p-2}{4\alpha}\right)\frac{2}{p}\|f\|_p^p,$$

where $c_0 = \frac{p-2}{p}\lambda\delta\big(1 + \frac{p-2}{4\alpha}\big)$. It is now clear that one can find a sufficiently large $\mu_0 = \mu_0(d,p,\delta) > 0$ so that, for all $\mu > \mu_0$, $(*)$ (with $b = b_n$) holds with

$$c_{\delta,p}^p = \delta\frac{p^2}{4}\frac{\left(1 + \frac{p-2}{4\alpha}\right)\frac{2}{p}}{p-1-(p-2)\left(\alpha + \frac{1}{4\alpha}\frac{p(p-2)}{4}\delta\right) - \frac{p(p-2)}{4}\delta} \qquad \Big(\text{we select } \alpha = \frac{p}{4}\sqrt{\delta}\Big)$$
$$= \frac{\frac{p}{2}\delta + \frac{p-2}{2}\sqrt{\delta}}{p-1-(p-1)\frac{p-2}{2}\sqrt{\delta} - \frac{p(p-2)}{4}\delta},$$

as claimed. Finally, we pass to the limit $n \to \infty$ using Fatou's Lemma. The proof of $(*)$ is completed.

Remark 2. It is seen that $\sqrt{\delta} < \frac{2}{p} \Rightarrow c_{\delta,p} < 1$. We also note that the above choice of α is the best possible.

(**b**) Set $u = (\mu - \Delta)^{-1}f$, $0 \leq f \in L^p$. Then

$$\|G_p f\|_p^p = \|b^{\frac{2}{p}} \cdot \nabla u\|_p^p$$
$$\text{(we argue as in (\textbf{a}))}$$
$$\leq \delta\big(\lambda\|\nabla u\|_p^p + \|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\big),$$

where, clearly, $\|\nabla u\|_p^p \leq \mu^{-\frac{p}{2}}\|f\|_p^p$. In turn, arguing as in (**a**), we arrive at $\mu\|w\|_p^p + I_p + (p-2)J_p = \langle f, -\nabla\cdot(w|w|^{p-2}\rangle$ $(w = \nabla u)$,

$$\mu\|w\|_p^p + (p-1)J_p \leq \langle f^2, |w|^{p-2}\rangle + (p-2)\langle f, |w|^{p-3}w\cdot\nabla|w|\rangle),$$

$$\mu\|w\|_p^p + (p-1)J_p \leq \langle f^2, |w|^{p-2}\rangle + (p-2)\big(\varepsilon J_p + \frac{1}{4\varepsilon}\langle f^2, |w|^{p-2}\rangle\big), \quad \varepsilon > 0.$$

Selecting ε sufficiently small, we obtain

$$J_p \leq C_0\|w\|_p^{p-2}\|f\|_p^2.$$

Now, applying $\|w\|_p \leq \mu^{-\frac{1}{2}}\|f\|_p$, we arrive at $\|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2 \leq C\mu^{-\frac{p}{2}+1}\|f\|_p^p$. Hence, $\|G_p f\|_p \leq C_1\mu^{-\frac{1}{2}+\frac{1}{p}}\|f\|_p$ for all $\mu > \mu_0$.

(**c**) Set $u = (\mu - \Delta)^{-1}|b|^{1-\frac{2}{p}}f$ $(= Q_p f)$, $0 \leq f \in L^p$. Then, multiplying $(\mu - \Delta)u = |b|^{1-\frac{2}{p}}f$ by u^{p-1}, we obtain

$$\mu\|u\|_p^p + \frac{4(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|_2^2 = \langle |b|^{1-\frac{2}{p}}f, u^{p-1}\rangle,$$

where we estimate the RHS using Young's inequality:

$$\langle |b|^{1-\frac{2}{p}}u^{\frac{p}{2}-1}, fu^{\frac{p}{2}}\rangle \leq \varepsilon^{\frac{2p}{p-2}}\frac{p-2}{2p}\langle |b|^2u^p\rangle + \varepsilon^{-\frac{2p}{p+2}}\frac{p+2}{2p}\langle f^{\frac{2p}{p+2}}u^{\frac{p^2}{p+2}}\rangle \quad \varepsilon > 0.$$

