Holomorphic almost periodic functions on coverings of complex manifolds

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Abstract. In this paper we discuss some results of the theory of holomorphic almost periodic functions on coverings of complex manifolds, recently developed by the authors. The methods of the proofs are mostly sheaf-theoretic which allows us to obtain new results even in the classical setting of H. Bohr's holomorphic almost periodic functions on tube domains.

1. Introduction

In the 1920s H. Bohr [Bo] created his famous theory of almost periodic functions which shortly acquired numerous applications to various areas of mathematics, from harmonic analysis to differential equations. Two branches of this theory were particularly rich on deep and interesting results: holomorphic almost periodic functions on a complex strip [Bo, Lv] (and later on tube domains [FR]), and J. von Neumann's almost periodic functions on groups [N]. Holomorphic almost periodic functions on a tube domain $T = \mathbb{R}^n + i\Omega$, where $\Omega \subset \mathbb{R}^n$ is open and convex, arise as uniform limits on subdomains $T' = \mathbb{R}^n + i\Omega'$, $\Omega' \Subset \Omega$, of exponential polynomials

$$z \mapsto \sum_{k=1}^{m} c_k e^{i\langle \lambda_k, z\rangle}, \quad z \in T, \quad c_k \in \mathbb{C}, \quad \lambda_k \in (\mathbb{R}^n)^* = \mathbb{R}^n;$$

here $\langle \lambda_k, \cdot\rangle$ is a complex linear functional defined by λ_k.

A classical approach to the study of such functions employs the fact that T is the trivial bundle with base Ω and fibre $\mathbb{R}^n$ (cf. characterization of almost periodic functions in terms of their Jessen functions defined on Ω [JT, Rn, FR], proofs using some results on almost periodic functions on $\mathbb{R}$ [Ln], etc.). In the present paper we consider T as a regular covering $p : T \to T_0$ with the deck transformation group $\mathbb{Z}^n$ over a relatively complete Reinhardt domain T_0 (e.g., complex strip covering an annulus if $n = 1$). It turns out that the holomorphic almost periodic functions on T are precisely those holomorphic functions which are von Neumann almost periodic on each fibre $p^{-1}(z) \cong \mathbb{Z}^n$, $z \in T_0$, and bounded on each subset $p^{-1}(U_0)$, $U \Subset T$. The latter enables us to regard holomorphic almost periodic functions on T as

(a) holomorphic sections of a certain holomorphic Banach vector bundle on T_0;

(b) 'holomorphic' functions on the fibrewise Bohr compactification of the covering, a topological space sharing some properties of a complex manifold.

2000 *Mathematics Subject Classification.* Primary 32A38, Secondary 42A75.

Key words and phrases. Holomorphic almost periodic function, regular covering, Cartan Theorem B.

Research of the first author is supported in part by NSERC.

This is an extended version of authors' talk at the conference "Recent Advances in Function-related Operator Theory", Puerto Rico, 2010.

This point of view allows us to enrich the variety of techniques used in the theory of holomorphic almost periodic functions (such as Fourier analysis-type arguments for $n = 1$, cf. [Bo, Ln, Lv], arguments based on the properties of Monge-Amperé currents for $n > 1$, cf. [F2R], etc.) by methods of Banach-valued complex analysis [Bu1, L] and by algebraic-geometric methods such as Cartan Theorem B for coherent sheaves. Using them we obtain new results on holomorphic almost periodic extensions from almost periodic complex submanifolds, Hartogs-type theorems, recovery of almost periodicity of a holomorphic function from that for its trace to real periodic hypersurfaces, etc.

The above equivalent definitions of almost periodicity on a tube domain suggest a natural way to define holomorphic almost periodic functions on a regular covering $p : X \to X_0$ with a deck transformation group G of a complex manifold X_0 as those holomorphic functions on X that are von Neumann almost periodic on each fibre $p^{-1}(z) \cong G$, $z \in X_0$, and bounded on each subset $p^{-1}(U) \subset X$, $U \Subset X_0$. Many of the results known for holomorphic almost periodic functions on tube domains, e.g., Bohr's approximation theorem (Theorem 1.1), on some properties of almost periodic divisors, are valid also for holomorphic almost periodic functions on regular coverings of Stein manifolds. In fact, some of these results can be carried over with practically the same proofs to certain algebras of holomorphic functions on X invariant with respect to the action of the deck transformation group G.

Earlier similar methods based on the Stone-Čech compactification of fibres of a regular covering of a complex manifold and on an analogous Banach vector bundle construction were developed in [Br1] – [Br4] in connection with corona-type problems for some algebras of bounded holomorphic functions on coverings of bordered Riemann surfaces and integral representation of holomorphic functions of slow growth on coverings of Stein manifolds. The Bohr compactification $b\mathbb{R}^n + i\Omega$ of tube domain $\mathbb{R}^n + i\Omega$ was used in [Fav1, Fav2] in the context of the problem of a characterization of zero sets of holomorphic almost periodic functions among all almost periodic divisors. It is interesting to note that already in his monograph [Bo] H. Bohr uses equally often the aforementioned "trivial fibre bundle" and "regular covering" points of view on a complex strip.

Let us recall the definitions of holomorphic almost periodic functions on tube domains and almost periodic functions on groups.

Following S. Bochner, a holomorphic function f on a tube domain $\mathbb{R}^n + i\Omega$ is called almost periodic if the family of its translates $\{z \mapsto f(z+s)\}_{s\in\mathbb{R}}$ is relatively compact in the topology of uniform convergence on tube subdomains $\mathbb{R}^n + i\Omega'$, $\Omega' \Subset \Omega$. The Frechet algebra of holomorphic almost periodic functions endowed with the above topology is denoted by $APH(\mathbb{R}^n + i\Omega)$. Analogously, one defines holomorphic almost periodic functions on a tube domain with an open relatively compact base $\Omega^c \subset \mathbb{R}^n$ as holomorphic functions continuous in the closure $\mathbb{R}^n + i\bar{\Omega}^c$ and such that their families of translates are relatively compact in the topology of uniform convergence on $\mathbb{R}^n + i\bar{\Omega}^c$. The Banach algebra of such holomorphic almost periodic functions is denoted by $APH(\mathbb{R}^n + i\Omega^c)$.

The following result, called the *approximation theorem*, is the cornerstone of Bohr's theory.

Theorem 1.1 (H. Bohr)**.** The exponential polynomials

$$APH_0(\mathbb{R}^n + i\Omega) := \mathrm{span}_{\mathbb{C}}\{e^{i\langle\lambda,z\rangle},\ z \in \mathbb{R}^n + i\Omega\}_{\lambda\in\mathbb{R}^n},$$

$$APH_0(\mathbb{R}^n + i\Omega^c) := \mathrm{span}_{\mathbb{C}}\{e^{i\langle\lambda,z\rangle},\ z \in \mathbb{R}^n + i\Omega^c\}_{\lambda\in\mathbb{R}^n}$$

are dense in $APH(\mathbb{R}^n + i\Omega)$ and $APH(\mathbb{R}^n + i\Omega^c)$, respectively.

J. von Neumann's almost periodic functions on a topological group G, originally introduced in connection with Hilbert's fifth problem on characterization of Lie groups among all topological groups, are obtained as limits of linear combinations of the form

$$t \mapsto \sum_{k=1}^{m} c_k \sigma_{ij}^k(t), \quad t \in G, \quad c_k \in \mathbb{C}, \quad \sigma^k = (\sigma_{ij}^k),$$

where σ^k $(1 \le k \le m)$ are finite-dimensional irreducible unitary representations of G.

The intrinsic definition of almost periodic functions on G is as follows. Let $C_b(G)$ be the algebra of bounded continuous $\mathbb{C}$-valued functions on G endowed with sup-norm. A function $f \in C_b(G)$ is called almost periodic if its translates $\{t \mapsto f(st)\}_{s\in G}$ and $\{t \mapsto f(ts)\}_{s\in G}$ are relatively compact in $C_b(G)$. It was later proved in [Ma] that the relative compactness of either the left of the right family of translates already gives the von Neumann definition of almost periodicity.

Let $AP(G) \subset C_b(G)$ be the uniform subalgebra of all almost periodic functions on G, and $AP_0(G)$ be the linear hull over $\mathbb{C}$ of matrix entries of finite-dimensional irreducible unitary representations of G.

Theorem 1.2 (J. von Neumann)**.** $AP_0(G)$ is dense in $AP(G)$.

In what follows, we assume that finite-dimensional irreducible unitary representations of G separate points of G. Following von Neumann, such groups are called *maximally almost periodic*. Equivalently, G is maximally almost periodic iff it admits a monomorphism into a compact topological group. Any residually finite group, i.e., a group such that the intersection of all its finite index normal subgroups is trivial, belongs to this class. In particular, finite groups, free groups, finitely generated nilpotent groups, pure braid groups, fundamental groups of three dimensional manifolds are maximally almost periodic.

In the paper we study the holomorphic almost periodic functions on a regular covering $p : X \to X_0$ of a complex manifold X_0 whose deck transformation group $\pi_1(X_0)/p_*\pi_1(X)$ is maximally almost periodic; here $\pi_1(X_0)$ stands for the fundamental group of X_0. We will prove only several basic results related to the above characterizations (a) and (b) of holomorphic almost periodic functions on regular coverings of complex manifolds, to the Liouville property for holomorphic almost periodic functions on regular coverings of ultraliouville complex manifolds, and to extensions of holomorphic almost periodic functions from almost periodic complex submanifolds in the case of almost periodic complex hypersurfaces. We refer to [BrK2] for the complete exposition with detailed proofs. Some results of this paper are presented in [BrK3].

2. Holomorphic almost periodic functions

Let X be a complex manifold with a free and properly discontinuous left holomorphic action of a discrete maximally almost periodic group G. It follows that the orbit space $X_0 := X/G$ is also a complex manifold and the projection $p : X \to X_0$ is holomorphic. In particular, if X is connected, then $p : X \to X_0$ is a regular covering with deck transformation group G.

We denote by $\mathcal{O}(X)$ the algebra of holomorphic functions on X.

Definition 2.1. A function $f \in \mathcal{O}(X)$ is called almost periodic if for each point X there exists a G-invariant open subset U containing this point, such that the family of translates $\{z \mapsto f(g \cdot z), z \in U\}_{g \in G}$ is relatively compact in the topology of uniform convergence on U.

This definition is a variant of definition in [W] (with G being the group of all biholomorphic automorphisms of X). An equivalent definition of a holomorphic almost periodic function is as follows.

Let $F_x := p^{-1}(x)$. Since F_x is discrete and G acts of F_x freely and transitively, we can define the algebra $AP(F_x)$ of continuous almost periodic functions on F_x so that $AP(F_x) \cong AP(G)$ (cf. the above cited result of [Ma]).

Definition 2.2. A function $f \in \mathcal{O}(X)$ is called almost periodic if each $x \in X_0$ has a neighbourhood $U_x \Subset X_0$ such that the restriction $f|_{p^{-1}(U_x)}$ is bounded, and $f|_{F_x} \in AP(F_x)$.

