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A NEW APPROACH TO THE L^p-THEORY OF $-\Delta + b\cdot\nabla$, AND ITS APPLICATIONS TO FELLER PROCESSES WITH GENERAL DRIFTS

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Abstract. We develop a detailed regularity theory of $-\Delta + b\cdot\nabla$ in $L^p(\mathbb{R}^d)$, for a wide class of vector fields. The L^p-theory allows us to construct associated strong Feller process in $C_\infty(\mathbb{R}^d)$. Our starting object is an operator-valued function, which, we prove, coincides with the resolvent of an operator realization of $-\Delta + b\cdot\nabla$, the generator of a holomorphic C_0-semigroup on $L^p(\mathbb{R}^d)$. Then the very form of the operator-valued function yields crucial information about smoothness of the domain of the generator.

1. Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d, \mathcal{L}^d)$ and $W^{1,p} = W^{1,p}(\mathbb{R}^d, \mathcal{L}^d)$ the standard Lebesgue and Sobolev spaces, $C^{0,\gamma} = C^{0,\gamma}(\mathbb{R}^d)$ the space of Hölder continuous functions $(0<\gamma<1)$, $C_b = C_b(\mathbb{R}^d)$ the space of bounded continuous functions endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity, $\mathcal{W}^{\alpha,p}$, $\alpha>0$, the Bessel space endowed with norm $\|u\|_{p,\alpha} := \|g\|_p$, $u = (1-\Delta)^{-\frac{\alpha}{2}} g$, $g \in L^p$, and $\mathcal{W}^{-\alpha,p}$ the dual of $\mathcal{W}^{\alpha,p}$. We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between complex Banach spaces $X \to Y$, endowed with operator norm $\|\cdot\|_{X\to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Set $\|\cdot\|_{p\to q} := \|\cdot\|_{L^p \to L^q}$.

For each $p \geqslant 1$ and $p' = p/(p-1)$, by $\langle u, v\rangle$ we denote the $(L^p, L^{p'})$ pairing, so that

$$\langle u, v\rangle = \langle u\bar{v}\rangle := \int_{\mathbb{R}^d} u\bar{v}\, d\mathcal{L}^d \qquad (u \in L^p, v \in L^{p'}).$$

2. Consider the following classes of vector fields.

(1) We say that a $b : \mathbb{R}^d \to \mathbb{C}^d$ belongs to $\mathbf{F}_\delta$, the class of form-bounded vector fields, and write $b \in \mathbf{F}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \sqrt{\delta}.$$

(2) We say that a $b : \mathbb{R}^d \to \mathbb{C}^d$ belongs to the Kato class $\mathbf{K}^{d+1}_\delta$, and write $b \in \mathbf{K}^{d+1}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta.$$

(3) We say that a $b : \mathbb{R}^d \to \mathbb{C}^d$ belongs to $\mathbf{F}^{\frac{1}{2}}_\delta$, the class of *weakly* form-bounded vector fields, and write $b \in \mathbf{F}^{\frac{1}{2}}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \sqrt{\delta}.$$

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Simple examples show: $\mathbf{F}_{\delta_1} - \mathbf{K}^{d+1}_{\delta} \neq \varnothing$ and $\mathbf{K}^{d+1}_{\delta_1} - \mathbf{F}_{\delta} \neq \varnothing$ for any $\delta, \delta_1 > 0$;

$$\mathbf{K}^{d+1}_{\delta} \subsetneq \mathbf{F}^{\frac12}_{\delta}, \qquad \mathbf{F}_{\delta_1} \subsetneq \mathbf{F}^{\frac12}_{\delta} \text{ for } \delta = \sqrt{\delta_1} \text{ by Heinz inequality [He]}^1;$$

$$\left(b \in \mathbf{F}_{\delta_1} \text{ and } \mathsf{f} \in \mathbf{K}^{d+1}_{\delta_2}\right) \Longrightarrow \left(b + \mathsf{f} \in \mathbf{F}^{\frac12}_{\delta},\ \sqrt{\delta} = \sqrt[4]{\delta_1} + \sqrt{\delta_2}\right).$$

Denote

$$m_d := \pi^{\frac12}(2e)^{-\frac12} d^{\frac d2}(d-1)^{\frac{1-d}{2}}, \qquad c_p := pp'/4.$$

The main results of our paper:

Theorem 1 (L^p-theory)**.** *Let $d \geqslant 3$ and $b : \mathbb{R}^d \to \mathbb{C}^d$. Assume that $b \in \mathbf{F}^{\frac12}_{\delta}$, $m_d\delta < 1$. Then, for every $p \in \mathcal{I} := \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right)$, there exists a C_0-semigroup $e^{-t\Lambda_p(b)}$ in L^p such that*

(*i*) *The resolvent set $\rho(-\Lambda_p(b))$ contains the half-plane $\mathcal{O} := \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \geqslant \kappa_d\lambda_\delta\}$, $\kappa_d := \frac{d}{d-1}$, and the resolvent admits the representation:*

$$(\zeta + \Lambda_p(b))^{-1} = \Theta_p(\zeta, b), \quad \zeta \in \mathcal{O},$$

where

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - Q_p(1 + T_p)^{-1}G_p, \tag{1}$$

the operators $Q_p, G_p, T_p \in \mathcal{B}(L^p)$,

$$\|G_p\|_{p\to p} \leqslant C_1|\zeta|^{-\frac{1}{2p'}},\ \|Q_p\|_p \leqslant C_2|\zeta|^{-\frac12-\frac{1}{2p}},\ \|T_p\|_p \leqslant m_dc_p\delta < 1,$$

$$G_p \equiv G_p(\zeta, b) := b^{\frac1p}\cdot\nabla(\zeta - \Delta)^{-1}, \quad b^{\frac1p} := |b|^{\frac1p - 1}b,$$

Q_p, T_p are the extensions by continuity of densely defined (on $\mathcal{E} := \bigcup_{\epsilon>0} e^{-\epsilon|b|}L^p$) operators

$$Q_p|_{\mathcal{E}} \equiv Q_p(\zeta, b)|_{\mathcal{E}} := (\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}}, \qquad T_p|_{\mathcal{E}} \equiv T_p(\zeta, b)|_{\mathcal{E}} := b^{\frac1p}\cdot\nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}}.$$

(*ii*) *$e^{-t\Lambda_p(b)}$ is holomorphic: there is a constant C_p such that*

$$\|(\zeta + \Lambda_p(b))^{-1}\|_{p\to p} \leqslant C_p|\zeta|^{-1}, \quad \zeta \in \mathcal{O}.$$

(*iii*) *For each $1 \leqslant r < p < q$ and $\zeta \in \mathcal{O}$, define*

$$G_p(r) \equiv G_p(r, \zeta, b) := b^{\frac1p}\cdot\nabla(\zeta - \Delta)^{-\frac12-\frac{1}{2r}}, \quad G_p(r) \in \mathcal{B}(L^p),$$

$$Q_p(q) \equiv Q_p(q, \zeta, b) := (\zeta - \Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}} \text{ on } \mathcal{E}.$$

The extension of $Q_p(q)$ by continuity we denote again by $Q_p(q)$. Then

$$\Theta_p(\zeta, b) = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac12-\frac{1}{2q}}Q_p(q)(1 + T_p)^{-1}G_p(r)(\zeta - \Delta)^{-\frac{1}{2r'}}, \quad \zeta \in \mathcal{O};$$

$$\Theta_p(\zeta, b) \text{ extends by continuity to an operator in } \mathcal{B}\big(\mathcal{W}^{-\frac{1}{r'},p},\ \mathcal{W}^{1+\frac1q,p}\big).$$

[1]In fact, at least for $b \in L^{d,\infty}$ ($d \geqslant 3$), $\delta < \sqrt{\delta_1}$, see [KPS, Corollary 2.9].

(iv) $D\big(\Lambda_p(b)\big) \subset \mathcal{W}^{1+\frac{1}{q},p}$ *($q>p$). In particular, if* $m_d\delta < 4\frac{d-2}{(d-1)^2}$*, there exists* $p\in\mathcal{I}$*,* $p>d-1$*, so* $D\big(\Lambda_p(b)\big)\subset C^{0,\gamma}$*,* $\gamma<1-\frac{d-1}{p}$*.*

(v) Let $u\in D(\Lambda_p(b))$*. Then*

$$\begin{aligned}&\langle \Lambda_p(b)u, v\rangle = \langle u, -\Delta v\rangle + \langle b\cdot\nabla u, v\rangle, \qquad v\in C_c^\infty(\mathbb{R}^d);\\ &u\in\mathcal{W}^{2,1}_{\rm loc}.\end{aligned}$$

(vi) $e^{-t\Lambda_p(b_n)} \xrightarrow{s} e^{-t\Lambda_p(b)}$ *in* L^p*,* $\quad t>0$*,*

where $b_n := b$ *if* $|b|\leqslant n$*,* $b_n := n|b|^{-1}b$ *if* $|b|>n$*, and* $\Lambda_p(b_n) := -\Delta + b_n\cdot\nabla$*,* $D(\Lambda_p(b_n)) = W^{2,p}$*.*

(vii) If b *is real-valued, then* $e^{-t\Lambda_p(b)}$ *is positivity preserving.*

(viii) $\|e^{-t\Lambda_p(b)}\|_{p\to r} \leqslant c_{p,r} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}$*,* $0<t\leqslant 1$*,* $p<r$*.*

Let

$$\eta(x) := \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1,\\ 0, & \text{if } |x|\geqslant 1,\end{cases}$$

where c is adjusted to $\int_{\mathbb{R}^d}\eta(x)dx = 1$. Define the standard mollifier

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right), \quad \varepsilon>0, \ \ x\in\mathbb{R}^d.$$