Using $b \in \mathbf{F}_\delta$ and selecting $\varepsilon > 0$ sufficiently small, we obtain that for any $\mu_1 > 0$ there exists $C > 0$ such that

$$(\mu - \mu_1)\|u\|_p^p \leq C\langle f^{\frac{2p}{p+2}} u^{\frac{p^2}{p+2}}\rangle, \qquad \mu > \mu_1.$$

Therefore, $(\mu - \mu_1)\|u\|_p^p \leq C\langle f^p\rangle^{\frac{2}{p+2}}\langle u^p\rangle^{\frac{p}{p+2}}$, so $\|u\|_p \leq C_2\mu^{-\frac{1}{2}-\frac{1}{p}}\|f\|_p$. The proof of (j) is completed.

(jj) Below we use the following formula: For every $0 < \alpha < 1$, $\mu > 0$,

$$(\mu - \Delta)^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t + \mu - \Delta)^{-1}dt.$$

We have

$$\begin{aligned}
\|Q_p(q)f\|_p \leq\ & \|(\mu - \Delta)^{-\frac{1}{2}+\frac{1}{q}}|b|^{1-\frac{2}{p}}|f|\|_p \\
&\leq k_q\int_0^\infty t^{-\frac{1}{2}+\frac{1}{q}}\|(t + \mu - \Delta)^{-1}|b|^{1-\frac{2}{p}}|f|\|_p dt \\
&\text{(we use } (\mathbf{c})) \\
&\leq\ k_qC_2\int_0^\infty t^{-\frac{1}{2}+\frac{1}{q}}(t + \mu)^{-\frac{1}{2}-\frac{1}{p}}dt\ \|f\|_p = K_{2,q}\|f\|_p, \quad f \in \mathcal{E},
\end{aligned}$$

where, clearly, $K_{2,q} < \infty$ due to $q > p$.

It suffices to consider the case $r > 2$. We have

$$\begin{aligned}
\|G_p(r)f\|_p &\leq k_r\int_0^\infty t^{-\frac{1}{2}-\frac{1}{r}}\|b^{\frac{2}{p}} \cdot \nabla(t + \mu - \Delta)^{-1}f\|_p dt \\
&\text{(we use } (\mathbf{b})) \\
&\leq k_rC_1\int_0^\infty t^{-\frac{1}{2}-\frac{1}{r}}(t + \mu)^{-\frac{1}{2}+\frac{1}{p}}dt\ \|f\|_p = K_{1,r}\|f\|_p, \quad f \in \mathcal{E},
\end{aligned}$$

where, clearly, $K_{1,r} < \infty$ due to $r < p$.

The proof of (jj) is completed. □

Remark 3. Proposition 1 is valid for b_n, $n = 1, 2, \ldots$, with the same constants.

Proposition 2. *The operator-valued function $\Theta_p(\mu, b_n)$ is a pseudo-resolvent on $\mu > \mu_0$, i.e.*

$$\Theta_p(\mu, b_n) - \Theta_p(\nu, b_n) = (\nu - \mu)\Theta_p(\mu, b_n)\Theta_p(\nu, b_n), \quad \mu, \nu > \mu_0.$$

Proof. The proof proceeds by direct calculation, cf. [Ki, proof of Prop. 2.4]. □

Proposition 3. *For every $n = 1, 2, \ldots$,*

$$\mu\Theta_p(\mu, b_n) \to 1 \text{ strongly in } L^p \text{ as } \mu \uparrow \infty \quad (\text{uniformly in } n).$$