We establish equivalence of Definitions 2.1 and 2.2 in Proposition 5.4.

The algebra of almost periodic functions on X, endowed with the topology of uniform convergence on subsets $p^{-1}(U_0)$, $U_0 \Subset X_0$, is denoted by $\mathcal{O}_{AP}(X)$. This is a Frechet algebra (i.e., it is complete). Note also that if G is finite, then $\mathcal{O}_{AP}(X) = \mathcal{O}(X)$.

In what follows, the complex manifold X is assumed to be connected.

If the complex manifold X_0 is Stein, then $\mathcal{O}_{AP}(X)$ separates the points on X. (The latter follows from the maximal almost periodicity of group G and Theorem 3.6 below.) In contrast, we have the following result.

The complex manifold X_0 is called *ultraliouville* if there are no non-constant bounded continuous plurisubharmonic functions on X_0 [Lin]. (Recall that an upper-semicontinuous function $f : X_0 \to [-\infty, \infty)$ is called plurisubharmonic if for any holomorphic map $q : \mathbb{D} \to Y$, where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk, the pullback $q^* f$ is subharmonic on $\mathbb{D}$.) In particular, compact complex manifolds and their Zariski open subsets are ultraliouville.

Theorem 2.3. (1) Suppose that the complex manifold X_0 is ultraliouville. Then all *bounded* holomorphic almost periodic functions in $\mathcal{O}_{AP}(X)$ are constant. In particular, if X_0 is a compact complex manifold, then $\mathcal{O}_{AP}(X_0) \cong \mathbb{C}$.

Let $n := \dim X_0 \geq 2$, $p : X \to X_0$ be a covering as above and $\tilde{X}_0 := X_0 \setminus D_0$, where $D_0 \Subset X_0$ is a (possibly empty) subdomain such that $\tilde{X} := p^{-1}(\tilde{X}_0)$ is connected.

(2) Suppose that X_0 is ultraliouville and D_0 has a connected piecewise smooth boundary and is contained in an open Stein submanifold of X_0.

Then all bounded holomorphic almost periodic functions in $\mathcal{O}_{AP}(\tilde{X})$ are constant.

For instance, consider the universal covering $p:\mathbb{D}\to\mathbb{C}\setminus\{0,1\}$ of doubly punctured complex plane (here the deck transformation group is free group with two generators). Although there are plenty of non-constant bounded holomorphic functions on $\mathbb{D}$, all bounded holomorphic almost periodic functions on $\mathbb{D}$ corresponding to this covering are constant because $\mathbb{C}\setminus\{0,1\}$ is ultraliouville.

One can deduce from Theorem 2.3 that the algebra $\mathcal{O}_{AP}(X)$ for X_0 a holomorphically convex manifold is obtained as the pullback of the algebra $\mathcal{O}_{AP}(Y)$, where Y is a regular covering of the Stein reduction Y_0 of X_0 with a maximally almost periodic deck transformation group which is a factor-group of the deck transformation group of the covering $X\to X_0$. (Recall that Y_0 is a normal Stein space and there exists a proper surjective holomorphic map $X_0\to Y_0$ with connected fibres. The algebra $\mathcal{O}_{AP}(Y)$ in this case is defined akin to Definitions 2.1 or 2.2.) Thus it is natural to study holomorphic almost periodic functions on regular coverings X of normal Stein spaces X_0. In what follows we assume that X_0 is a Stein manifold. Some of our results are also valid for X_0 being a normal Stein space.

We also consider almost periodic functions on closed sets of the form $\bar{D}:=p^{-1}(\bar{D}_0)\subset X$, where D_0 is a relatively compact subdomain of X_0.

Definition 2.4. A function $f\in\mathcal{O}(D)\cap C(\bar{D})$ is almost periodic if it is bounded on D and $f|_{F_x}\in AP(F_x)$ for all $x\in\bar{D}_0$.

The algebra of almost periodic holomorphic functions on $\bar{D}$, endowed with supnorm is denoted by $\mathcal{A}_{AP}(D)$. It follows from the definition that $\mathcal{A}_{AP}(D)$ is a Banach algebra.

Unless specified otherwise, we assume that D_0 is strictly pseudoconvex; here and below any strictly pseudonconvex domain is assumed to have a C^2-smooth boundary (we refer to [GR] for the corresponding definitions). Since for each point $x\in\partial D$ there exist a neighbourhood $U\subset X$ of x and a neighbourhood $U_0\subset X_0$ of $p(x)\in\partial D_0$ such that $p|_U:U\to U_0$ is a biholomorphism, the set $D=p^{-1}(D_0)$ is strictly pseudoconvex in X.

Example 2.5. (1) In the classical case of holomorphic almost periodic functions on a tube domain $T:=\mathbb{R}^n+i\Omega\subset\mathbb{C}^n$ with $\Omega\subset\mathbb{R}^n$ open and convex, we consider T as a regular covering with deck transformation group $\mathbb{Z}^n$ of a domain $T_0\Subset\mathbb{C}^n$; here the covering map $p:T\to T_0\,(:=p\,(T))$ is defined by the formula

$$p(z):=\left(e^{2\pi i z_1},\dots,e^{2\pi i z_n}\right),\quad z=(z_1,\dots,z_n)\in T.$$

Then T_0 is a pseudoconvex subset of $\mathbb{C}^n$ (i.e., T_0 is a Stein manifold).

Analogously, we define covering $p:T^s\to T_0^s\,(:=p\,(T^s))$, where $T^s:=\mathbb{R}^n+i\Omega^c$, $\Omega^c\Subset\mathbb{R}^n$ is open and strictly convex with C^2-smooth boundary. Then T_0^s is a strictly pseudoconvex subset of $\mathbb{C}^n$.

It is also easy to see that T_0 and T_0^s are relatively complete Reinhardt domains in $\mathbb{C}^n$ (see [S]).

Further, inclusions $APH(\mathbb{R}^n+i\Omega)\subset\mathcal{O}_{AP}(T)$ and $APH(\mathbb{R}^n+i\Omega^c)\subset\mathcal{A}_{AP}(T^s)$ are obvious. The opposite inclusions follow, e.g., from Theorem 3.1 and Example 3.3 below. (See also [BrK1] for their proof.)

(2) Let X_0 be a non-compact Riemann surface, $p:X\to X_0$ be a regular covering with a maximally almost periodic deck transformation group G (for instance, X_0 is hyperbolic, then $X=\mathbb{D}$ is its universal covering, and $G=\pi_1(X_0)$ is a free (not necessarily finitely generated) group). The functions in $\mathcal{O}_{AP}(X)$ arise, e.g.,

as linear combinations over $\mathbb{C}$ of matrix entries of fundamental solutions of certain linear differential equations on X. Indeed, a unitary representation $\sigma : G \to U_n$ can be obtained as the monodromy of the system $dF = \omega F$ on X_0, where ω is a holomorphic 1-form on X_0 with values in the space of $n \times n$ complex matrices $M_n(\mathbb{C})$ (see, e.g., [For]). In particular, the pulled back system $dF = (p^*\omega)F$ on X admits a global solution $F \in \mathcal{O}\big(X, GL_n(\mathbb{C})\big)$ such that $F \circ g^{-1} = F\sigma(g)$ $(g \in G)$. By definition, a linear combination of matrix entries of F is an element of $\mathcal{O}_{AP}(X)$.

3. Approximation theorems. Extension from periodic sets

In this section we study function-theoretic properties of holomorphic almost periodic functions. Our proofs are based on an equivalent description of holomorphic almost periodic functions on X as *holomorphic sections* of a holomorphic Banach vector bundle over the Stein manifold X_0 (see Section 3.3 below).

3.1. First, we describe an extension of Bohr's approximation theorem (Theorem 1.1).

Let R_G be the set of finite dimensional irreducible unitary representations of group G. For a given $\sigma \in R_G$ we denote by $\mathcal{O}_\sigma(X)$ and $\mathcal{A}_\sigma(D)$ the $\mathbb{C}$-linear hulls of coordinates of vector-valued functions f in $\mathcal{O}(X, \mathbb{C}^n)$ and $\mathcal{A}(D, \mathbb{C}^n)$, respectively, having the property that $f_g = \sigma(g) f$ for all $g \in G$. (Here $f_g(z) := f(g \cdot z)$, $z \in X$.) Further, let $\mathcal{O}_0(X)$ and $\mathcal{A}_0(D)$ be $\mathbb{C}$-linear hulls of spaces $\mathcal{O}_\sigma(X)$ and $\mathcal{A}_\sigma(D)$, respectively, with σ varying over the set R_G.

Theorem 3.1. $\mathcal{O}_0(X)$ is dense in $\mathcal{O}_{AP}(X)$.

Recall that a Banach space A is said to have the *approximation property* if for any $\varepsilon > 0$ and any relatively compact subset $K \subset A$ there exists a finite rank bounded linear (*approximating*) operator $S = S_{\varepsilon,K} \in \mathcal{L}(A, A)$ such that $\|x - Sx\|_A < \varepsilon$ for all $x \in K$.

For example, algebra $AP(G)$ has the approximation property with approximating operators in $\mathcal{L}(AP(G), AP_0(G))$.

If T is a compact Hausdorff topological space, A is a closed subspace of $C(T)$, B is a Banach space (here and below all Banach spaces are complex) and $A_B \subset C_B(T)$ is the space of continuous B-valued functions f such that for any $\varphi \in B^*$ one has $\varphi(f) \in A$, then A has the approximation property if and only if $B \otimes A$ is dense in A_B, see [G].

Theorem 3.2. The space $\mathcal{A}_{AP}(D)$ has the approximation property with approximating operators in $\mathcal{L}\big(\mathcal{A}_{AP}(D), \mathcal{A}_0(D)\big)$. In particular, $\mathcal{A}_0(D)$ is dense $\mathcal{A}_{AP}(D)$.

In the following example we show that for almost periodic holomorphic functions on a tube domain Theorems 3.1 and 3.2 imply Theorem 1.1.