Theorem 2 (C_∞-theory)**.** *Let* $d\geqslant 3$*. Assume that*

$$b:\mathbb{R}^d\to\mathbb{R}^d, \quad b\in\mathbf{F}_\delta^{\frac{1}{2}}, \quad m_d\delta < 4\frac{d-2}{(d-1)^2}.$$

Then for every $\tilde\delta>\delta$ *satisfying* $m_d\tilde\delta < 4\frac{d-2}{(d-1)^2}$ *there exist* $\varepsilon_n>0$*,* $\varepsilon_n\downarrow 0$*, such that*

$$\tilde b_n := \eta_{\varepsilon_n} * b_n \in C^\infty(\mathbb{R}^d,\mathbb{R}^d)\cap\mathbf{F}_{\tilde\delta}^{\frac{1}{2}}, \quad n=1,2,\dots,$$

and

(i)

$$e^{-t\Lambda_{C_\infty}(b)} := s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(\tilde b_n)}, \quad t>0,$$

determines a positivity preserving contraction C_0*-semigroup on* C_∞*,*
where $\Lambda_{C_\infty}(\tilde b_n) := -\Delta + \tilde b_n\cdot\nabla$*,* $D(\Lambda_{C_\infty}(\tilde b_n)) = C^2\cap C_\infty$*.*

(ii) (Strong Feller property) $(\mu+\Lambda_{C_\infty}(b))^{-1}[L^p\cap C_\infty]\subset C^{0,\alpha}$*,*
$\mu>0$*,* $p\in\big(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\big)$*,* $\alpha<1-\frac{d-1}{p}$*.*

(iii) The integral kernel $e^{-t\Lambda_{C_\infty}(b)}(x,y)$ $(x,y\in\mathbb{R}^d)$ *of* $e^{-t\Lambda_{C_\infty}(b)}$ *determines the (sub-Markov) transition probability function of a strong Feller process.*

Remarks. 1. Theorem 1 allows us to move the problem of convergence in C_∞ (in Theorem 2*(i)*) to L^p, a space having much weaker topology (locally).

2. In place of $\tilde b_n$ in Theorem 2, one can take (without changing the proof)

$$\hat b_n := \eta_{\varepsilon_n} * (\mathbf{1}_n b) \in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)\cap\mathbf{F}_{\tilde\delta}^{\frac{1}{2}}, \quad m_d\tilde\delta < 4\frac{d-2}{(d-1)^2}, \quad n=1,2,\dots,$$

where $\mathbf{1}_n$ is the characteristic function of $\{x \in \mathbb{R}^d : |b(x)| \leqslant m_n, |x| \leqslant n\}$, $m_n < n$ for all n, $m_n \uparrow \infty$ as $n \uparrow \infty$.

3. **On the existing results prior to our work.** First, it had been known for a long time, see [KS], that, for $b : \mathbb{R}^d \to \mathbb{R}^d$, $d \geqslant 3$, and $b \in \mathbf{F}_\delta$,

(i) *(The basic fact) $D(\Lambda_p(b)) \subset W^{1,jp}$ for every $p \in (d-2, 2/\sqrt{\delta})$, $j = \frac{d}{d-2}$,* provided that $0 < \delta < \min\{1, (\frac{2}{d-2})^2\}$.

(ii) *If, in addition to the assumptions in (i), $|b| \in L^2 + L^\infty$, then*

$$s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(b_n)}$$

exists uniformly in each finite interval of $t \geqslant 0$, and hence determines a strongly Feller semigroup on C_∞.

Remark. The additional (to $|b| \in L^2_{\text{loc}}$) assumption $|b| \in L^2 + L^\infty$ in (ii) was removed in [Ki].

Theorem 3 (Yu. A. Semenov)**.** *Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $d \geqslant 3$.*

a) *If $b \in \mathbf{K}_\delta^{d+1}$, $m_d\delta < 1$, then, for each $p \in [1, \infty)$, s-L^p-$\lim_n e^{-t\Lambda_p(b_n)}$ exists uniformly on each finite interval of $t \geqslant 0$, and hence determines a C_0-semigroup $e^{-t\Lambda_p(b)}$.*

$e^{-t\Lambda_p(b)}$ is a quasi-bounded positivity preserving L^∞-contraction C_0- semigroup;

$$\|e^{-t\Lambda_r(b)}\|_{r\to q} \leqslant c_{d,\delta}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})} \text{ for all } 0 < t \leqslant 1,\ 1 \leqslant r < q \leqslant \infty;$$

The resolvent set $\rho(-\Lambda_p(b))$ contains the half-plane $\mathcal{O}$,

$$(\zeta + \Lambda_p(b))^{-1} = \Theta_p(\zeta, b),\ \ \zeta \in \mathcal{O},$$

$$\begin{aligned}
&\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2}} S_p(1 + T_p)^{-1} G_p,\\
&S_p := (\zeta - \Delta)^{-\frac{1}{2}} |b|^{\frac{1}{p'}}, \quad G_p := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}, \quad T_p := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}};\\
&\Theta_p(\zeta, b) \in \mathcal{B}(L^p, \mathcal{W}^{1,p});\\
&D(\Lambda_p(b)) \subset \mathcal{W}^{1,\,p}. \textit{ In particular, for } p > d,\ D(\Lambda_p(b)) \subset C^{0,\,\alpha},\ \alpha = 1 - \frac{d}{p};\\
&\langle \Lambda_p(b) f, g\rangle = \langle \nabla f, \nabla g\rangle + \langle b \cdot \nabla f, g\rangle, \qquad f \in D(\Lambda_p(b)),\ g \in C_c^\infty(\mathbb{R}^d).
\end{aligned}$$

b) *If $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, then, for each $p \in [2, \infty)$, s-L^p-$\lim_n e^{-t\Lambda_p(b_n)}$ exists uniformly on each finite interval of $t \geqslant 0$, and hence determines a C_0-semigroup $e^{-t\Lambda_p(b)}$.*

$e^{-t\Lambda_p(b)}$ is a quasi-bounded positivity preserving L^∞-contraction C_0- semigroup.

$\|e^{-t\Lambda_r(b)}\|_{r\to q} \leqslant c_{d,\delta}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}$ *for all $0 < t \leqslant 1$, $2 \leqslant r < q \leqslant \infty$.*

$D(\Lambda_2(b)) \subset W^{\frac{3}{2},2}$.

$\langle \Lambda_2(b) f, g\rangle = \langle \nabla f, \nabla g\rangle + \langle b \cdot \nabla f, g\rangle, \qquad f \in D(\Lambda_2(b)),\ g \in C_c^\infty(\mathbb{R}^d)$.

We outline the proof of Semenov's results.

Proof. **a**) Indeed, for all ζ with Re $\zeta > 0$,

$$|\nabla(\zeta - \Delta)^{-1}(x, y)| \leqslant m_d(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}}(x, y) \text{ pointwise on } \mathbb{R}^d \times \mathbb{R}^d$$

(see (A.1) in the Appendix). Therefore, for $b \in \mathbf{K}_\delta^{d+1}$,

$$\|b\cdot\nabla(\zeta-\Delta)^{-1}\|_{1\to 1} \leqslant m_d\delta, \qquad \mathrm{Re}\ \zeta \geqslant \kappa_d\lambda,$$

and so by the Miyadera perturbation theorem, the operator $-\Lambda_1(b) := \Delta - b\cdot\nabla$ of domain $D(\Lambda_1(b)) = \mathcal{W}^{2,1}$ is the generator of a quasi-bounded C_0 semigroup on L^1 *whenever* $m_d\delta < 1$.

Clearly $b_n \in \mathbf{K}_\delta^{d+1}$, $\|b_n\cdot\nabla(\zeta-\Delta)^{-1}\|_{1\to 1} \leqslant m_d\delta$, and, for $m_d\delta < 1$ and every $f \in D(\Lambda_1(b))$, $\Lambda_1(b_n)f \overset{s}{\to} \Lambda_1(b)f$ by the Dominated Convergence Theorem. (See, if needed, (A.0).) The latter easily implies the strong resolvent and the semigroup convergence of $\Lambda_1(b_n)$ to $\Lambda_1(b)$.

Then, for each $n = 1, 2, \dots$, the semigroups $e^{-t\Lambda_1(b_n)}, t > 0$, are positivity preserving L^∞-contractions, and so is $e^{-t\Lambda_1(b)}$. The bounds

$$\|e^{-t\Lambda_1(b)}\|_{1\to 1} \leqslant Me^{t\omega},\ \omega = \kappa_d\lambda,\ \text{and}\ \|e^{-t\Lambda_1(b)}f\|_\infty \leqslant \|f\|_\infty,\ f \in L^1\cap L^\infty,$$

yield via the Riesz interpolation theorem

$$\|e^{-t\Lambda_1(b)}f\|_p \leqslant M^{1/p}e^{t\omega/p}\|f\|_p,\ \ f \in L^1\cap L^\infty.$$

Therefore, we obtain a family $\{e^{-t\Lambda_p(b)}\}_{1\leqslant p<\infty}$ of consistent C_0-semigroups by setting $e^{-t\Lambda_p(b)} :=$ the extension by continuity in L^p of $e^{-t\Lambda_1(b)} \upharpoonright L^1\cap L^\infty$.