Proof. The proof repeats [Ki, proof of Prop. 2.5(ii)]. Since $\mu(\mu - \Delta)^{-1} \to 1$ strongly in L^p, it suffices to show that $\mu\Theta_p - \mu(\mu - \Delta)^{-1} \to 0$ strongly in L^p. By Proposition 1, $\mu\Theta_p$ is uniformly (in μ) bounded in $\mathcal{B}(L^p)$, so it suffices to prove the convergence on C_c^∞. We have ($h \in C_c^\infty$)

$$\Theta_p h - (\mu + A_p)^{-1}h = -Q_p\big(1 + T_p\big)^{-1}G_p h$$

where, by Proposition 1(j), $\|Q_p\|_{p\to p} \le C_2 \mu^{-\frac{1}{2}-\frac{1}{p}}$, $\|(1+T_p)^{-1}\|_{p\to p} < 1$, and

$$\|G_p h\|_p = \|b_n^{\frac{2}{p}} \cdot \nabla(\nu-\Delta)^{-1}(\mu-\Delta)^{-1}(\nu-\Delta)h\|_p \qquad (\nu > \mu_0 \text{ is fixed})$$
$$\leqslant \|b_n^{\frac{2}{p}} \cdot \nabla(\nu-\Delta)^{-1}\|_{p\to p}\|(\mu-\Delta)^{-1}(\nu-\Delta)h\|_p \leqslant C\mu^{-1}\|(\nu-\Delta)h\|_p,$$

and so

$$\|\Theta_p h - (\mu-\Delta)^{-1}h\|_p \le C_0 \mu^{-\frac{3}{2}-\frac{1}{p}}\|(\nu-\Delta)h\|_p \to 0 \quad \text{ as } \mu\to\infty, \quad C_0 \neq C_0(n).$$

□

Proposition 4. *We have $\{\mu : \mu > \mu_0\} \subset \rho(-\Lambda_p(b_n))$, the resolvent set of $-\Lambda_p(b_n)$. The operator-valued function $\Theta_p(\mu, b_n)$ is the resolvent of $-\Lambda_p(b_n)$:*

$$\Theta_p(\mu, b_n) = (\mu + \Lambda_p(b_n))^{-1}, \quad \mu > \mu_0.$$

Proof. By the Hille Perturbation Theorem, $\Theta_p(\mu_n, b_n) = (\mu_n + \Lambda_p(b_n))^{-1}$ for all sufficiently large μ_n $(= \mu(\|b_n\|_\infty))$. Now, by a theorem of T. Kato [Ka2], in reflexive space L^p the pseudo-resolvent $\Theta_p(\mu, b_n)$ (Proposition 2) satisfying $\mu\Theta_p(\mu, b_n) \overset{s}{\to} 1$ in L^p as $\mu \uparrow \infty$ (Proposition 3) is the resolvent of a densely defined closed operator on L^p. This operator coincides with $-\Lambda_p(b_n)$. □

Proposition 5. *We have, for all $n = 1, 2, \dots$,*

$$\|(\mu + \Lambda_p(b_n))\|_{p\to p} \le (\mu - \mu_0)^{-1}, \quad \mu > \mu_0$$

(replacing, if necessary, μ_0 by $\max\{\mu_0, \frac{\lambda\delta}{2(p-1)}\}$).

Proof. See [KS, Theorem 1]. □

Proposition 6. *For every $\mu > \mu_0$,*

$$\Theta_p(\mu, b_n) \to \Theta_p(\mu, b) \text{ strongly in } L^p.$$

Proof. The proof proceeds by applying carefully the Dominated Convergence Theorem to operators $Q_p(b_n)$, $T_p(b_n)$, $G_p(b_n)$ in the definition of $\Theta_p(\mu, b_n)$, cf. [Ki, proof of Prop. 2.8]. □

Now, by the Trotter Approximation Theorem [Ka, IX.2.5], $\Theta_p(\mu, b) = (\mu + \Lambda_p(b))^{-1}$, $\mu > \mu_0$, where $\Lambda_p(b)$ is the generator of a quasi contraction C_0 semigroup in L^p. (i) follows. (ii) follows from Proposition 1(jj). (iii) is Proposition 6. The proof of Theorem 1 is completed.

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