Example 3.3. (1) For a tube domain $T \subset \mathbb{C}^n$ consider the covering $p : T \to T_0$ of Example 2.5(1). Let us show that the subspace of exponential polynomials

$$z \mapsto \sum_{k=1}^{m} c_k e^{i\langle \lambda_k, z\rangle}, \ z \in T, \ c_k \in \mathbb{C}, \ \lambda_k \in \mathbb{R}^k,$$

is dense in $\mathcal{O}_0(T)$ (a similar result with an analogous proof is valid for algebra $\mathcal{A}_{AP}(D)$). Indeed, since group $G := \mathbb{Z}^n$ is free Abelian, all irreducible unitary representations of G are one-dimensional and given by the formulas $\sigma_\lambda(l) := e^{i\langle\lambda,l\rangle}$, $l \in \mathbb{Z}^n$ for $\lambda \in \mathbb{R}^n$. We define

$$e_\lambda(z) := e^{i\langle\lambda,z\rangle}, \quad z \in T.$$

Then $e_\lambda \in \mathcal{O}_{\sigma_\lambda}(T)$ and for any $h \in \mathcal{O}_{\sigma_\lambda}(T)$ there exists a function $\tilde{h} \in \mathcal{O}(T_0)$ such that $h/e_\lambda = p^*\tilde{h}$. By definition, for an $f \in \mathcal{O}_0(T)$ there exist functions $h_k \in \mathcal{O}_{\sigma_{\lambda_k}}(T)$ $(1 \le k \le m)$ such that

$$f(z) = \sum_{k=1}^{m} c_k h_k(z), \quad z \in T,\ c_k \in \mathbb{C}.$$

Therefore,

$$f(z) = \sum_{k=1}^{m} c_k (p^*\tilde{h}_k)(z) e_{\lambda_k}(z), \quad z \in T. \tag{3.1}$$

Since the base T_0 of the covering is a relatively complete Reinhardt domain, functions $\tilde{h}_k$ admit expansions into Laurent series (see, e.g., [S])

$$\tilde{h}_k(z) = \sum_{|\ell|=-\infty}^{\infty} b_\ell z^\ell, \quad z \in T_0, \quad b_\ell \in \mathbb{C},$$

where $\ell = (\ell_1, \dots, \ell_n)$, $|\ell| = \ell_1 + \cdots + \ell_n$. Therefore, $p^*\tilde{h}_k$ admit approximations by finite sums

$$\sum_{|\ell|=-M}^{M} b_j e^{2\pi i\langle\ell,z\rangle}, \quad z \in T,$$

uniformly on subsets $p^{-1}(W_0) \subset T$, $W_0 \Subset T_0$. Together with (3.1) this implies the required.

(2) In the setting of Example 2.5(2), given an irreducible unitary representation $\sigma : G \to U_n$ we consider a function $F \in \mathcal{O}(X, GL_n(\mathbb{C}))$ such that $F \circ g^{-1} = F\sigma(g)$, $g \in G$, and a function $f \in \mathcal{O}(X, \mathbb{C}^n)$ such that $f \circ g = \sigma(g) f$ (i.e., with coordinates lying in $\mathcal{O}_\sigma(X)$). It follows that there exists a function $\tilde{f} \in \mathcal{O}(X_0, \mathbb{C}^n)$ such that $Ff = p^*\tilde{f}$. Therefore all functions in $\mathcal{O}_\sigma(X)$ are obtained as $\mathbb{C}$-linear combinations of coordinates of vector-valued functions of the form $F^{-1}(p^*\tilde{f})$ with $\tilde{f} \in \mathcal{O}(X_0, \mathbb{C}^n)$. Note that entries of F^{-1} are the same as for $(F^\top)^{-1}$ satisfying $(F^\top)^{-1} \circ g^{-1} = (F^\top)^{-1}(\sigma(g)^\top)^{-1}$, $g \in G$, where $(\sigma^\top)^{-1}$ is an irreducible unitary representation of G as well.

Let $[R_G]$ be the set of equivalence classes of irreducible representations $G \to U_n$, $n \in \mathbb{N}$. For each class $[\sigma] \in [R_G]$ representing $\sigma : G \to U_n$ we fix an $F_{[\sigma]} \in \mathcal{O}(X, GL_n(\mathbb{C}))$ satisfying $F_{[\sigma]} \circ g^{-1} = F_{[\sigma]}\sigma(g)$, $g \in G$. Further, consider an at most countable subset $A \subset \mathcal{O}(X_0)$ such that the complex algebra generated by A is dense in $\mathcal{O}(X_0)$ (e.g., in the case $X_0 = \{z \in \mathbb{C} : |w(z)| < 1,\ w$ is holomorphic in a neighbourhood of $X_0\} \Subset \mathbb{C}$ is an analytic polyhedron, we may take $A = \{w\}$). Then the products of matrix entries of $F_{[\sigma]}$, $[\sigma] \in [R_G]$, with functions in p^*A may be viewed as analogs of exponential polynomials of Example 3.3(1). The $\mathbb{C}$-linear hull generated by these functions is dense in $\mathcal{O}_\sigma(X)$. (A similar result is valid for $\mathcal{A}_{AP}(D)$.)

3.2. Next, we formulate our results on holomorphic almost periodic extension.

Definition 3.4. A subset $Y \subset X$ of the form $Y = p^{-1}(Y_0)$, where $Y_0 \subset X_0$, is called *periodic*.

(This terminology comes from Example 2.5(1) where a periodic subset of a tube domain $T \subset \mathbb{C}^n$ is $(2\pi, \dots, 2\pi)$-periodic with respect to the natural action of group $\mathbb{R}^n$ on T by translations.)

We consider a complete path metric d on X_0 with an associated $(1,1)$-form obtained, e.g., as the restriction of the Hermitian $(1,1)$-form on $\mathbb{C}^{2n+1}$ to X_0, where X_0 is regarded as a closed submanifold of $\mathbb{C}^{2n+1}$ [St]. For a simply connected subset $W_0 \subset X_0$ we naturally identify $p^{-1}(W_0)$ with $W_0 \times G$.

Definition 3.5. A continuous function f on a periodic set $Y \subset X$ is called almost periodic if it is bounded and uniformly continuous on subsets $p^{-1}(V_0) \cong V_0 \times G$, where $V_0 := U_0 \cap Y_0$ and $U_0 \Subset Y_0$ is a simply connected coordinate chart on X_0, with respect to the semi-metric d' on $p^{-1}(U_0)$ defined by

$$d'((z_1, g), (z_2, g)) := d(z_1, z_2), \qquad (z_1, g),\ (z_2, g) \in p^{-1}(U_0)\,(\cong U_0 \times G),$$

and $f|_{p^{-1}(x)} \in AP(F_x)$ for all $x \in Y_0$.

If, in addition $Y_0 := p(Y)$ is a complex analytic subspace of X_0 and $f \in \mathcal{O}(Y)$ satisfies the above properties, it is called holomorphic almost periodic.

We denote by $C_{AP}(Y)$ and $\mathcal{O}_{AP}(Y)$ the algebras of continuous and holomorphic almost periodic functions on Y.

Theorem 3.6. Let M_0 be a closed complex submanifold of X_0 (so that $M := p^{-1}(M_0)$ is a closed complex submanifold of X).

(1) For every function $f \in \mathcal{O}_{AP}(M)$ there exists a function $F \in \mathcal{O}_{AP}(X)$ such that $F|_M = f$.

Suppose that M_0 is a complex submanifold of a neighbourhood of the closure of a strictly pseudoconvex domain $D_0 \Subset \mathbb{C}^n$. Let $\mathcal{A}_{AP}(D \cap M)$, $D := p^{-1}(D_0)$, be the algebra of holomorphic almost periodic functions in $D \cap M$ continuous on $\bar{D} \cap M$ endowed with the sup-norm.

(2) For every function $f \in \mathcal{A}_{AP}(D \cap M)$ there is a function $F \in \mathcal{A}_{AP}(D)$ such that $F|_{D \cap M} = f$.

The following result shows that almost periodicity of a holomorphic function can be recovered from that for its trace to real periodic hypersurfaces.

Theorem 3.7. Let $S_0 \subset X_0$, $S := p^{-1}(S_0)$, be a piecewise smooth real hypersurface. Suppose $f \in \mathcal{O}(X)$ is such that $f|_S \in C_{AP}(S)$ and f is bounded on subsets $W = p^{-1}(W_0)$, $W_0 \Subset X_0$. Then $f \in \mathcal{O}_{AP}(X)$.

The classical result of Bohr's theory of holomorphic almost periodic functions on complex strip $T := \mathbb{R} + i(a, b) \subset \mathbb{C}$ states that if a function $f \in \mathcal{O}(T)$ is bounded on closed substrips and is almost periodic on a horizontal line in T, then $f \in \mathcal{O}_{AP}(\Sigma)$ (see, e.g, [Ln]). It follows from Theorem 3.7 that the horizontal line can be replaced here by any real periodic piecewise smooth curve. In fact, a similar result holds if S_0 is a uniqueness set for the space $\mathcal{O}(X_0)$, e.g., if S_0 is a generic CR submanifold of X_0 of real codimension $\leq n$, see [BrK2].

We use a result in [Br3] and an argument similar to the one in the proof of Theorem 3.7 to obtain the following Hartogs-type theorem. Suppose $n = \dim(X_0) \geq 2$. Let $S_0 \subset X_0$ be a smooth real hypersurface, $S := p^{-1}(S_0)$. We say that a function $f \in C_{AP}(S)$ satisfies tangential CR equations on S if

$$\int_S f\ \bar{\partial}\omega = 0$$

for all C^∞-smooth $(n, n-2)$-forms ω on X having compact supports in X_0.

Theorem 3.8. Let $n = \dim(X_0) \geq 2$, and $D_0 \Subset X_0$ be a subdomain with a connected piecewise smooth boundary S_0; then $S := p^{-1}(S_0)$ is the boundary of D. Suppose that $f \in C_{AP}(S)$ satisfies tangential CR equations on S. Then there exists a function $F \in \mathcal{A}_{AP}(D)$ such that $F|_S = f$.

The result can be applied, e.g., to tube domains $T := \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$, $n \geq 2$, where $\Omega \Subset \mathbb{R}^n$ is a domain with piecewise-smooth boundary $\partial\Omega$, and to continuous almost periodic functions on the boundary $\partial T := \mathbb{R}^n + i\partial\Omega$ of T satisfying tangential CR equations there. Then every such a function admits a continuous extension to a function from $\mathcal{A}_{AP}(T)$.

3.3. Our proofs of the results above are based on an equivalent presentation of holomorphic almost periodic functions on X as holomorphic sections of a holomorphic Banach vector bundle on the manifold X_0 (see Proposition 3.9) defined as follows.

The regular covering $p : X \to X_0$ is a principal fibre bundle with structure group G. By definition, given a cover $(U_\gamma)_{\gamma\in\Gamma}$ of X_0 by simply connected open sets $U_\gamma \Subset X_0$ there exists a locally constant cocycle $\{c_{\delta\gamma} : U_\gamma \cap U_\delta \to G,\ \gamma, \delta \in \Gamma\}$, so that the covering $p : X \to X_0$ can be obtained from the disjoint union $\sqcup_{\gamma\in\Gamma} U_\gamma \times G$ by the identification

$$U_\delta \times G \ni (x, g) \sim (x, g \cdot c_{\delta\gamma}(x)) \in U_\gamma \times G \quad \text{ for all } x \in U_\gamma \cap U_\delta, \quad \gamma, \delta \in \Gamma.$$

Here projection p is induced by the projections $U_\gamma \times G \to U_\gamma$.

The algebra $AP(G)$ is isomorphic to the algebra $C(bG)$ of continuous functions on bG, a compact group called the Bohr compactification of G. Since G is maximally almost periodic, G is a dense subgroup of bG (see Section 4.2 for details).