Next, for each $p \in [1,\infty)$ and all $f \in \mathcal{E} := \bigcup_{\epsilon>0} e^{-\epsilon|b|}L^p$, the inequality

$$\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}f\|_p \leqslant \delta\|f\|_p$$

as well as inequality

$$\|(|b|+\sqrt{\lambda})^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}(|b|+\sqrt{\lambda})^{\frac{1}{p'}}f\|_p \leqslant (1+\delta)\|f\|_p$$

follow from the very definition of $\mathbf{K}_\delta^{d+1}$ using Hölder's inequality. Note that the last inequality clearly implies that

$$\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}\|_{p\to p} \leqslant (1+\delta)\lambda^{-\frac{1}{2p'}},$$

and the first inequality implies that, for every $\zeta \in \mathcal{O}$, $p \in [1,\infty)$ and all $f \in \mathcal{E}$,

$$\|b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}\ |b|^{\frac{1}{p'}}f\|_p \leqslant m_d\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\|_p \leqslant m_d\delta\|f\|_p.$$

Now, for every $p \in [1,\infty)$ and $\zeta \in \mathcal{O}$, we define operators G_p, S_p, T_p acting in L^p by

$$G_p = b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1},\ S_p = (\zeta-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}},\ T_p = b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}\ |b|^{\frac{1}{p'}}.$$

It is seen that G_p is bounded:

$$\|G_p\|_{p\to p} \leqslant m_d\|b^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}\|_{p\to p} \leqslant m_d(1+\delta)\lambda^{-\frac{1}{2p'}}.$$

S_p and T_p are densely defined (on $\mathcal{E}$) and, for all $f \in \mathcal{E}$,

$$\|S_pf\|_p \leqslant (1+\delta)^{-1}\lambda^{-\frac{1}{2p}}\|f\|_p \text{ and } \|T_pf\|_p \leqslant m_d\delta\|f\|_p.$$

Their extensions by continuity we denote again by S_p, T_p.

Next, we define an operator function $\Theta_p(\zeta, b)$ in L^p by

$$\Theta_p(\zeta,b) := (\zeta-\Delta)^{-1} - (\zeta-\Delta)^{-\frac{1}{2}}S_p\ (1+T_p)^{-1}\ G_p \qquad \zeta \in \mathcal{O}.$$

Obviously

$$\Theta_p(\zeta,b)\in\mathcal{B}(L^p) \text{ and } \Theta_p(\zeta,b)\in\mathcal{B}(L^p,W^{1,p}).$$

It is also seen that

$$(\zeta+\Lambda_1(b))^{-1}=\Theta_1(\zeta,b),\ \ (\zeta+\Lambda_p(b))^{-1}\mid L^1\cap L^p=\Theta_p(\zeta,b)\mid L^1\cap L^p, \text{ and so}$$

$$(\zeta+\Lambda_p(b))^{-1}=\Theta_p(\zeta,b),\qquad \zeta\in\mathcal{O}.$$

The latter implies that $D(\Lambda_p(b))\subset W^{1,p}$, for all $p\in[1,\infty)$. The main assertion is proved.

b) Let $b\in\mathbf{F}_\delta^{\frac12}$, $\delta<1$. Define $H=|b|^{\frac12}(\zeta-\Delta)^{-\frac14}$, $S=b^{\frac12}\cdot\nabla(\zeta-\Delta)^{-\frac34}$ and

$$\begin{aligned}\Theta_2(\zeta,b):=&(\zeta-\Delta)^{-\frac34}(1+H^*S)^{-1}(\zeta-\Delta)^{-\frac14} \qquad (*)\\ =&(\zeta-\Delta)^{-1}-(\zeta-\Delta)^{-\frac34}H^*(1+SH^*)^{-1}S((\zeta-\Delta)^{-\frac14},\ \ \mathrm{Re}\,\zeta\geqslant\lambda.\end{aligned}$$

We have

$$\|H^*S\|_{2\to2}\leqslant\|H\|_{2\to2}\|S\|_{2\to2}\leqslant\|H\|^2_{2\to2}\|\nabla(\zeta-\Delta)^{-\frac12}\|_{2\to2}\leqslant\delta \text{ and } \|\Theta_2(\zeta,b)\|_{2\to2}\leqslant(1-\delta)^{-1}|\zeta|^{-1}.$$

Note that $D(\Lambda_2(b_n))=W^{2,2}$ and, for all Re $\zeta\geqslant\lambda$, by the first representation of $\Theta_2(\zeta,b_n)$,

$$\Theta_2(\zeta,b_n)^{-1}|W^{2,2}=(\zeta+\Lambda_2(b_n))|W^{2,2},\quad \Theta_2(\zeta,b_n)=(\zeta+\Lambda_2(b_n))^{-1},$$

$$\zeta\Theta_2(\zeta,b_n)\xrightarrow{s}1 \text{ as } \zeta\uparrow\infty \text{ by the second representation of } \Theta_2(\zeta,b_n).$$

Therefore, $\Theta_2(\zeta,b_n)$ *is the resolvent of* $-\Lambda_2(b_n)$.

Since $\|\Theta_2(\zeta,b_n)\|_{2\to2}\leqslant(1-\delta)^{-1}|\zeta|^{-1}$, the semigroups $e^{-t\Lambda_2(b_n)}$ are holomorphic and equi-bounded.

Finally, it is seen that $\Theta_2(\zeta,b_n)\xrightarrow{s}\Theta_2(\zeta,b)$ in L^2 on Re $\zeta\geqslant\lambda$, and $\mu\,\Theta_2(\mu,b_n)\xrightarrow{s}1$ in L^2 as $\mu\uparrow\infty$ uniformly in n. Therefore, by the Trotter approximation theorem s-L^2-$\lim_n e^{-t\Lambda_2(b_n)}$ exists and determines a C_0-semigroup in L^2. It is also clear that this semigroup is holomorphic and L^∞-contractive. □

4. **Comments.** 1. The fact that $b:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $\mathbf{K}_\delta^{d+1}$ or $\mathbf{F}_\delta$ allows us to construct operator realizations of the formal differential operator $-\Delta+b\cdot\nabla$ as (minus) generators of strongly continuous semigroups in L^p for some or all $p\in[1,\infty)$, C_∞ and/or C_b, by means of general tools of the standard perturbation theory (e.g. theorems of Miyadera [Vo] or Phillips [Ph], respectively).

2. Concerning the class $\mathbf{F}_\delta^{\frac12}$ one can not appeal to the standard perturbation theory (in contrast to $\mathbf{K}_\delta^{d+1}$ and $\mathbf{F}_\delta$) in order to properly characterize the domain of the generator $\Lambda_p(b)$. Indeed, the arguments in [Se, p. 413-416] (repeated above in the proof of Theorem 3b) say nothing about $\mathcal{W}^{\alpha,p}$-smoothness of $D(\Lambda_p(b))$ for $p\neq2$. These arguments rely on $(*)$ that implies that $\Theta_2(\zeta,b)$ is invertible. The natural analogue of $(*)$ in L^p is valid only for considerably smaller class of vector fields: $|b|\in L^{d,\infty}$, although leading to a slightly better smoothness result: $D(\Lambda_p(b))\subset\mathcal{W}^{1+\frac1p,p}$, cf. Theorem 1($iv$).

3. Theorem 3 is a special case of our Theorem 1. Indeed, the constraints on p and δ in Theorem 1 come solely from the estimate on $\|T_p\|_{p\to p}$. Now, if $b\in\mathbf{F}_\delta^{\frac12}$, $\delta<1$, then

$$\|T_2\|_{2\to2}\leqslant\|H\|_{2\to2}\|H^*\|_{2\to2}\|\nabla(\zeta-\Delta)^{-\frac12}\|_{2\to2}\leqslant\|H\|^2_{2\to2}\leqslant\delta<1.$$

And if $b \in \mathbf{K}^{d+1}_\delta$, $m_d\delta < 1$, then $\|T_p\|_{p\to p} < 1$ for *all* $p \in [1,\infty)$, so that the interval $\mathcal{I} \ni p$ transforms into $[1,\infty)$, and a possible causal dependence of the properties of $D(\Lambda_p(b))$ on δ gets lost. The latter indicates the smallness of $\mathbf{K}^{d+1}_\delta$ as a subclass of $\mathbf{F}^{\frac{1}{2}}_\delta$.

4. Both proofs of Theorem 1 and Theorem 3 are based on similar operator-valued functions, although the arguments involved differ considerably.

5. Note that for $b \in \mathbf{K}^{d+1}_\delta$, $u \in D(\Lambda_1(b))$ satisfies $u \in \mathcal{W}^{2,1}$, for $b \in \mathbf{F}_\delta$ $u \in \mathcal{W}^{2,2}_{\rm loc}$ [KS], while in Theorem 1(v) $u \in D(\Lambda_p(b))$, $p \in \mathcal{I}$, satisfies $u \in \mathcal{W}^{2,1}_{\rm loc}$.

6. Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $b \in \mathbf{F}^{\frac{1}{2}}_\delta$, $m_d\delta < 1$. Theorem 1(i) and the argument in the proof of Theorem 3a (using the Riesz interpolation theorem) yields a consistent family of C_0-semigroups $e^{-t\Lambda_p(b)}$ on L^p, for all $p \in (\frac{2}{1+\sqrt{1-m_d\delta}}, \infty)$.

7. The author considers the assertion (iv) of Theorem 1 (the $\mathcal{W}^{1+\frac{1}{q},p}$-smoothness) as the main result of the paper. Theorem 1, compared to [KS] and Theorem 3a, covers the larger class of vector fields, and at the same time establishes stronger smoothness properties of $D(\Lambda_p(b))$: $D(\Lambda_p(b)) \subset \mathcal{W}^{1+\frac{1}{q},p}$, $p \in \mathcal{I}$ $(q > p)$, while in [KS] $D(\Lambda_p(b)) \subset W^{1,jp}$, $jp \in (d, 2j/\sqrt{\delta})$, and in Theorem 3a $D(\Lambda_p(b)) \subset \mathcal{W}^{1,p}$, $p \in [1,\infty)$.

8. The C_∞-theory of operator $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}^{\frac{1}{2}}_\delta$ (Theorem 2) follows almost automatically from the L^p-theory (Theorem 1) (with $p > d-1$), in contrast to [KS], where the C_∞-theory is obtained from the L^p-theory by running a specifically tailored Moser-type iterative procedure (see also [Ki]).

9. To the author's knowledge, Theorem 1 and Theorem 2 are the first results where b can combine different kinds of singularities, e.g. $(|x|-1)^{-\beta}$, $\beta < 1$, and $|x|^{-1}$ (originally, the main motivation for this work).

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1. Proof of Theorem 1

The method of proof. We start with operator-valued function

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - Q_p(1+T_p)^{-1}G_p, \quad \zeta \in \mathcal{O},$$

defined in L^p for each p from the interval

$$\mathcal{I} := \left]\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right[, \quad m_d\delta < 1,$$

and step by step prove that, for $n = 1, 2, \dots,$

$$\|\Theta_p(\zeta, b_n)\|_{p\to p},\ \|\Theta_p(\zeta, b)\|_{p\to p} \leqslant c|\zeta|^{-1};$$

$$\Theta_p(\zeta, b_n) \text{ is a pseudo-resolvent;}$$

$$\Theta_p(\zeta, b_n) \text{ coincides with the resolvent } R(\zeta, -\Lambda_p(b_n)) = (\zeta + \Lambda_p(b_n))^{-1} \text{ on } \mathcal{O};$$

$$\Theta_p(\zeta, b_n) \overset{s}{\to} \Theta_p(\zeta, b) \text{ in } L^p \text{ as } n \uparrow \infty;$$

$$\mu\,\Theta_p(\mu,b_n) \overset{s}{\to} 1 \text{ as } \mu\uparrow\infty \text{ in } L^p \text{ uniformly in } n.$$

All this combined leads to the conclusion: for each $p\in\mathcal{I}$ there is a holomorphic semigroup $e^{-t\Lambda_p(b)}$ in L^p such that the resolvent $R(\zeta,-\Lambda_p(b))$ on $\zeta\in\mathcal{O}$ has the representation $\Theta_p(\zeta,b)$;

$\Theta_p(\zeta,b)$ can be written as ABC, where $C\in\mathcal{B}(\mathcal{W}^{-\frac{1}{r'},p},L^p)$, $B\in\mathcal{B}(L^p)$, $A\in\mathcal{B}(L^p,\mathcal{W}^{1+\frac{1}{q},p})$, $r<p<q$, $r'=r/(r-1)$.

Propositions 1-4 below constitute the core of the proof of Theorem 1.

Proposition 1. *Let $p\in\mathcal{I}$.*

(i) *For every $1\leqslant r<p<q\leqslant\infty$ and $\zeta\in\mathcal{O}$ $(=\{\zeta\in\mathbb{C}:\operatorname{Re}\zeta\geqslant\kappa_d\lambda\},\ \lambda=\lambda_\delta)$ define operators on L^p*

$$Q_p(q)=(\zeta-\Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}},\quad G_p(r)=b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-\frac12-\frac{1}{2r}},\quad T_p=b^{\frac1p}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}}.$$

Then $G_p(r)$ is bounded: $\|G_p(r)\|_{p\to p}\leqslant K_{1,r}$. $Q_p(q)$ and T_p are densely defined (on $\mathcal{E}$), and for all $f\in\mathcal{E}$,

$$\|Q_p(q)f\|_p\leqslant K_{2,q}\|f\|_p,$$

$$\|T_pf\|_p\leqslant m_dc_p\delta\|f\|_p,\quad m_dc_p\delta<1,\quad c_p=pp'/4. \tag{2}$$

Their extensions by continuity we denote again by $Q_p(q)$, T_p.

(ii) *Set $G_p=b^{\frac1p}\cdot\nabla(\zeta-\Delta)^{-1}$, $Q_p=(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}}$, $P_p=|b|^{\frac1p}(\zeta-\Delta)^{-1}$. The operator Q_p is densely defined on $\mathcal{E}$. There exist constants C_i, $i=1,2,3$, such that*

$$\|G_p\|_{p\to p}\leqslant C_1|\zeta|^{-\frac{1}{2p'}},\quad \|P_p\|_{p\to p}\leqslant C_3|\zeta|^{-\frac12-\frac{1}{2p'}},\quad \|Q_pf\|_p\leqslant C_2|\zeta|^{-\frac12-\frac{1}{2p}}\|f\|_p\quad(f\in\mathcal{E}),\quad \zeta\in\mathcal{O}. \tag{3}$$

The extension of Q_p by continuity we denote again by Q_p.

(iii) $\|\Theta_p(\zeta,b_n)\|_{p\to p}\leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$.

Remark. The proof of Proposition 1 uses ideas in [BS], [LS], and appeals to the L^p-inequalities between operator $(\lambda-\Delta)^{\frac12}$ and "potential" $|b|$.

Proof. (i) Let $r\in(1,\infty)$. Then

(**a**) $\mu\geqslant\lambda\ \Rightarrow\ \||b|^{\frac1r}(\mu-\Delta)^{-\frac12}\|_{r\to r}\leqslant C_{r,\delta}\mu^{-\frac{1}{2r'}}$, $C_{r,\delta}=(c_r\delta)^{\frac1r}$, $c_r=rr'/4$.

Indeed, define in L^2 $A=(\mu-\Delta)^{\frac12}$, $D(A)=W^{1,2}$. Then $-A$ is a symmetric Markov generator. Therefore (see e.g. [LS]), for any $r\in(1,\infty)$,

$$0\leqslant u\in D(A_r)\Rightarrow v:=u^{\frac r2}\in D(A^{\frac12})\text{ and }c_r^{-1}\|A^{\frac12}v\|_2^2\leqslant\langle A_ru,u^{r-1}\rangle.$$

Now let u be the solution of $A_ru=|f|$, $f\in L^r$. Note that $A\geqslant\sqrt{\mu}$ and $\|u\|_r\leqslant\mu^{-\frac12}\|f\|_r$.

Since $b\in\mathbf{F}_\delta^{\frac12}$, we have

$$(c_r\delta)^{-1}\||b|^{\frac12}v\|_2^2\leqslant\langle A_ru,u^{r-1}\rangle,$$

and so $\||b|^{\frac1r}u\|_r^r\leqslant c_r\delta\|f\|_r\|u\|_r^{r-1}$, $\||b|^{\frac1r}A^{-1}|f|\|_r^r\leqslant c_r\delta\mu^{-\frac{r-1}{2}}\|f\|_r^r$. (**a**) is proved.

(**b**) $\mu\geqslant\lambda\ \Rightarrow\ \||b|^{\frac1r}(\mu-\Delta)^{-\frac12}|b|^{\frac{1}{r'}}f\|_r\leqslant c_r\delta\|f\|_r$, $f\in\mathcal{E}$.

Indeed, let u be the solution of $Au = |b|^{\frac{1}{r'}}|f|$, $f \in \mathcal{E}$. Then, obviously,

$$\||b|^{\frac{1}{r}}u\|_r^r \leqslant c_r\delta\|f\|_r\||b|^{\frac{1}{r}}u\|_r^{r-1},$$

or $\||b|^{\frac{1}{r}}u\|_r \leqslant c_r\delta\|f\|_r$. (**b**) is proved.

(**c**) $\mu \geqslant \lambda \;\Rightarrow\; \|(\mu-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{r'}}f\|_r \leqslant C_{r',\delta}\,\mu^{-\frac{1}{2r}}\|f\|_r$, $f \in \mathcal{E}$.

Indeed, (**c**) follows from (**a**) by duality.

Let us prove (2). Let $\zeta \in \mathcal{O}$. Using (A.1) + (**b**) with $r = p \in \mathcal{I}$, $\mu = \lambda$, we obtain:

$$\|T_pf\|_p \leqslant m_d\|b^{\frac{1}{p}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\|_p \leqslant m_dc_p\delta\|f\|_p, \quad f \in \mathcal{E}.$$

$m_dc_p\delta < 1$ since $p \in \mathcal{I}$.

Next, we estimate $\|Q_p(q)\|_{p\to p}$, $\|G_p(r)\|_{p\to p}$. Let $\mathrm{Re}\;\zeta \geqslant \lambda$, $p < q$. We obtain:

$$\begin{aligned}\|Q_p(q)f\|_p \leqslant\; & \|(\mathrm{Re}\;\zeta-\Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}|f|\|_p\\ &\leqslant \|(\lambda-\Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}|f|\|_p\\ &\quad\text{(here we are using (A.5))}\\ &\quad\leqslant k_q\int_0^\infty t^{-1+\frac{1}{2q}}\|(t+\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\|_p dt\\ &\quad\text{(here we are using (}\mathbf{c}\text{) with } r = p\in\mathcal{I},\ \mu = t+\lambda)\\ &\quad\leqslant\; k_qC_{p',\delta}\int_0^\infty t^{-1+\frac{1}{2q}}(t+\lambda)^{-\frac{1}{2p}}dt\;\|f\|_p = K_{2,q}\|f\|_p, \quad f\in\mathcal{E}.\end{aligned}$$

Let $\zeta \in \mathcal{O}$, $1 \leqslant r < p$. Using (A.2), we obtain:

$$\begin{aligned}\|G_p(r)f\|_p &\leqslant m_{r,d}\||b|^{\frac{1}{p}}(\kappa_d^{-1}\mathrm{Re}\;\zeta-\Delta)^{-\frac{1}{2r}}|f|\|_p\\ &\leqslant m_{r,d}\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2r}}|f|\|_p\\ &\text{(here we are using (A.5) with } q' := r)\\ &\leqslant m_{r,d}k_{r'}\int_0^\infty t^{-1+\frac{1}{2r'}}\||b|^{\frac{1}{p}}(t+\lambda-\Delta)^{-\frac{1}{2}}|f|\|_p dt\\ &\text{(here we are using (}\mathbf{a}\text{) with } r = p\in\mathcal{I},\ \mu = t+\lambda)\\ &\leqslant m_{r,d}k_{r'}C_{p,\delta}\int_0^\infty t^{-1+\frac{1}{2r'}}(t+\lambda)^{-\frac{1}{2p'}}dt\;\|f\|_p = K_{1,r}\|f\|_p, \quad f\in\mathcal{E}.\end{aligned}$$

The proof of (i) is completed.