We define a holomorphic Banach vector bundle $\tilde{p} : CX \to X_0$ to be the associated bundle to the principal fibre bundle $p : X \to X_0$, having fibre $C(bG)$. By definition, CX is obtained from the disjoint union $\sqcup_{\gamma\in\Gamma} U_\gamma \times C(bG)$ by the identification

$$U_\delta \times C(bG) \ni (x, f(\omega)) \sim (x, f(\omega \cdot c_{\delta\gamma}(x))) \in U_\gamma \times C(bG) \quad \text{ for all } x \in U_\gamma \cap U_\delta,$$

where $\gamma, \delta \in \Gamma$. The projection $\tilde{p}$ is induced by projections $U_\gamma \times C(bG) \to U_\gamma$.

Let $\mathcal{O}(CX)$ be the set of (global) holomorphic sections of CX. It forms a Frechet algebra with respect to the usual pointwise operations and the topology of uniform convergence on compact subsets of X_0.

Analogously, we define the Banach vector bundle CD, and denote by $\mathcal{A}(CD)$ the algebra of continuous sections of CD over $\bar{D}_0$ holomorphic in D_0. It forms a Banach algebra with respect to the usual pointwise operations and sup-norms on the base and fibres.

Proposition 3.9. $\mathcal{O}_{AP}(X) \cong \mathcal{O}(CX)$, $\mathcal{A}_{AP}(D) \cong \mathcal{A}(CD)$.

For example, to prove Theorem 3.6(1), let us note that algebra $\mathcal{O}_{AP}(M)$ is isomorphic to the algebra $\mathcal{O}(CX)|_{M_0}$ of holomorphic sections of the bundle CX over M_0. By the result in [L] there exist holomorphic Banach vector bundles $p_1 : E_1 \to X_0$ and $p_2 : E_2 \to X_0$ with fibres B_1 and B_2, respectively, such that $E_2 = E_1 \oplus CX$ (the Whitney sum) and E_2 is holomorphically trivial, i.e., $E_2 \cong X_0 \times B_2$. Thus, any holomorphic section of E_2 can be naturally identified with a B_2-valued holomorphic function on X_0. By $q : E_2 \to CX$ and $\iota : CX \to E_2$ we denote the corresponding quotient and embedding homomorphisms of the bundles so that $q \circ \iota = \mathrm{Id}$. (Similar identifications hold for bundle CD.) For a given function $f \in \mathcal{O}(CX)|_{M_0}$ consider its image $\tilde{f} := \iota(f)$, a B_2-valued holomorphic function on M_0, and apply to it the Banach-valued extension result in [Bu2] asserting existence of a function $\tilde{F} \in \mathcal{O}(X_0, B_2)$ such that $\tilde{F}|_{M_0} = \tilde{f}$. Finally, we define $F := q(\tilde{F})$.

3.4. The results of Section 3.2 are valid for some subsets of X that are *almost periodic* (in the sense that will be made precise later). For instance, suppose that we are in the setting of Example 2.5(1) of holomorphic almost periodic functions on the covering $p : T \to T_0$, where T is a tube domain. If we consider a different covering map $p_\lambda : T \to T_{0\lambda}$ $(:= p_\lambda(T))$,

$$p_\lambda(z_1, \dots, z_n) := \left(e^{\lambda i z_1}, \dots, e^{\lambda i z_n}\right), \quad z = (z_1, \dots, z_n) \in T, \quad \lambda > 0, \tag{3.2}$$

then we obtain the same algebra $\mathcal{O}_{AP}(T)$. Thus, the assertion of Theorem 3.6 is true also for those almost periodic subsets of T that are periodic (in the sense of the definition of Section 3.2) with respect to covering maps p_λ, $\lambda > 0$. (In fact it is true even for sufficiently small almost periodic holomorphic perturbations of such sets, see details in [BrK2].)

In the next section we develop an approach that allows us to extend Theorem 3.6 to a subclass of almost periodic complex submanifolds of X. Unlike periodic submanifolds of X, the almost periodic submanifolds are defined in terms of algebra $\mathcal{O}_{AP}(X)$.

4. Almost periodic complex submanifolds

4.1. The main result of this section is Theorem 4.1.

We start with the definition of a certain subclass of almost periodic complex submanifolds in X (defined in full generality in Section 4.2).

A complex submanifold $Y \subset X$ of codimension $k \le n$ is called *cylindrical almost periodic* if for each point $x \in X_0$ there exist a simply connected coordinate chart $U_0 \subset X_0$ of $p(x)$ and functions $h_1, \dots, h_k \in \mathcal{O}_{AP}(U)$, $U := p^{-1}(U_0) \cong U_0 \times G$, such that

(1) $Y \cap U = \{y \in U : h_1(y) = \cdots = h_k(y) = 0\}$;

(2) the maximum of moduli of determinants of all $k \times k$ submatrices of the Jacobian matrix $\left(\partial h_i(z,g)/\partial z_j\right)_{1 \le i \le k,\, 1 \le j \le n}$ with $(z, g) \in U$ of the map $h = (h_1, \dots, h_k) : U \to \mathbb{C}^n$ are uniformly bounded away from zero on $Y \cap U$; here $z = (z_1, \dots, z_n)$ are local coordinates on U_0.

The simplest example of such Y is the zero set of a function $h \in \mathcal{O}_{AP}(X)$ such that $|\nabla h(x)| \ge \delta$, $x \in X$, for some $\delta > 0$; here ∇h is defined by differentiation with respect to local coordinates lifted from X_0. Other examples include periodic complex submanifolds of X and their sufficiently small almost periodic holomorphic perturbations and, in the case of a tube domain T, finite unions of non-intersecting complex submanifolds periodic with respect to the action of group $\mathbb{R}^n$ on T by translations and having different periods.

We say that a function f on Y is holomorphic almost periodic if $f \in \mathcal{O}(Y)$ and admits an extension to a continuous almost periodic function on X. The algebra of holomorphic almost periodic on Y functions is denoted by $\mathcal{O}_{AP}(Y)$ (see Definition 4.11 for an intrinsic definition of functions from $\mathcal{O}_{AP}(Y)$).

For instance, for a function $f \in \mathcal{O}(Y)$ assume that there exist an open cover $(U_{0\alpha})_{\alpha\in\Lambda}$ of X_0 and functions $f_\alpha \in \mathcal{O}_{AP}(U_\alpha)$, $U_\alpha := p^{-1}(U_{0\alpha})$, such that $f|_{U_\alpha\cap Y} = f_\alpha|_{U_\alpha\cap Y}$. Then $f \in \mathcal{O}_{AP}(Y)$.

Theorem 4.1. Suppose that $Y \subset X$ is a cylindrical almost periodic submanifold. Let $f \in \mathcal{O}_{AP}(Y)$. Then there exists a function $F \in \mathcal{O}_{AP}(X)$ such that $F|_Y = f$.

For instance, in the setting of Example 2.5(1) of holomorphic almost periodic functions on a tube domain T, suppose that $Y_1, Y_2 \subset T$ are non-intersecting smooth complex hypersurfaces that are periodic with respect to the action of $\mathbb{R}^n$ on T by translations and have different periods, and $f_1 \in \mathcal{O}(Y_1)$, $f_2 \in \mathcal{O}(Y_2)$ are holomorphic periodic with respect to these periods functions. Then there exists a holomorphic almost periodic function $F \in \mathcal{O}_{AP}(T)$ such that $F|_{Y_i} = f_i$, $i = 1, 2$.

Our proof of Theorem 4.1 uses an equivalent presentation of holomorphic almost periodic functions as 'holomorphic' functions on the *fibrewise Bohr compactification* bX of the covering $p : X \to X_0$ introduced in the next section. In Section 5 we outline the proof of Theorem 4.1 for cylindrical almost periodic *hypersurfaces* (see [BrK2] for the proof of the general case).

4.2. In this section we define the fibrewise Bohr compactification $p_b : bX \to X_0$ of the regular covering $p : X \to X_0$.

1. Preliminaries on the Bohr compactification of a group.

The Bohr compactification bG of a topological group G is a compact topological group together with a homomorphism $j : G \to bG$ determined by the universal property

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[d, "\nu"'] & bG \arrow[dl, dotted, "\eta"] \\ H & \end{tikzcd}$$

here H is a compact topological group, and ν is a (continuous) homomorphism. Applying this to $H := U_n$, $n \geq 1$, we obtain that G is maximally almost periodic iff j is an embedding.

The universal property implies that there exists a bijection between sets of finite-dimensional irreducible unitary representations of G and bG. It turn, the Peter-Weyl theorem for $C(bG)$ and von Neumann's approximation theorem for $AP(G)$ (Theorem 1.2) yield that $AP(G) \cong C(bG)$. Thus bG is homeomorphic to the maximal ideal space of algebra $AP(G)$, and $j(G)$ is dense in bG.

Example 4.2. The Bohr compactification $b\mathbb{Z}$ of integers $\mathbb{Z}$ is the inverse limit of a family of compact Abelian Lie groups $\mathbb{T}^k \times \oplus_{l=1}^m \mathbb{Z}/(n_l\mathbb{Z})$, $k, m, n_l \in \mathbb{N}$, where $\mathbb{T}^k := (\mathbb{S}^1)^k$ is the real k-torus. In particular, $b\mathbb{Z}$ is disconnected and has infinite covering dimension.

The projections (homomorphisms) $b\mathbb{Z} \to \mathbb{T}^k \times \oplus_{l=1}^m \mathbb{Z}/(n_l\mathbb{Z})$ are defined by finite families of characters $\mathbb{Z} \to \mathbb{S}^1$. For instance, let λ_1, $\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ be linearly independent over $\mathbb{Q}$ and $\chi_{\lambda_i} : \mathbb{Z} \to \mathbb{S}^1$, $\chi_{\lambda_i}(n) := e^{2\pi i \lambda_i n}$, $i = 1, 2$, be the corresponding characters. Then the map $(\chi_{\lambda_1}, \chi_{\lambda_2}) : \mathbb{Z} \to \mathbb{T}^2$ is extended by continuity to a continuous surjective homomorphism $b\mathbb{Z} \to \mathbb{T}^2$. If λ_1, λ_2 are linearly dependent over $\mathbb{Q}$, then the corresponding extended homomorphism has image in $\mathbb{T}^2$ isomorphic to $\mathbb{S}^1 \times \mathbb{Z}/(m\mathbb{Z})$ for some $m \in \mathbb{N}$.

The right action of G on itself extends uniquely to a (right) action r of G on bG given by the formula $r(g)(\omega) := \omega \cdot j(g^{-1})$, $\omega \in bG$, $g \in G$.