(ii) Let $\mathrm{Re}\;\zeta \geqslant \lambda$. Using (A.4) + (**c**) with $r = p \in \mathcal{I}$, $\mu = |\zeta|$, we obtain:

$$\|Q_p(2\zeta,b)f\|_p \leqslant C_{p',\delta}2^{-\frac{d}{4}+\frac{1}{4}}|\zeta|^{-\frac{1}{2p}}\|f\|_p, \quad f\in\mathcal{E}.$$

Now, using the identity $(\zeta-\Delta)^{-1} = \big(1+\zeta(\zeta-\Delta)^{-1}\big)(2\zeta-\Delta)^{-1}$, we obtain:

$$\begin{aligned}\|Q_p(\zeta,b)f\|_p &\leqslant \|1+\zeta(\zeta-\Delta)^{-1}\|_{p\to p}\|Q_p(2\zeta,b)f\|_p\\ &\leqslant 2C_{p',\delta}2^{-\frac{d}{4}+\frac{1}{4}}|\zeta|^{-\frac{1}{2p}}\|f|_p\\ &= C_2|\zeta|^{-\frac{1}{2p}}\|f\|_p, \quad f\in\mathcal{E}.\end{aligned}$$

Let Re $\zeta \geqslant \lambda$. Using (A.4), we obtain:

$$\begin{aligned}\|P_p(2\zeta,b)\|_{p\to p} &\leqslant \||b|^{\frac{1}{p}}(2\zeta-\Delta)^{-\frac{1}{2}}\|_{p\to p}\|(2\zeta-\Delta)^{-\frac{1}{2}}\|_{p\to p} \\ &\leqslant 2^{\frac{d}{4}+\frac{1}{4}}\||b|^{\frac{1}{p}}(|\zeta|-\Delta)^{-\frac{1}{2}}\|_{p\to p}(2|\zeta|)^{-\frac{1}{2}} \\ &\quad \text{(here we are using } (\mathbf{a}) \text{ with } r=p\in\mathcal{I},\ \mu=|\zeta|) \\ &\leqslant C_{p,\delta}2^{\frac{d}{4}+\frac{1}{4}}2^{-\frac{1}{2}}|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}.\end{aligned}$$

Now, using the identity $(\zeta-\Delta)^{-1}=(2\zeta-\Delta)^{-1}\big(1-\zeta(\zeta-\Delta)^{-1}\big)$, we obtain:

$$\begin{aligned}\|P_p(\zeta,b)\|_{p\to p} &\leqslant 2C_{p,\delta}2^{\frac{d}{4}+\frac{1}{4}}2^{-\frac{1}{2}}|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}} \\ &= C_3|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}.\end{aligned}$$

Let $\zeta\in\mathcal{O}$. Using (A.3) + $(\mathbf{a})$ with $r=p\in\mathcal{I}$, $\mu=|\zeta|$, we obtain:

$$\|G_p(2\kappa_d\zeta,b)\|_{p\to p}\leqslant m_dC_{p,\delta}2^{\frac{d}{4}}|\zeta|^{-\frac{1}{2p'}}.$$

Now, using the identity $(\zeta-\Delta)^{-1}=(2\kappa_d\zeta-\Delta)^{-1}\big(1-(2\kappa_d-1)\zeta(\zeta-\Delta)^{-1}\big)$, we obtain:

$$\begin{aligned}\|G_p(\zeta,b)\|_{p\to p} &\leqslant 2\kappa_dm_dC_{p,\delta}2^{\frac{d}{4}}|\zeta|^{-\frac{1}{2p'}} \\ &= C_1|\zeta|^{-\frac{1}{2p'}}.\end{aligned}$$

The proof of (ii) is completed.

(iii) By the definition of $\Theta_p(\zeta,b)$, see (1), for every $\zeta\in\mathcal{O}$,

$$\begin{aligned}\|\Theta_p(\zeta,b)\|_{p\to p} &\leqslant \|(\zeta-\Delta)^{-1}\|_{p\to p}+\|Q_p\|_{p\to p}\|(1+T_p)^{-1}\|_{p\to p}\|G_p\|_{p\to p} \\ &\quad \text{(here we are using (2), (3))} \\ &\leqslant |\zeta|^{-1}+C_2|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}(1-m_dc_p\delta)^{-1}C_1|\zeta|^{-\frac{1}{2p}} \\ &\leqslant C_p|\zeta|^{-1}, \quad C_p:=1+C_1C_2(1-m_dc_p\delta)^{-1}.\end{aligned}$$

□

Remark. Since $|b_n|\leqslant|b|$ a.e., Proposition 1 is valid for b_n, $n=1,2,\dots$, with the same constants.

Proposition 2. *For every $p\in\mathcal{I}$, and $n=1,2,\dots$, the operator-valued function $\Theta_p(\zeta,b_n)$ is a pseudo-resolvent on $\mathcal{O}$, i.e.*

$$\Theta_p(\zeta,b_n)-\Theta_p(\eta,b_n)=(\eta-\zeta)\Theta_p(\zeta,b_n)\Theta_p(\eta,b_n),\quad \zeta,\eta\in\mathcal{O}.$$

Proof. Define $S_\zeta^k:=(-1)^k(\zeta-\Delta)^{-1}b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots b_n\cdot\nabla(\zeta-\Delta)^{-1}$, $k:=\#\ b_n$'s. Obviously,

$$\begin{aligned}\Theta_p(\zeta,b_n) &:= (\zeta-\Delta)^{-1}-Q\left(1+T\right)^{-1}G \\ &= (\zeta-\Delta)^{-1}-Q\sum_{k=0}^{\infty}(-1)^kT^k\,G=\sum_{k=0}^{\infty}S_\zeta^k \quad \text{(absolutely convergent in } L^p),\end{aligned}$$

$$\Theta_p(\zeta, b_n)\Theta_p(\eta, b_n) = \sum_{\ell=0}^{\infty}\sum_{i=0}^{\ell} S_\zeta^i S_\eta^{\ell-i}, \quad \zeta, \eta \in \mathcal{O}. \tag{4}$$

Define

$$\begin{aligned} I_{j,m}^k(\zeta,\eta) := (\zeta-\Delta)^{-1} b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots b_n\cdot\nabla(\zeta-\Delta)^{-1} \\ b_n\cdot\nabla(\eta-\Delta)^{-1} b_n\cdot\nabla(\eta-\Delta)^{-1}\dots b_n\cdot\nabla(\eta-\Delta)^{-1}, \\ j := \#\zeta\text{'s}, \quad m := \#\eta\text{'s}, \quad k := \#b_n\text{'s}. \end{aligned}$$

Substituting the identity $(\zeta-\Delta)^{-1}(\eta-\Delta)^{-1} = (\eta-\zeta)^{-1}\big((\zeta-\Delta)^{-1} - (\eta-\Delta)^{-1}\big)$ inside the product

$$\begin{aligned} &S_\zeta^k S_\eta^j \\ &= (-1)^{k+j}(\zeta-\Delta)^{-1} b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots\ b_n\cdot\nabla \underbrace{(\zeta-\Delta)^{-1}(\eta-\Delta)^{-1}}_{(\eta-\zeta)^{-1}((\zeta-\Delta)^{-1}-(\eta-\Delta)^{-1})} b_n\cdot\nabla(\eta-\Delta)^{-1}\dots\ b_n\cdot\nabla(\eta-\Delta)^{-1}, \end{aligned}$$

we obtain $S_\zeta^k S_\eta^j = (\eta-\zeta)^{-1}(-1)^{k+j}\big[I_{k+1,j}^{k+j} - I_{k,j+1}^{k+j}\big]$. Therefore,

$$\begin{aligned} \sum_{i=0}^{\ell} S_\zeta^i S_\eta^{\ell-i} &= (\eta-\zeta)^{-1}(-1)^\ell \Big[I_{1,\ell}^\ell - I_{0,\ell+1}^\ell + I_{2,\ell-1}^\ell - I_{1,\ell}^\ell + \dots + I_{\ell+1,0}^\ell - I_{\ell,1}^\ell\Big] \\ &= (\eta-\zeta)^{-1}(-1)^\ell\big(I_{\ell+1,0}^\ell - I_{0,\ell+1}^\ell\big). \end{aligned}$$

Substituting the last identity in the right-hand side of (4), we obtain

$$\Theta_p(\zeta,b_n)\Theta(\eta,b_n) = (\eta-\zeta)^{-1}\sum_{\ell=0}^{\infty}(-1)^\ell\big(I_{\ell+1,0}^\ell - I_{0,\ell+1}^\ell\big) = (\eta-\zeta)^{-1}\big(\Theta_p(\zeta,b_n)h - \Theta_p(\eta,b_n)\big).$$

□

Proposition 3. *For every $p \in \mathcal{I}$, and $n = 1, 2, \dots,$*

(i) $\mu\Theta_p(\mu, b_n) \overset{s}{\to} 1$ *in L^p as $\mu \uparrow \infty$,*

(ii) $\|\Theta_p(\zeta, b_n)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$, *for a constant C_p independent of n.*

Proof. Proof of (i). Put $\Theta_p \equiv \Theta_p(\mu,b_n)$, $Q_p \equiv Q_p(\mu,b_n)$, $T_p \equiv T_p(\mu,b_n)$, $P_p \equiv P_p(\mu,b_n)$. Since $\mu(\mu-\Delta)^{-1} \overset{s}{\to} 1$, it suffices to show that $\mu\Theta_p - \mu(\mu-\Delta)^{-1} \overset{s}{\to} 0$ in L^p. Since $\Theta_p \in \mathcal{B}(L^p)$, and C_c^∞ is dense in L^p, it suffices to show that $\mu\Theta_p h - \mu(\mu-\Delta)^{-1}h \to 0$ in L^p for every $h \in C_c^\infty$. Write

$$\Theta_p h - (\mu-\Delta)^{-1}h = -Q_p\big(1+T_p\big)^{-1} b_n^{\frac{1}{p}}\cdot(\mu-\Delta)^{-1}\nabla h.$$

By (2), $\quad \|\big(1+T_p\big)^{-1}\|_{p\to p} \leqslant \dfrac{1}{1-\|T_p\|_{p\to p}} \leqslant \dfrac{1}{1-m_d c_p \delta} < \infty$, $\quad$ by (3), $\quad \|Q_p\|_{p\to p} \leqslant C_2\mu^{-\frac{1}{2}-\frac{1}{2p}}$.

Again, by (3),

$$\begin{aligned}\|b_n^{\frac{1}{p}}\cdot(\mu-\Delta)^{-1}\nabla h\|_p &\leqslant \||b_n|^{\frac{1}{p}}(\mu-\Delta)^{-1}|\nabla h|\|_p\\ &\leqslant \|P_p\|_{p\to p}\|\nabla h\|_p\\ &\leqslant C_3\mu^{-\frac{1}{2}-\frac{1}{2p'}}\|\nabla h\|_p.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Theta_p h-(\mu-\Delta)^{-1}h\|_p &\leqslant \|Q_p\|_{p\to p}\|\big(1+T_p\big)^{-1}\|_{p\to p}\|b_n^{\frac{1}{p}}\cdot(\mu-\Delta)^{-1}\nabla h\|_p\\ &\leqslant C_0\mu^{-\frac{3}{2}}\|\nabla h\|_p,\end{aligned}$$

which clearly implies (i).

Proof of (ii). This is Proposition 1(*iii*). □

Proposition 4. *For every $p\in\mathcal{I}$, and $n=1,2,\dots$, we have $\mathcal{O}\subset\rho(-\Lambda_p(b_n))$, the resolvent set of $-\Lambda_p(b_n)$. The operator-valued function $\Theta_p(\zeta,b_n)$ is the resolvent of $-\Lambda_p(b_n)$:*

$$\Theta_p(\zeta,b_n)=(\zeta+\Lambda_p(b_n))^{-1},\quad \zeta\in\mathcal{O},$$

and

$$\|(\zeta+\Lambda_p(b_n))^{-1}\|_{p\to p}\leqslant C_p|\zeta|^{-1},\quad \zeta\in\mathcal{O}.$$

Proof. By definition, we need to verify that, for every $\zeta\in\mathcal{O}$, $\Theta_p(\zeta,b_n)$ has dense image, and is the left and the right inverse of $\zeta+\Lambda_p(b_n)$. Indeed, Proposition 3(i) implies that $\Theta_p(\zeta,b_n)$ has dense image. $\Lambda_p(b_n):=-\Delta+b_n\cdot\nabla$, $D(\Lambda_p(b_n))=W^{2,p}$, is the generator of a C_0-semigroup $e^{-t\Lambda_p(b_n)}$ on L^p. Clearly, $\Theta_p(\zeta_n,b_n)=(\zeta_n+\Lambda_p(b_n))^{-1}$ for all sufficiently large ζ_n ($=\zeta(\|b_n\|_\infty)$), therefore, by Proposition 2,

$$\Theta_p(\zeta,b_n)=(\zeta_n+\Lambda_p(b_n))^{-1}\big(1+(\zeta_n-\zeta)\Theta_p(\zeta,b_n)\big),\quad \zeta\in\mathcal{O},$$

so $\Theta_p(\zeta,b_n)L^p\subset D(\Lambda_p(b_n))=W^{2,p}$, and $(\zeta+\Lambda_p(b_n))\Theta_p(\zeta,b_n)g=g$, $g\in L^p$, i.e. $\Theta_p(\zeta,b_n)$ is the right inverse of $\zeta+\Lambda_p(b_n)$ on $\mathcal{O}$. Similarly, it is seen that $\Theta(\zeta,b_n)$ is the left inverse of $\zeta+\Lambda_p(b_n)$ on $\mathcal{O}$.

Remark. Alternatively, we could verify conditions of the Kato theorem [Ka2]: in the reflexive space L^p, the pseudo-resolvent $\Theta_p(\zeta,b_n)$ (see Proposition 2) satisfying $\mu\Theta_p(\mu,b_n)\xrightarrow{s}1$ in L^p as $\mu\uparrow\infty$ (see Proposition 3(ii)) is the resolvent of a densely defined closed operator on L^p. This operator coincides with $-\Lambda_p(b_n)$ (since $\Theta_p(\zeta_n,b_n)=(\zeta_n+\Lambda_p(b_n))^{-1}$ for all large ζ_n).

Now, $\|(\zeta+\Lambda_p(b_n))^{-1}\|_{p\to p}\leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$, follows from Proposition 3(ii). □

Proposition 5. *For every $\zeta\in\mathcal{O}$ and $p\in\mathcal{I}$,*

$$\Theta_p(\zeta,b_n)\xrightarrow{s}\Theta_p(\zeta,b)\ \textit{in}\ L^p,$$

Proof. Put $\Theta_p(b)\equiv\Theta_p(\zeta,b)$, $Q_p(b)\equiv Q_p(\zeta,b)$, $T_p(b)\equiv T_p(\zeta,b)$, $G_p(b)\equiv G_p(\zeta,b)$ (similarly for b_n's). It suffices to prove that

$$Q(b_n)(1+T(b_n))^{-1}G(b_n)\xrightarrow{s}Q(b)(1+T(b))^{-1}G(b).$$

Thus it suffices to prove consecutively that

$$G(b_n) \xrightarrow{s} G(b),\ (1+T(b_n))^{-1} \xrightarrow{s} (1+T(b))^{-1},\ Q(b_n) \xrightarrow{s} Q(b).$$

In turn, since $(1+T(b_n))^{-1}-(1+T(b))^{-1} = (1+T(b_n))^{-1}(T(b)-T(b_n))(1+T(b))^{-1}$, it suffices to prove that $T(b_n) \xrightarrow{s} T(b)$. Finally,

$$T(b_n)-T(b) = T(b_n) - b_n^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} + b_n^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} - T(b),$$

and hence we have to prove that

$$b_n^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} - T(b) := J_n^{(1)} \xrightarrow{s} 0 \text{ and } T(b_n) - b_n^{s}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} := J_n^{(2)} \xrightarrow{s} 0.$$

Now, by the Dominated Convergence Theorem (cf. the argument in the proof of (A.0)), $G(b_n) \xrightarrow{s} G(b)$, $J_n^{(1)}|_{\mathcal{E}} \xrightarrow{s} 0$. Also

$$\begin{aligned}\|J_n^{(2)}f\|_p =& \|G(b_n)(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}})f\|_p \\ &\leqslant \|G(b_n)\|_{p\to p}\|(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}})f\|_p \\ &\leqslant m_d(1+\delta)|\zeta|^{-\frac{1}{2p'}}\|(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}})f\|_p, \quad (f\in\mathcal{E}).\end{aligned}$$

Thus, $J_n^{(2)}|_{\mathcal{E}} \xrightarrow{s} 0$. Since $\|J_n^{(2)}\|_{p\to p}, \|J_n^{(1)}\|_{p\to p} \leqslant m_d\delta$, we conclude that $T(b_n) \xrightarrow{s} T(b)$. It is clear now that $Q(b_n) \xrightarrow{s} Q(b)$. □

Now we are going to prove Theorem 1 using the Trotter approximation theorem [Ka1, IX.2.6]. Recall its conditions (in terms of $\Theta_p(\zeta, b_n)$ on the base of Proposition 4):

1) $\sup_{n\geqslant 1}\|\Theta_p(\zeta,b_n)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$.

2) $\mu\Theta_p(\mu,b_n) \xrightarrow{s} 1$ in L^p as $\mu\uparrow\infty$ uniformly in n.

3) There exists s-L^p- $\lim_n \Theta_p(\zeta,b_n)$ for some $\zeta\in\mathcal{O}$.

Now, 1) is the content of Proposition 3(ii). The proof of Proposition 3(i), in fact, yields 2). Proposition 5 implies 3).

Therefore, by the Trotter approximation theorem, $\Theta_p(\zeta,b) = (\zeta+\Lambda_p(b))^{-1}$, $\zeta\in\mathcal{O}$, where $\Lambda_p(b)$ is the generator of the holomorphic C_0-semigroup $e^{-t\Lambda_p(b)}$ on L^p.

Hence, the assertions (i), (vi) of Theorem 1 follow. (ii) follows from Proposition 3(ii) and Proposition 5. (iii) is obvious from the definitions of the operators involved, cf. Proposition 1. $(iii) \Rightarrow (iv)$.