Let $\Upsilon \subset bG$ be a set of representatives of equivalence classes $bG/j(G)$. Each element $\xi \in \Upsilon$ determines a homomorphism (monomorphism if G is maximally almost periodic) $j_\xi : G \to bG$, $j_\xi(g) := \xi \cdot j(g^{-1})$, $g \in G$. In particular, if $\xi = 1$, then $j_\xi = j$. Since $j(G)$ is dense in bG and bG is a group, $j_\xi(G)$ is dense in bG for all $\xi \in \Upsilon$; it is also clear that $j_\xi(G) \cap j_{\xi'}(G) = \varnothing$ for $\xi \neq \xi'$, and bG is covered by subsets $j_\xi(G)$, $\xi \in \Upsilon$.

2. Fibrewise Bohr compactification of the covering.

We retain the notation of Section 3.3. Consider the fibre bundle $p_b : bX \to X_0$ with fibre bG associated to the bundle $p : X \to X_0$. By definition, bX is obtained from the disjoint union $\sqcup_{\gamma\in\Gamma} U_\gamma \times bG$ by the identification of $(x, \omega) \in U_\gamma \times bG$ with $(x, \omega \cdot c_{\delta\gamma}(x)) \in U_\delta \times bG$. The bundle bX will be called the fibrewise Bohr compactification of X.

Let $\xi \in \Upsilon$. Each embedding j_ξ (recall that G is maximally almost periodic) induces local embeddings $U_\gamma \times G \hookrightarrow U_\gamma \times bG$ which, in turn, induce a global embedding $\iota_\xi : X \hookrightarrow bX$. Thus $bX = \sqcup_{\xi\in\Upsilon} \iota_\xi(X)$ and each $\iota_\xi(X)$ is dense in bX.

We define the fibrewise Bohr compactification $b\bar{D}$ of the covering $p : \bar{D} \to \bar{D}_0$, $D_0 \Subset X_0$ is a domain, analogously. Spaces bX and $b\bar{D}$ with a strictly pseudoconvex D_0 are maximal ideal spaces of algebras $\mathcal{O}_{AP}(X)$ and $\mathcal{A}_{AP}(D)$, respectively, see [BrK2] for details.

Definition 4.3. A function $f \in C(bX)$ is called holomorphic if $\iota_\xi^* f \in \mathcal{O}(X)$ for all $\xi \in \Upsilon$. We denote by $\mathcal{O}(bX)$ the algebra of holomorphic functions on bX endowed with the topology of uniform convergence on compact subsets of bX.

A function $f \in C(b\bar{D})$ is called holomorphic if $\iota_\xi^* f \in \mathcal{O}(D) \cap C(\bar{D})$ for all $\xi \in \Upsilon$. The algebra of these functions equipped with sup-norm is denoted by $\mathcal{A}(bD)$.

In fact, we prove that it suffices to require that the above f satisfies $\iota_\xi^* f \in \mathcal{O}(X)$ or $\mathcal{O}(D) \cap C(\bar{D})$ for some $\xi \in \Upsilon$ to get $f \in \mathcal{O}(bX)$ or $\mathcal{A}(bD)$.

The next proposition gives another characterization of holomorphic almost periodic functions.

Proposition 4.4. $\mathcal{O}_{AP}(X) \cong \mathcal{O}(bX)$, $\mathcal{A}_{AP}(D) \cong \mathcal{A}(bD)$.

Since each ι_ξ is a continuous map, $\iota_\xi^{-1}(U) \subset X$ is open for an open $U \subset bX$.

Definition 4.5. A function $f \in C(U)$ is called *holomorphic* if $\iota_\xi^* f \in \mathcal{O}\big(\iota_\xi^{-1}(U)\big)$ for all $\xi \in \Upsilon$.

Let $\mathcal{O}(U)$ denote the algebra of holomorphic on U functions. Clearly a function $f \in C(bX)$ belongs to $\mathcal{O}(bX)$ if and only if each point in bX has a neighbourhood U such that $f|_U \in \mathcal{O}(U)$.

For $U \subset bX$ open, by ${}_U\mathcal{O}$ we denote the sheaf of germs of holomorphic functions on U.

Given a sheaf of modules $\mathcal{F}$ on bX over the sheaf of rings ${}_{bX}\mathcal{O}$ by $H^i(bX, \mathcal{F})$, $i \geq 0$, we denote the Čech cohomology groups of bX with values in $\mathcal{F}$. As usual, $H^0(bX, \mathcal{F})$ is naturally identified with the module $\Gamma(bX, \mathcal{F})$ of global sections of $\mathcal{F}$. We say that the sheaf $\mathcal{F}$ is *coherent* if each point in bX has a neighbourhood of the form $U := p_b^{-1}(U_0)$ with $U_0 \subset X_0$ open over which there exists an exact sequence of sheaves

$$0 \to ({}_{bX}\mathcal{O})^{m_k}|_U \to \cdots \to ({}_{bX}\mathcal{O})^{m_1}|_U \to ({}_{bX}\mathcal{O})^{m_0}|_U \to \mathcal{F}|_U \to 0. \tag{4.3}$$

Theorem 4.6 (Cartan Theorem B)**.** If $\mathcal{F}$ is coherent, then $H^i(bX, \mathcal{F}) = 0$, $i \geq 1$.

We require also the following

Definition 4.7. A continuous rank k complex vector bundle E on bX is called holomorhic if $\iota_\xi^* E$ is a holomorphic vector bundle on X for each $\xi \in \Upsilon$.

In fact, in this definition it suffices to take only one such $\xi \in \Upsilon$. Also, one easily shows that each such E is determined on an open cover of bX by a holomorphic 1-cocycle (in the sense of Definition 4.5) with values in $GL_k(\mathbb{C})$.

Similarly one defines holomorphic sections and (holomorphic) homomorphisms of holomorphic bundles on bX.

In the following definition for a simply connected open set $U_0 \subset X_0$ we naturally identify $p_b^{-1}(U_0)$ with $U_0 \times bG$.

Definition 4.8. A closed subset $Z \subset bX$ is called an *almost periodic complex submanifold* of codimension k if for each point $x \in bX$ there exist its neighbourhood $U = U_0 \times K \subset bX$, where $U_0 \subset X_0$ is open simply connected and $K \subset bG$ is open, and functions $h_1, \dots, h_k \in \mathcal{O}(U)$ such that

(1) $Z \cap U = \{x \in U : h_1(x) = \cdots = h_k(x) = 0\}$;

(2) for each $\omega \in K$ the rank of the map $z \mapsto \big(h_1(z, \omega), \dots, h_k(z, \omega)\big)$ is k at each point of U_0.

In the case the codimension of Z is $k = 1$, then Z is called an *almost periodic complex hypersurface*.

Any almost periodic complex submanifold $Z \subset bX$ satisfies the following properties:

(i) For each $\xi \in \Upsilon$ the set $\iota_\xi^{-1}\big(Z \cap \iota_\xi(X)\big) \subset X$ is a (smooth) complex submanifold;

(ii) For each $\xi \in \Upsilon$ the set $Z \cap \iota_\xi(X)$ is dense in Z.

Definition 4.9. An almost periodic complex submanifold Z is called *cylindrical* if the sets U in Definition 4.8 have form $U := p_b^{-1}(U_0)$.

Let $Z \subset bX$ be an almost periodic complex hypersurface. Then it suffices to require that at least one set U in Definition 4.8 has form $U = p_b^{-1}(U_0)$ for some open $U_0 \subset X_0$ in order for the almost periodic hypersurface Z to be cylindrical (see Theorem 4.18 below).

A subset $Y := \iota_\xi^{-1}\big(Z \cap \iota_\xi(X)\big) \subset X$, where $Z \subset bX$ is an almost periodic complex submanifold, will be called an almost periodic complex submanifold of X. If Z is cylindrical, then Y will be called a cylindrical almost periodic submanifold of X. (This definition is equivalent to the one given in Section 4.1).) Note that even a complex strip contains non-cylindrical almost periodic complex submanifolds, see Section 4.4.

Definition 4.10. Let $Z \subset bX$ be an almost periodic complex submanifold. A function $f \in C(Z)$ is called holomorphic if the functions $\iota_\xi^* f|_{\iota_\xi^{-1}(Z \cap \iota_\xi(X))}$, $\xi \in \Upsilon$, are holomorphic in the usual sense (cf. (i) above).

The algebra of functions holomorphic on Z is denoted by $\mathcal{O}(Z)$.

Definition 4.11. For a given $\xi \in \Upsilon$ we set $Y := \iota_\xi^{-1}(Z) \subset X$, and define the algebra of holomophic almost periodic functions on Y as

$$\mathcal{O}_{AP}(Y) := \iota_\xi^* \mathcal{O}(bX).$$

One can easily see (using a normal family argument and the Tietze-Urysohn extension theorem) that this definition is equivalent to the one given in Section 4.1).

Our proof of Theorem 4.1 is based on Theorem 4.6 and the fact that the sheaf of germs of holomorphic functions vanishing on a cylindrical almost periodic complex submanifold of bX is coherent. It is not yet clear whether the assertion of Theorem 4.1 holds for all almost periodic complex submanifolds of X.

Remark added in November 2010. In the revised version of [BrK2] the authors proved that the statement of Theorem 4.1 is valid for all almost periodic complex submanifolds of X by establishing Cartan Theorem B for 'generalized' coherent sheaves on bX defined as in (4.3) with arbitrary open $U \subset bX$.

4.3. In this section we provide a sufficient condition for an almost periodic complex hypersurface to be determined by a function in $\mathcal{O}_{AP}(X)$ (see Theorem 4.17). We formulate this condition for a wider class of objects, *almost periodic divisors*. Almost periodic divisors were studied intensively (see, e.g., [Lv, FRR, Fav1, F2R] and references therein), originally in the case $n = 1$, in connection with problems of distribution of zeros of entire functions. Using the developed technique, we extend some of the results in [FRR, Fav1, F2R].

Definition 4.12. An (effective) almost periodic (Cartier) divisor D on bX is given by an open cover (U_α) of bX and a collection of (non-identically zero) holomorphic functions $f_\alpha \in \mathcal{O}(U_\alpha)$ such that $f_\alpha / f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

Here and below $\mathcal{O}^*(U)$ stands for the set of nowhere zero holomorphic functions defined on an open subset $U \subset bX$.

The set of almost periodic divisors on bX is denoted by $\mathrm{Div}(bX)$.

Divisors $D := (U_\alpha, f_\alpha)$ and $G := (W_\gamma, g_\gamma)$ from $\mathrm{Div}(bX)$ are called *equivalent* if there exists a refinement $\{V_\beta\}$ of both covers (U_α) and (W_γ) and holomorphic functions $c_\beta \in \mathcal{O}^*(V_\beta)$ such that

$$f_\alpha|_{V_\beta} = c_\beta \cdot g_\gamma|_{V_\beta} \qquad \text{for} \quad V_\beta \subset U_\alpha \cap W_\gamma.$$

If $D = (U_\alpha, f_\alpha)$, then the 1-cocycle $\{f_\alpha/f_\beta\}$ determines a holomorphic line bundle E_D on bX. The family $\{f_\alpha\}$ (uniquely) determines a holomorphic section s_D of E_D. Therefore there exists a bijective correspondence between $\mathrm{Div}(bX)$ and the set of holomorphic sections of holomorphic line bundles on bX. Then divisors D, G are equivalent if and only if there exists an isomorphism $\eta : E_D \to E_G$ of holomorphic line bundles such that $\eta^*(s_G) = s_D$.

Given a divisor $D \in \mathrm{Div}(bX)$ and $\xi \in \Upsilon$ the pullback $D' := \iota_\xi^* D$ is a (standard) divisor on X. Using the Hurwitz theorem one can show that for a fixed $\xi \in \Upsilon$ the divisor D can be uniquely recovered from D'.

Definition 4.13. A divisor on X of the form $\iota_\xi^* D$, $\xi \in \Upsilon$, $D \in \mathrm{Div}(bX)$, is called *almost periodic*.

By $\mathrm{Div}_{AP}(X)$ we denote the collection of all almost periodic divisors on X. The equivalence relation on $\mathrm{Div}_{AP}(X)$ is defined as follows: divisors $D_1' := \iota_\xi^* D_1$ and $D_2' := \iota_\xi^* D_2$ from $\mathrm{Div}_{AP}(X)$ are equivalent iff D_1 and D_2 are equivalent in $\mathrm{Div}(bX)$.

An equivalent definition of almost periodic divisors on X can be given in terms of currents. In the setting of Example 2.5(1), i.e., when X is a tube domain, this definition coincides with the classical one (cf. [Lv, F2R]).

We assume without loss of generality that the open sets U_α in Definition 4.12 are chosen so that $U_\alpha = p_b^{-1}(U_0) \cong U_{0,\alpha} \times K$, where $U_0 \Subset X_0$ is a simply connected coordinate chart and $K \subset bG$ is open. As before, we identify U_α with $U_{0,\alpha} \times K$.

Definition 4.14. A divisor $D \in \mathrm{Div}(bX)$ is called *smooth* if for all $(z, \eta) \in U_\alpha = U_{0,\alpha} \times K$ moduli of gradients with respect to z of functions f_α in Definition 4.12 satisfy $|\nabla_z f_\alpha(z, \eta)| > 0$.

A divisor in $\mathrm{Div}_{AP}(X)$ is called smooth if it has the form $\iota_\xi^* D$, where D is smooth.

Definition 4.15. A divisor $D \in \mathrm{Div}(bX)$ is called *cylindrical* if the open sets U_α in Definition 4.12 have the form $U_\alpha := p_b^{-1}(U_{0,\alpha})$, $U_{0,\alpha} \subset X_0$.

A divisor in $\mathrm{Div}_{AP}(X)$ is called cylindrical if it has the form $\iota_\xi^* D$, where D is cylindrical.

If in Definition 4.12 the open cover consists of a single set bX, then the divisor D is called *principal*. It follows that a principal almost periodic divisor is determined by a single (nonzero) function on bX (in particular, each principal almost periodic divisor is cylindrical). Accordingly, an almost periodic divisor in $\mathrm{Div}_{AP}(X)$ is called principal if it is determined by a single function of the form $\iota_\xi^* f$ with $f \in \mathcal{O}(bX)$.

Any divisor $D' := \iota_\xi^* D \in \mathrm{Div}_{AP}(X)$ is defined by the holomorphic section $\iota_\xi^* s_D$ of the complex vector bundle $\iota_\xi^* E_D$ on X. The set of zeros of $\iota_\xi^* s_D$ is called the support of the divisor D' and is denoted by $\mathrm{supp}(D') \subset X$. (Similarly we introduce the definition of support for elements of $\mathrm{Div}(bX)$.)

Note that the equivalence relation on $\mathrm{Div}_{AP}(X)$ preserves supports: if D, D' are equivalent, then $\mathrm{supp}(D) = \mathrm{supp}(D')$. The same is valid for divisors in $\mathrm{Div}(bX)$.

The next statement relates the notions of an almost periodic complex hypersurface (cf. Definition 4.8) and of an almost periodic divisor.

Proposition 4.16. Let $Z \subset bX$ be an almost periodic complex hypersuface. Let U_α and $h_\alpha \in \mathcal{O}(U_\alpha)$ be open sets and holomorphic functions determining Z in Definition 4.8.

(1) The collection (U_α, h_α) determines a smooth almost periodic divisor $D_Z \in \mathrm{Div}(bX)$, such that $Z = \mathrm{supp}(D_Z)$.
(2) If D_1, $D_2 \in \mathrm{Div}(bX)$ are smooth almost periodic divisors such that
$$\mathrm{supp}(D_1) = \mathrm{supp}(D_2),$$
then D_1, D_2 are equivalent. In particular, D_Z is determined by the hypersurface Z uniquely up to equivalence.
(3) The support $Z := \mathrm{supp}(D)$ of a smooth almost periodic divisor $D \in \mathrm{Div}(bX)$ is an almost periodic hypersurface in bX, and any representing divisor D_Z is equivalent to D.
(4) The hypersurface Z is cylindrical if and only if D_Z is equivalent to a smooth cylindrical almost periodic divisor.

Similar to (1)–(4) correspondences hold between almost periodic complex hypersurfaces and almost periodic divisors in X.

The proof of Proposition 4.16 uses the fact that if $U \subset bX$ is open, $h \in \mathcal{O}(U)$ is the defining function for $Z \cap U$ as in Definition 4.8, and a function $f \in \mathcal{O}(U)$ vanishes on $Z \cap U$, then $f/h \in \mathcal{O}(U)$ (see details in [BrK2]).

Theorem 4.17. Suppose that $X_0' \subset X_0$ is an open submanifold homotopically equivalent to X_0 and $X' := p^{-1}(X_0')$. Let $D \in \mathrm{Div}_{AP}(X)$. If the restriction $D|_{X'}$ is a principal almost periodic divisor, then D is equivalent to a principal almost periodic divisor.

In particular, if $\mathrm{supp}(D) \cap X' = \varnothing$, then D is equivalent to a principal almost periodic divisor.

Particular cases of this result for X, X' tube domains (cf. Example 2.5(1)) were proved in [FRR] ($n = 1$), and [Fav1] ($n \geq 1$). The proof in [FRR] uses Arakelyan's theorem and gives an explicit construction of the almost periodic function f determining the principal divisor. The proof in [Fav1] uses a sheaf-theoretic argument; our proof employs a similar argument.

Theorem 4.18. Let $D \in \mathrm{Div}(bX)$ be an almost periodic divisor. Then the following is true:

(1) If any divisor equivalent to D is non-cylindrical, then the projection of $\mathrm{supp}(D)$ onto X_0 coincides with X_0.
(2) If one of the sets U_α in Definition 4.12 for D has form $U_\alpha := p_b^{-1}(U_{0,\alpha})$, then D is equivalent to a cylindrical divisor.

The converse to assertion (1) of Theorem 4.18 is not true (e.g., one can modify the construction in Section 4.4 to produce an example of a cylindrical divisor $D \in \mathrm{Div}(bX)$ such that the projection of $\mathrm{supp}(D)$ onto X_0 coincides with X_0).

The following problem is considered in [FRR]: describe the class $\mathcal{C}$ of holomorphic functions on T such that the set of elements of equivalence classes of all *principal* divisors determined by functions from $\mathcal{C}$ coincides with $\mathrm{Div}_{AP}(X)$. It was proved in [FRR] that $\mathcal{C} = \{f \in \mathcal{O}(T) : |f| \in C_{AP}(T)\}$. We extend this result as follows.

Let $D_f \in \mathrm{Div}(X)$ denote the divisor determined by a function $f \in \mathcal{O}(X)$.

Theorem 4.19. Suppose that X_0 is a non-compact Riemann surface. For each $D \in \mathrm{Div}_{AP}(X)$ there exists a function $f \in \mathcal{O}(X)$ with $|f| \in C_{AP}(X)$ and a divisor $D' \in \mathrm{Div}(X)$ equivalent to D such that $D' = D_f$.

Conversely, for a complex manifold X_0 (of any dimension $n \geq 1$) the divisor D_f defined by a function $f \in \mathcal{O}(X)$ with $|f| \in C_{AP}(X)$ belongs to $\mathrm{Div}_{AP}(X)$.

In the case $n > 1$ the first assertion of the theorem is no longer true, at least in tube domains (see [Fav2] where also an explicit necessary and sufficient condition for validity of this assertion is obtained in the case of a tube domain). The proofs in [FRR, Fav2] use some properties of almost periodic currents (distributions). Our proof is mostly sheaf-theoretic, see [BrK2].

4.4. In this section we construct a smooth non-cylindrical almost periodic hypersurface Z in a regular covering $p : X \to X_0$ of a Riemann surface of finite type X_0, having the deck transformation group $\mathbb{Z}$. We assume that X_0 is a relatively compact subdomain of a larger (non-compact) Riemann surface $\tilde{X}_0$ whose fundamental group satisfies $\pi_1(\tilde{X}_0) \cong \pi_1(X_0)$.

In general, since a regular covering $X' \to X_0$ with a maximally almost periodic deck transformation group whose commutator subgroup has infinite index (e.g., the universal covering of X_0) can be factorized via a regular covering $p : X \to X_0$ with the deck transformation group $\mathbb{Z}$, the pullback of Z by the factorizing covering map is a smooth non-cylindrical hypersurface in X'.

Note that the covering of Example 2.5(1) with $n = 1$, i.e., a complex strip covering an annulus, is the regular covering of the above form.

Let us briefly describe this construction. Instead of dealing with bundle bX we work with bundle $b_{\mathbb{T}^2}X$ defined as follows. Choose two characters $\chi_1, \chi_2 : \mathbb{Z} \to \mathbb{S}^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$ such that the homomorphism $(\chi_1, \chi_2) : \mathbb{Z} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is an embedding with dense image. Consider the fibre bundle $b_{\mathbb{T}^2}X$ over X_0 with fibre $\mathbb{T}^2$ associated with the homomorphism (χ_1, χ_2). The bundle $b_{\mathbb{T}^2}X$ is embedded into a holomorphic fibre bundle $b_{(\mathbb{C}^*)^2}X$ with fibre $(\mathbb{C}^*)^2$, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, associated with the composite of the embedding homomorphism $\mathbb{T}^2 \hookrightarrow (\mathbb{C}^*)^2$ and (χ_1, χ_2). Moreover, the covering X of X_0 admits an injective C^∞ map into $b_{\mathbb{T}^2}X$ with dense image and the composite of this map with the embedding of $b_{\mathbb{T}^2}X$ into $b_{(\mathbb{C}^*)^2}X$ is an injective holomorphic map $X \to b_{(\mathbb{C}^*)^2}X$. Further, the bundle $b_{(\mathbb{C}^*)^2}X$ admits a holomorphic trivialization $\eta : b_{(\mathbb{C}^*)^2}X \to X_0 \times (\mathbb{C}^*)^2$. We choose $\chi_1(1)$ and $\chi_2(1)$ so close to $1 \in \mathbb{S}^1$ that the image $\eta(b_{\mathbb{T}^2}X) \subset X_0 \times (\mathbb{C}^*)^2$ is sufficiently close to $X_0 \times \mathbb{T}^2$. Thus identifying X (by means of holomorphic injection $X \hookrightarrow b_{(\mathbb{C}^*)^2}X \stackrel{\eta}{\to} X_0 \times (\mathbb{C}^*)^2$) with a subset of $X_0 \times (\mathbb{C}^*)^2$, we obtain that X is sufficiently close to $X_0 \times \mathbb{T}^2$.