(v) Let $\zeta\in\mathcal{O}$. By Proposition 5, $\Lambda_p(b_n)(\zeta+\Lambda_p(b_n))^{-1} \xrightarrow{s} \Lambda_p(b)(\zeta+\Lambda_p(b))^{-1}$ in L^p. Put $Q_p(b)\equiv Q_p(\zeta,b)$, $T_p(b)\equiv T_p(\zeta,b)$, $G_p(b)\equiv G_p(\zeta,b)$ (similarly for b_n's). Since $(\zeta+\Lambda_p(b))^{-1} = (\zeta-\Delta)^{-1} - Q_p(b)(1+T_p(b))^{-1}G_p(b)$, we have

$$b^{\frac{1}{p}}\cdot\nabla(\zeta+\Lambda_p(b))^{-1} = G_p(b) - T_p(b)(1+T_p(b))^{-1}G_p(b)$$

(similarly for b_n's). Since $G_p(b_n) \xrightarrow{s} G_p(b)$, $T_p(b_n) \xrightarrow{s} T_p(b)$ in L^p (see the proof of Proposition 5),

$$b_n^{\frac{1}{p}}\cdot\nabla(\zeta+\Lambda_p(b_n))^{-1} \xrightarrow{s} b^{\frac{1}{p}}\cdot\nabla(\zeta+\Lambda_p(b))^{-1} \text{ in } L^p. \qquad (**)$$

Now, given $u \in D(\Lambda_p(b))$, we have $u = (\zeta + \Lambda_p(b))^{-1}g$ for some $g \in L^p$, and so, for every $v \in C_c^\infty$,

$$\begin{aligned}
\langle \Lambda_p(b)u, v\rangle &= \langle \Lambda_p(b)(\zeta+\Lambda_p(b))^{-1}g, v\rangle \\
&= \lim_n \langle \Lambda_p(b_n)(\zeta+\Lambda_p(b_n))^{-1}g, v\rangle \\
&= \lim_n \langle (\zeta+\Lambda_p(b_n))^{-1}g, -\Delta v\rangle + \lim_n \langle b_n^{\frac{1}{p}}\cdot\nabla(\zeta+\Lambda_p(b_n))^{-1}g, |b_n|^{\frac{1}{p'}}v\rangle \\
&\quad \text{(here we are using } (**) \text{ and the fact that } |b_n|^{\frac{1}{p'}}v \to |b|^{\frac{1}{p'}}\, v \text{ in } L^{p'}) \\
&= \langle (\zeta+\Lambda_p(b))^{-1}g, -\Delta v\rangle + \langle b^{\frac{1}{p}}\cdot\nabla(\zeta+\Lambda_p(b))^{-1}g, |b|^{\frac{1}{p'}}\, v\rangle \\
&= \langle u, -\Delta v\rangle + \langle b^{\frac{1}{p}}\cdot\nabla u, |b|^{\frac{1}{p'}}v\rangle.
\end{aligned}$$

Next, by the definition of class $\mathbf{F}_\delta^{\frac{1}{2}}$, $|b| \in L^1_{\rm loc}$. Since for $u \in D(\Lambda_p(b))$, $b^{\frac{1}{p}}\cdot\nabla u \in L^p$, it follows that $b\cdot\nabla u = |b|^{\frac{1}{p'}}\, b^{\frac{1}{p}}\cdot\nabla u \in L^1_{\rm loc}$. Also, $\Lambda_p(b)u \in L^p$, and hence $\langle \Lambda_p(b)u, v\rangle = \langle u, -\Delta v\rangle + \langle b\cdot\nabla u, v\rangle$. Therefore, Δu (understood in the sense of distributions) $= -\Lambda_p(b)u + b\cdot\nabla u \in L^1_{\rm loc}$, i.e. $u \in \mathcal{W}^{2,1}_{\rm loc}$. The proof of (v) is completed.

For the proof of $(viii)$ see the argument in [Se, p. 415-416].

The proof of Theorem 1 is completed.

2. Proof of Theorem 2

It is easily seen that, due to the strict inequality $m_d\delta < 4\frac{d-2}{(d-1)^2}$, for every $\tilde{\delta} > \delta$ such that $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$ there exist $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$, such that

$$\tilde{b}_n := \eta_{\varepsilon_n} * b_n \in \mathbf{F}_{\tilde{\delta}}^{\frac{1}{2}}, \quad n = 1, 2, \dots$$

(i) We verify conditions of the Trotter approximation theorem:

1°) $\sup_n \|(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}\|_{\infty\to\infty} \leqslant 1$, $\mu \geqslant \kappa_d\lambda$.

2°) $\mu(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1} \to 1$ in C_∞ as $\mu \uparrow \infty$ uniformly in n.

3°) There exists s-C_∞-$\lim_n(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}$ for some $\mu \geqslant \kappa_d\lambda$.

The condition 1°) is immediate. In view of 1°), it suffices to verify 2°), 3°) on $\mathcal{S}$, the L. Schwartz space of test functions. Fix $p \in \mathcal{I}$, $p > d-1$ (such p exists since $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$).

Proposition 6. *For every $\mu \geqslant \kappa_d\lambda$, $n = 1, 2, \dots,$ $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$, and*

$$(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}|_{\mathcal{S}} = \Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}.$$

Proof. The inclusion $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$ is clear. Clearly, $\Theta_p(\mu_n, \tilde{b}_n)|_{\mathcal{S}} = (\mu_n + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}|_{\mathcal{S}}$ for all sufficiently large μ_n $(= \mu(\|\tilde{b}_n\|_\infty))$. By $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$ and Proposition 2, $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ satisfies the resolvent identity on $\mu \geqslant \kappa_d\lambda$,

$$\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}} = (\mu_n + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}\big(1 + (\mu_n - \mu)\Theta_p(\mu, \tilde{b}_n)\big)|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d\lambda,$$

so $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ is the right inverse of $\mu + \Lambda_{C_\infty}(\tilde{b}_n)|_{\mathcal{S}}$ on $\mu \geqslant \kappa_d\lambda$. Similarly, it is seen that $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ is the left inverse of $\mu + \Lambda_{C_\infty}(\tilde{b}_n)|_{\mathcal{S}}$ on $\mu \geqslant \kappa_d\lambda$. □

Proposition 7. *For every $\mu \geqslant \kappa_d\lambda$, $\Theta_p(\mu, b)\mathcal{S} \subset C_\infty$, and*

$$\Theta_p(\mu, \tilde{b}_n) \overset{s}{\to} \Theta_p(\mu, b) \text{ in } C_\infty.$$

Proof. By Theorem 1(iv), since $p > d-1$, $\Theta_p(\mu,b)L^p \subset C_\infty$. Put

$$Q_p(q,b) \equiv Q_p(q,\mu,b), \quad T_p(b) \equiv T_p(\mu,b), \quad G_p(b) \equiv G_p(\mu,b).$$

To establish the required convergence, it suffices to prove that

$$(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q(q,\tilde{b}_n)(1+T(\tilde{b}_n))^{-1}G(\tilde{b}_n) \overset{s}{\to} (\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q(q,b)(1+T(b))^{-1}G(b) \text{ in } C_\infty.$$

We choose q ($> p$) close to $d-1$ so that $(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}L^p \hookrightarrow C_\infty$. Thus it suffices to prove that

$$G(\tilde{b}_n) \overset{s}{\to} G(b),\ (1+T(\tilde{b}_n))^{-1} \overset{s}{\to} (1+T(b))^{-1},\ Q(q,\tilde{b}_n) \overset{s}{\to} Q(q,b) \text{ in } L^p,$$

which can be done by repeating the arguments in the proof of Proposition 5. □

Proposition 8.

$$\mu\Theta_p(\mu,\tilde{b}_n) \overset{s}{\to} 1 \text{ as } \mu\uparrow\infty \text{ in } C_\infty \text{ uniformly in } n. \tag{5}$$

Proof. Put $\Theta_p \equiv \Theta_p(\mu,\tilde{b}_n)$, $T_p \equiv T_p(\mu,\tilde{b}_n)$. Since $\mu(\mu-\Delta)^{-1} \overset{s}{\to} 1$ in C_∞, and $\mathcal{S}$ is dense in C_∞, it suffices to show that $\|\mu\Theta_p f - \mu(\mu-\Delta)^{-1}f\|_\infty \to 0$ for every $f\in\mathcal{S}$. For each $f\in\mathcal{S}$ there is $h\in\mathcal{S}$ such that $f=(\lambda-\Delta)^{-\frac{1}{2}}h$, where $\lambda=\lambda_\delta>0$. Let $q>p$. Write

$$\Theta_p f - (\mu-\Delta)^{-1}f = -(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q_p(q)\big(1+T_p\big)^{-1}b^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}\cdot(\mu-\Delta)^{-1}\nabla h.$$

Now, arguing as in the proof of Proposition 3(ii), but using estimates

$$\|(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}\|_{p\to\infty} \leqslant c\mu^{-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}}, \quad c<\infty, \quad \text{and} \quad \|Q_p(q)\|_{p\to p} \leqslant \tilde{K}_{2,q} < \infty \quad (\text{see } (3)),$$

we obtain

$$\|\Theta_p f - (\mu-\Delta)^{-1}f\|_\infty \leqslant C\mu^{-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}}\mu^{-1}\|\nabla h\|_p.$$

Since $p>d-1$, choosing q sufficiently close to p, we obtain

$$-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}-1<-1,$$

so $\mu\Theta_p - \mu(\mu-\Delta)^{-1} \overset{s}{\to} 0$ in C_∞, as needed. □

Now, Proposition 7 verifies condition 3°), and Proposition 8 verifies condition 2°). The assertion (i) of Theorem 2 now follows from the Trotter approximation theorem.

Assertion (ii) of Theorem 2 follows from Theorem 1(iii).

The proof of assertion (iii) is standard, and is omitted.

Remark. We could construct $e^{-t\Lambda_{C_\infty}(b)}$ alternatively as follows:

$$e^{-t\Lambda_{C_\infty}(b)} := \big(e^{-t\Lambda_p(b)}|_{C_\infty\cap L^p}\big)^{\text{clos}}_{C_\infty} \ \text{(after a change on a set of measure zero)}, \quad t>0,$$

where $p\in\big(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\big)$.

Appendix A.

Define $I_n := \|(b-b_n)\cdot\nabla(\zeta-\Delta)^{-1}f\|_1$.