Next, we construct a smooth complex hypersurface in $X_0 \times (\mathbb{C}^*)^2$ such that in each cylindrical coordinate chart $U_0 \times (\mathbb{C}^*)^2$ on $X_0 \times (\mathbb{C}^*)^2$ for $U_0 \Subset X_0$ simply connected it cannot be determined as the set of zeros of a holomorphic function on $U_0 \times (\mathbb{C}^*)^2$. Intersecting this hypersurface with X we obtain a non-cylindrical almost periodic hypersurface in X.

To construct such a hypersurface in $X_0 \times (\mathbb{C}^*)^2$, we determine a smooth divisor in $(\mathbb{C}^*)^2$ that has a non-zero Chern class (i.e., it cannot be given by a holomorphic function on $(\mathbb{C}^*)^2$), and whose support intersects the real torus $\mathbb{T}^2 \subset (\mathbb{C}^*)^2$ transversely. Then we take the pullback of this divisor with respect to the projection $X_0 \times (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ to get the desired hypersurface.

In the following two sections we provide the details of the construction.

I. Characters $\chi_1, \chi_2 : \mathbb{Z} \to \mathbb{S}^1 \subset \mathbb{C}^*$ in the definition of the fibre bundle $b_{\mathbb{T}^2}X$ can be chosen in the form $\chi_i := e^{i\lambda_i\varphi}$, $\lambda_i \in \mathbb{R}$, $i = 1, 2$, with $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ linearly independent over $\mathbb{Q}$. This guarantees that the homomorphism $\chi := (\chi_1, \chi_2) : \mathbb{Z} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is an embedding with dense image. Recall that $p_{b,\mathbb{T}^2} : b_{\mathbb{T}^2}X \to X_0$ is the bundle with fibre $\mathbb{T}^2$ associated with χ. The homomorphism χ induces a C^∞ injection ι_0^χ of the covering $p : X \to X_0$ into $b_{\mathbb{T}^2}X$ such that $\iota_0^\chi(X)$ is dense in $b_{\mathbb{T}^2}X$. Extending χ to a continuous surjective homomorphism $b\mathbb{Z} \to \mathbb{T}^2$ we extend by continuity ι_0^χ to a surjective homomorphism of fibre bundles $Q : bX \to b_{\mathbb{T}^2}X$. In particular, $\iota_0^\chi = Q \circ \iota_0$, cf. Section 4.2.

Next, we introduce notions of a holomorphic function and a divisor on $b_{\mathbb{T}^2}X$.

A function $f \in C(U)$ on an open subset $U \subset b_{\mathbb{T}^2}X$ is called *holomorphic* if its pullback $(\iota_0^\chi)^* f$ is holomorphic on $X \cap (\iota_0^\chi)^{-1}(U)$ (cf. Definition 4.3 and the remark after).

The definition of a divisor on $b_{\mathbb{T}^2}X$ (or on its open subset) is analogous to that of an almost periodic divisor on bX. Similarly, we define equivalence relation and smooth and cylindrical divisors on $b_{\mathbb{T}^2}X$.

Let $\mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$ denote the collection of divisors on $b_{\mathbb{T}^2}X$. We clearly have that the pullback by Q of a divisor from $\mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$ belongs to $\mathrm{Div}(bX)$. In particular, if $H \in \mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$, then $(\iota_0^\chi)^* H \in \mathrm{Div}_{AP}(X)$. We also have the following

Proposition 4.20. Let $H \in \mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$.

(1) If divisor H is smooth, then the almost periodic divisor $(\iota_0^\chi)^* H$ is smooth.

(2) If divisor H is cylindrical, then the almost periodic divisor $(\iota_0^\chi)^* H$ is cylindrical.

The next statement justifies the choice of the bundle $b_{\mathbb{T}^2}X$ (instead of bX) in our construction.

Proposition 4.21. Suppose that divisor $H \in \mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$ is not equivalent to a cylindrical divisor on $b_{\mathbb{T}^2}X$. Then the almost periodic divisor $(\iota_0^\chi)^* H \in \mathrm{Div}_{AP}(X)$ is not equivalent to a cylindrical almost periodic divisor on X.

It follows from Propositions 4.20(1) and 4.21 that it suffices to construct a smooth divisor $H \in \mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$ not equivalent to a cylindrical divisor on $b_{\mathbb{T}^2}X$. Then its pullback $D := (\iota_0^\chi)^* H$ is a smooth non-cylindrical almost periodic divisor on X. By Proposition 4.16(3) (cf. remark after), the support $Z := \mathrm{supp}(D)$ of divisor D is an almost periodic complex hypersurface in X. Moreover, Z is non-cylindrical, for otherwise, by Proposition 4.16(4) its representing divisor $D_Z \in \mathrm{Div}_{AP}(X)$ is equivalent to a (smooth) cylindrical divisor. However, by Proposition 4.16(3) divisor D is equivalent to divisor D_Z giving a contradiction.

II. Now we construct divisor H on $b_{\mathbb{T}^2}X$ with the required properties.

(A) First, note that the bundle $b_{\mathbb{T}^2}X$ is embedded into a holomorphic fibre bundle $p_{b,(\mathbb{C}^*)^2} : b_{(\mathbb{C}^*)^2}X \to X_0$ with fibre $(\mathbb{C}^*)^2$ associated with the composite of the embedding homomorphism $\mathbb{T}^2 \hookrightarrow (\mathbb{C}^*)^2$ and χ. Moreover, the composite of this embedding with ι_0^χ is a holomorphic injective map $X \to b_{(\mathbb{C}^*)^2}X$.

Proposition 4.22. For any neighbourhood $U \Subset (\mathbb{C}^*)^2$ of $\mathbb{T}^2$ there exist sufficiently small λ_1, λ_2 in the definition of χ and a holomorphic trivialization $\eta : b_{(\mathbb{C}^*)^2}X \to X_0 \times (\mathbb{C}^*)^2$ of the corresponding to χ bundle $b_{(\mathbb{C}^*)^2}X$ such that $\eta(b_{\mathbb{T}^2}X) \subset X_0 \times U$ (i.e., under a suitable definition, $\eta(b_{\mathbb{T}^2}X)$ is sufficiently close to $X_0 \times \mathbb{T}^2$).

In what follows, we assume that $\max\{\lambda_1, \lambda_2\} > 0$ and is sufficiently small.

(B) We define the equivalence relation on the set of divisors on $(\mathbb{C}^*)^2$ analogously to that for divisors on bX (cf. Section 4.3).

Let G be a divisor on $(\mathbb{C}^*)^2$ whose support intersects the real torus $\mathbb{T}^2 \subset (\mathbb{C}^*)^2$ transversely (cf. Proposition 4.24 below). Let $\pi : X_0 \times (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ be the natural projection. We define a smooth divisor E on the complex manifold $b_{(\mathbb{C}^*)^2}X$ as pullback with respect to the holomorphic map $\pi \circ \eta$ of the divisor G on $(\mathbb{C}^*)^2$. By Proposition 4.22, since $\mathrm{supp}(G)$ intersects $\mathbb{T}^2 \subset (\mathbb{C}^*)^2$ transversely, for all sufficiently small λ_1, $\lambda_2 > 0$ the support of the divisor E has a non-empty intersection with the subbundle $b_{\mathbb{T}^2}X$ of $b_{(\mathbb{C}^*)^2}X$.

We define divisor H as the restriction of divisor E to $b_{\mathbb{T}^2}X$. Specifically, if divisor G on $(\mathbb{C}^*)^2$ is determined by an open cover (U_α) and functions $\{f_\alpha \in \mathcal{O}(U_\alpha)\}$, then divisor H is determined by the open cover $\big(b_{\mathbb{T}^2}X \cap (\pi\circ\eta)^{-1}(U_\alpha)\big)$ and the functions $\big((\pi\circ\eta)^* f_\alpha\big)|_{b_{\mathbb{T}^2}X\cap(\pi\circ\eta)^{-1}(U_\alpha)}$.

Clearly, each set $b_{\mathbb{T}^2}X \cap (\pi\circ\eta)^{-1}(U_\alpha)$ is open in $b_{\mathbb{T}^2}X$, and the ratios of functions $(\pi\circ\eta)^* f_\alpha$ on intersections of their domains are nowhere zeros. However, to claim that H is a well-defined divisor on $b_{\mathbb{T}^2}X$ we must prove that these functions are holomorphic.

Proposition 4.23. (1) Functions $(\pi\circ\eta)^* f_\alpha$ are holomorphic on open subsets $b_{\mathbb{T}^2}X \cap (\pi\circ\eta)^{-1}(U_\alpha)$ of $b_{\mathbb{T}^2}X$. Thus, divisor H is well defined.

(2) If divisor $G \in \mathrm{Div}(\mathbb{C}^*)^2$ is smooth, then divisor $H \in \mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$ is also smooth.

(3) If divisor G is not equivalent to a principal divisor, then divisor H is not equivalent to a cylindrical divisor.

Proposition 4.24. There exists a divisor G on $(\mathbb{C}^*)^2$ which is smooth, not equivalent to a principal divisor, and whose support intersects the real torus $\mathbb{T}^2 \subset (\mathbb{C}^*)^2$ transversely in finitely many points.

It follows from Propositions 4.23 and 4.24 that there exists a smooth divisor H on $b_{\mathbb{T}^2}X$ which is not equivalent to a cylindrical divisor. This concludes our construction (cf. the discussion in the end of part I).

In the case $X := T$ is a tube domain, cf. Example 2.5(1), the construction of a non-cylindrical hypersurface is much simpler. In fact, one can show that there exists $\lambda > 0$ such that any periodic hypersurface $Y := p^{-1}(Y_0) \subset T$, $p = p_{2\pi}$, where Y_0 cannot be determined by a single function in $\mathcal{O}(T_0)$, is a non-cylindrical almost periodic hypersurface with respect to the projection $p_\lambda : T \to T_{0\lambda}$, see (3.2).

5. Proofs

Unless specified otherwise, in the proofs below a holomorphic function $f \in \mathcal{O}(X)$ is called almost periodic (written $f \in \mathcal{O}_{AP}(X)$) if it is almost periodic in the sense of Definition 2.2. We prove the equivalence of Definitions 2.1 and 2.2 in Proposition 5.4.