1. *Let $b\in\mathbf{K}_\delta^{d+1}$. For every $f\in L^1$ and $\operatorname{Re}\zeta \geqslant \kappa_d\lambda$,*

$$I_n \to 0 \text{ as } n\uparrow\infty. \tag{A.0}$$

Proof of (A.0). Since $I_n \leqslant 2m_d\||b|(\lambda-\Delta)^{-\frac12}|f|\|_1 \leqslant 2m_d\delta\|f\|_1$, it suffices to prove (A.0) for each $f\in L^1\cap L^\infty$. Let $f\in L^1\cap L^\infty, \lambda>0$ and b be fixed. Since $|b|(\lambda-\Delta)^{-\frac12}|f|\in L^1$, for a given $\epsilon>0$, there exists $\mathcal{K}$, a compact, such that

$$\|(\mathbf{1}-\mathbf{1}_{\mathcal{K}})|b|(\lambda-\Delta)^{-\frac12}|f|\|_1 \leqslant \epsilon,$$

where $\mathbf{1}_{\mathcal{K}}$ is the characteristic function of $\mathcal{K}$. Define $I_{\mathcal{K},n} := \|\mathbf{1}_{\mathcal{K}}|b-b_n|(\lambda-\Delta)^{-\frac12}|f|\|_1$. Clearly,

$$I_{\mathcal{K},n} \leqslant \lambda^{-\frac12}\|f\|_\infty\|\mathbf{1}_{\mathcal{K}}|b-b_n|\|_1.$$

Since $|b|\in L^1_{\rm loc}$ and $\mathcal{K}$ independent of $n=1,2,\dots,$

$$\|\mathbf{1}_{\mathcal{K}}|b-b_n|\|_1 \leqslant \|\mathbf{1}_{|b|\geqslant n}(\mathbf{1}_{\mathcal{K}}|b|)\|_1 \to 0 \text{ as } n\uparrow\infty.$$

Therefore, for a given ϵ, there exists $n_0=n_0(\epsilon)\geqslant 1$, such that $I_{\mathcal{K},n}\leqslant\epsilon$ whenever $n\geqslant n_0$, and so

$$I_n \leqslant 3m_d\epsilon \qquad \forall n\geqslant n_0.$$

□

We use the following pointwise estimates ($x,y\in\mathbb{R}^d$, $x\neq y$).

2. *For every $\operatorname{Re}\zeta>0$,*

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant m_d(\kappa_d^{-1}\operatorname{Re}\zeta-\Delta)^{-\frac12}(x,y), \tag{A.1}$$

where $m_d^2 := \pi(2e)^{-1}d^d(d-1)^{1-d}$, $\kappa_d := \frac{d}{d-1}$.

For every $r\in(1,\infty]$ there exists a constant $m_{r,d}<\infty$ such that for all $\operatorname{Re}\zeta>0$,

$$|\nabla(\zeta-\Delta)^{-1+\frac{1}{2r}}(x,y)| \leqslant m_{r,d}(\kappa_d^{-1}\operatorname{Re}\zeta-\Delta)^{-\frac12+\frac{1}{2r}}(x,y). \tag{A.2}$$

3. *For every $\operatorname{Re}\zeta>0$,*

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant 2^{\frac{d}{4}}m_d\Big(\kappa_d^{-1}2^{-\frac12}|\zeta|-\Delta\Big)^{-\frac12}(x,y), \tag{A.3}$$

$$|(\zeta-\Delta)^{-\frac12}(x,y)| \leqslant 2^{\frac{d}{4}+\frac14}\Big(2^{-\frac12}|\zeta|-\Delta\Big)^{-\frac12}(x,y). \tag{A.4}$$

Proof of (A.1). Let $\alpha\in(0,1)$. Set $c(\alpha) := \sup_{\xi>0}\xi e^{-(1-\alpha)\xi^2}\ \Big(=\frac{1}{\sqrt2}(1-\alpha)^{-\frac12}e^{-\frac12}\Big)$, so that

$$\xi e^{-\xi^2} \leqslant c(\alpha)e^{-\alpha\xi^2} \quad \text{for all } \xi>0. \tag{$\star$}$$

We use the well known formula

$$(\zeta-\Delta)^{-\frac{\gamma}{2}}(x,y) = \frac{1}{\Gamma(\frac{\gamma}{2})}\int_0^\infty e^{-\zeta t}t^{\frac{\gamma}{2}-1}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt, \quad 0<\gamma\leqslant 2,$$

first with $\gamma = 2$, and then with $\gamma = 1$, to obtain:

$$\begin{aligned}
|\nabla(\zeta-\Delta)^{-1}(x,y)| &\leqslant \int_0^\infty e^{-t\mathrm{Re}\,\zeta}(4\pi t)^{-\frac{d}{2}}\frac{|x-y|}{2t}e^{-\frac{|x-y|^2}{4t}}dt \\
&\leqslant c(\alpha)\int_0^\infty e^{-t\mathrm{Re}\,\zeta}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\alpha\frac{|x-y|^2}{4t}}dt \qquad \left(\text{By } (\star) \text{ with } \xi := \frac{|x-y|}{2\sqrt{t}}\right) \\
&\leqslant c(\alpha)\alpha^{-\frac{1}{2}-\frac{d}{2}+1}\int_0^\infty e^{-(\mathrm{Re}\,\zeta)\alpha t}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \qquad \left(\text{change } t/\alpha \text{ to } t\right) \\
&= c(\alpha)\alpha^{\frac{1}{2}-\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)\left(\alpha\mathrm{Re}\,\zeta-\Delta\right)^{-\frac{1}{2}}(x,y).
\end{aligned}$$

Now, we minimize $c(\alpha)\alpha^{\frac{1}{2}-\frac{d}{2}}\Gamma(\frac{1}{2})$ in $\alpha\in(0,1)$. The minimum is attained at $\alpha_d = \frac{d-1}{d}$ $(=: \kappa_d^{-1})$, and is equal to m_d.

The proof of (A.2) is similar. □

Proof of (A.3). First, suppose that Im $\zeta \leqslant 0$. By Cauchy theorem,

$$(\zeta-\Delta)^{-1}(x,y) = \int_0^\infty e^{-\zeta t}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt = \int_0^\infty e^{-\zeta r e^{i\frac{\pi}{4}}}e^{-i\frac{\pi}{4}\frac{d}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}e^{i\frac{\pi}{4}}dr,$$

(i.e. we have changed the contour of integration from $\{t : t\geqslant 0\}$ to $\{re^{i\frac{\pi}{4}} : r\geqslant 0\}$). Thus,

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant \int_0^\infty \left|e^{-\zeta re^{i\frac{\pi}{4}}}\right|(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|\left|e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}\right|dr.$$

We have

$$|e^{-\zeta re^{i\frac{\pi}{4}}}| \leqslant e^{-r\frac{1}{\sqrt{2}}(\mathrm{Re}\,\zeta-\mathrm{Im}\,\zeta)}, \quad \left|e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}\right| \leqslant e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}, \quad \mathrm{Re}\,\zeta-\mathrm{Im}\,\zeta\geqslant|\zeta|.$$

Therefore,

$$\begin{aligned}
|\nabla(\zeta-\Delta)^{-1}(x,y)| &\leqslant \int_0^\infty e^{-r\frac{1}{\sqrt{2}}|\zeta|}(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}dr \qquad \left(\text{change } r\sqrt{2} \text{ to } r\right) \\
&= 2^{\frac{d}{4}}\int_0^\infty e^{-r\frac{1}{2}|\zeta|}(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|e^{-\frac{|x-y|^2}{4r}}dr \\
&\leqslant \frac{2^{\frac{d}{4}}m_d}{\Gamma(\frac{1}{2})}\int_0^\infty e^{-r\kappa_d^{-1}\frac{1}{2}|\zeta|}(4\pi r)^{-\frac{d}{2}}r^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{4r}}dr \quad \left(\text{cf. proof of (A.1)}\right) \\
&= 2^{\frac{d}{4}}m_d\left(\kappa_d^{-1}2^{-1}|\zeta|-\Delta\right)^{-\frac{1}{2}}(x,y)
\end{aligned}$$

which yields (A.3) for Im $\zeta\leqslant 0$. The case Im $\zeta > 0$ is treated analogously. □

Proof of (A.4). First, suppose that Im $\zeta\leqslant 0$. By Cauchy theorem,

$$\begin{aligned}
(\zeta-\Delta)^{-\frac{1}{2}}(x,y) &= \int_0^\infty e^{-\zeta t}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \\
&= \int_0^\infty e^{-\zeta re^{i\frac{\pi}{4}}}r^{-\frac{1}{2}}e^{-i\frac{\pi}{8}}e^{-i\frac{\pi}{4}\frac{d}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}e^{i\frac{\pi}{4}}dr,
\end{aligned}$$

so we estimate as above:

$$
\begin{aligned}
|(\zeta-\Delta)^{-\frac{1}{2}}(x,y)| &\leqslant \int_0^\infty e^{-r\frac{1}{\sqrt{2}}|\zeta|} r^{-\frac{1}{2}} (4\pi r)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}} dr \\
&= 2^{\frac{d}{4}+\frac{1}{4}} \left(2^{-1}|\zeta| - \Delta\right)^{-\frac{1}{2}}(x,y).
\end{aligned}
$$

The case $\operatorname{Im}\zeta > 0$ is treated analogously. □

4. *In the proof of Proposition 1 we need the following formula: for every* $\operatorname{Re}\zeta > 0$, $q \in (1,\infty)$,

$$
(\zeta-\Delta)^{-\frac{1}{2q'}} = c_q \int_0^\infty t^{-1+\frac{1}{2q}} (t+\zeta-\Delta)^{-\frac{1}{2}} dt, \quad c_q := \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2q}\right)\Gamma\left(\frac{1}{2q'}\right)}, \tag{A.5}
$$

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