Proof of Proposition 3.9. Let us establish the first isomorphism. It is easy to see that any function $f \in \mathcal{O}_{AP}(X)$ is locally Lipschitz with respect to the semi-metric d' (see Section 3.2), i.e.,

$$|f(x_1,g) - f(x_2,g)| \le Cd'\big((x_1,g),(x_2,g)\big) := Cd(x_1,x_2) \tag{5.4}$$

for all (x_1,g), $(x_2,g) \in W_0 \times G \cong p^{-1}(W_0)$, where $W_0 \Subset X_0$ is a simply connected coordinate chart. (Here C depends on d and W_0 only.) We denote $f_{x_0} := f|_{p^{-1}(x_0)} \in AP(G)$, $x_0 \in X_0$, and define

$$\tilde{f}(x_0) := f_{x_0}, \quad x_0 \in X_0.$$

Then $\tilde{f}$ is a section of bundle CX. From (5.4) for any linear functional $\varphi \in (AP(G))^*$ we have $\varphi(\tilde{f}(x)(g)) := \varphi(f(x,g)) \in \mathcal{O}(W_0)$, $g \in G$, $x \in W_0 \Subset X_0$, a simply connected coordinate chart, cf. [Lin] for similar arguments. Thus $\tilde{f}$ is a holomorphic section of CX. Reversing these arguments we obtain that any holomorphic section of CX determines an almost periodic holomorphic function on X.

The proof of the second isomorphism is similar. □

Proof of Proposition 4.4. We conduct the proof for the first isomorphism only (for the second one it is analogous). Fix an element $\xi \in \Upsilon$ and define an algebra homomorphism $i : \mathcal{O}(bX) \to \mathcal{O}(X)$ by the formula

$$i(\hat{f}) := \iota_\xi^* \hat{f}, \quad \hat{f} \in \mathcal{O}(bX).$$

Since

$$i(\hat{f})|_{p^{-1}(x_0)} = j_\xi^*\big(\hat{f}|_{p_b^{-1}(x_0)}\big) \in AP(G), \quad x_0 \in X_0,$$

$i(\hat{f}) \in \mathcal{O}_{AP}(X)$, i.e., i maps $\mathcal{O}(bX)$ into $\mathcal{O}_{AP}(X)$. To define the inverse map i^{-1}, we extend a function $f \in \mathcal{O}_{AP}(X)$ from $\iota_\xi(X)$ to bX, and then show that the corresponding extension $\hat{f}$ belongs to $\mathcal{O}(bX)$. Since $\iota_\xi(X)$ is dense in bX (see Section 4.2), the required result will follow.

Given $f \in \mathcal{O}_{AP}(X)$ denote $f_{x_0} := f|_{p^{-1}(x_0)}$ and then define $\hat{f}_{x_0}$ to be the extension of f_{x_0} from $p^{-1}(x_0) \cong G$ to $p_b^{-1}(x_0) \cong bG$ so that $j_\xi^* \hat{f}_{x_0} = f_{x_0}$. The family of the extended functions over points of X_0 determines a function $\hat{f}$ on bX such that $\hat{f}(x) = \hat{f}_{x_0}(x)$ for $x_0 := p_b(x)$. Using a normal family argument one shows that $\hat{f} \in \mathcal{O}(bX)$, see, e.g., [Lin] or [BrK1, Lemma 2.3] for similar results. Clearly, f is such that $\iota_\xi^* \hat{f} = f$. Thus i is an isomorphism. □

Proof of Theorem 2.3. We prove the second assertion only (the proof of the first one is analogous). By Theorem 3.8, under the hypothesis of the theorem any function $f \in \mathcal{O}_{AP}(X)$ extends to a function $\tilde{f} \in \mathcal{O}_{AP}(\tilde{X})$; if f is bounded, then $\tilde{f}$ is bounded as well. Thus, without loss of generality we may assume that $\tilde{X}_0$ coincides with the ultraliouville manifold X_0. We have to show that if $f \in \mathcal{O}_{AP}(X)$ is bounded, then f is constant.

Our proof relies on the maximum modulus principle for holomorphic functions in $\mathcal{O}(bX)$. We use the isomorphism $\mathcal{O}_{AP}(X) \cong \mathcal{O}(bX)$ established in Proposition 4.4.

Proposition 5.1 (Maximum modulus principle)**.** Let X_0 be a connected complex manifold. Suppose that the modulus $|f|$ of a function $f \in \mathcal{O}(bX)$ attains its maximum at a point $y \in bX$. Then f is constant.

Proof. Let $\xi \in \Upsilon$ be such that $y \in \iota_\xi(X)$. Then the modulus of function $\iota_\xi^* f \in \mathcal{O}_{AP}(X)$ attains its maximum at $\iota_\xi^{-1}(y) \in X$, an interior point of X. By the usual maximum modulus principle for holomorphic functions $\iota_\xi^* f \equiv \text{const}$, so f is as well (because $\iota_\xi(X)$ is dense in bX). □

Next, define

$$\tilde{h}(y) := \sup_{g \in G}\{|f(g \cdot y)|\}, \quad y \in X.$$

Since $\{y \mapsto f(g \cdot y)\,,\, y \in X\}_{g \in G}$ is a family of uniformly bounded holomorphic functions, $\tilde{h}$ is a continuous plurisubharmonic function invariant with respect to the action of G on X, cf. [Lin] for a similar argument. Hence there exists a bounded continuous plurisubharmonic function h on X_0 such that $\tilde{h} = p^* h$. In particular, $\tilde{h} \equiv M\,(\geq 0)$, because X_0 is ultraliouville.

Note that $M = \sup_{x \in bX} |\hat{f}(x)|$, where $\hat{f} \in \mathcal{O}(bX)$ stands for the extension of f to bX. For otherwise, there exists a point $x_0 \in bX$ such that $|\hat{f}(x_0)| > M$. Let $z_0 := p_b(x_0) \in X_0$, then

$$M = h(z_0) := \sup_{x \in p^{-1}(z_0)} |f(x)| = \max_{x \in p_b^{-1}(z_0)} |\hat{f}(x)| \geq |\hat{f}(x_0)| > M,$$

a contradiction. Thus, $\hat{f}$ attains its maximum at a point in bX. By Proposition 5.1 $\hat{f}$ (and hence f) is constant. □

Sketch of the proof of Theorem 4.1 for cylindrical hypersurfaces. We use the following two statements, which are only formulated here, see [BrK2] for their proofs.

Let $Z \subset bX$ be a cylindrical complex almost periodic hypersurface (cf. Definition 4.9).

Proposition 5.2. Let $f \in \mathcal{O}(Z)$. Then for any point $z \in Z$ there exists its neighbourhood $V \subset bX$ and a function $F_V \in \mathcal{O}(V)$ such that $F|_{Z \cap V} = f|_{Z \cap V}$.

The proof of Proposition 5.2 employs an analogue of the inverse function theorem for holomorphic maps between open subsets of bX. Using this theorem, we also prove

Proposition 5.3. Let $U := p_b^{-1}(U_0)$, $U_0 \subset X_0$ be open, $h \in \mathcal{O}(U)$ be the defining function for $Z \cap U$ (cf. Definition 4.9). Suppose that a function $q \in \mathcal{O}(U)$ vanishes on $Z \cap U$, then $q = ph$ for some $p \in \mathcal{O}(U)$.

Let $\mathcal{F}_Z$ be the sheaf of germs of holomorphic functions on bX vanishing on Z. By Proposition 5.3 the sequence

$$0 \to {}_{bX}\mathcal{O}|_U \to \mathcal{F}_Z|_U \to 0,$$

where the second map is given by multiplication by $h \in \mathcal{O}(U)$, is exact, i.e., sheaf $\mathcal{F}_Z$ is coherent. By Proposition 5.2, for a function $f \in \mathcal{O}(Z)$ there exist an open cover (V_α) of bX and functions $F_{V_\alpha} \in \mathcal{O}(V_\alpha)$ such that $F|_{Z \cap V_\alpha} = f|_{Z \cap V_\alpha}$. We define a 1-cocycle with values in $\mathcal{F}_Z$ by the formulas $G_{\alpha\beta} := F_{V_\alpha} - F_{V_\beta}$ on $V_\alpha \cap V_\beta$. Applying to the cohomology class defined by this cocycle Theorem 4.6 and passing to a refinement of (V_α), if necessary, we may assume without loss of generality that $H^1\big((V_\alpha), \mathcal{F}_Z\big) = 0$. Thus, there exist functions $H_\alpha \in \mathcal{O}(V_\alpha)$ vanishing on $Z \cap V_\alpha$ such that $G_{\alpha\beta} = H_\alpha - H_\beta$ on $V_\alpha \cap V_\beta$. It follows that the family $\{F_{V_\alpha} - H_\alpha \in \mathcal{O}(V_\alpha)\}$ determines a function $F \in \mathcal{O}(bX)$ such that $F|_Z = f$. □

Proposition 5.4. Definitions 2.1 and 2.2 are equivalent.

Proof. Clearly, a function $f \in \mathcal{O}(X)$ almost periodic in the sense of Definition 2.1 is almost periodic in the sense of Definition 2.2. Conversely, suppose that $f \in \mathcal{O}(X)$ is almost periodic in the sense of Definition 2.2. A neighbourhood $U \subset X$ in Definition 2.1 has the form $p^{-1}(U_0) \cong U_0 \times G$, where $U_0 \Subset X_0$ is a simply connected coordinate chart on X_0. We naturally identify U with $U_0 \times G$. Then we must show that the family of translates $\{(z,h) \mapsto f(z,gh),\ z \in U_0,\ h \in G\}_{g \in G}$ is relatively compact. By Proposition 4.4, function f admits an extension $\hat{f} \in \mathcal{O}(bX)$. We prove the relative compactness of the family $\{(z,\xi) \mapsto \hat{f}(z,\eta\xi),\ z \in U_0,\ \xi \in bG\}_{\eta \in bG}$. Since $G \hookrightarrow bG$, this implies the required.

Indeed, the family $\{(z,\xi) \mapsto \hat{f}(z,\eta\xi),\ z \in U_0,\ \xi \in bG\}_{\eta \in bG}$ is uniformly bounded. Its equicontinuity follows from uniform continuity of $\hat{f}$, since $\bar{U}_0 \times bG$ is compact. Arzelà-Ascoli theorem now implies the required. □

Proof of Proposition 4.21. Our proof consists of three parts.

1. By definition, divisor H is not equivalent to a cylindrical divisor if there exists a cylindrical neighbourhood $U := p_{b,\mathbb{T}^2}^{-1}(U_0) \subset b_{\mathbb{T}^2}X$, where $U_0 \subset X_0$ is open simply connected, such that the restriction $H|_U$ of H to U is not equivalent to any principal divisor on U (i.e., a divisor defined by a single function in $\mathcal{O}(U)$).

We can reformulate the latter statement in terms of the Chern classes of divisors on U. By definition the Chern class $c_{U,\mathbb{T}^2}(D) \in H^2(U,\mathbb{Z})$ of a divisor D on U is the Chern class of the line bundle associated with D (i.e., constructed by transition functions which are the ratios of holomorphic functions on an open cover of U determining D). If D is principal, then the associated line bundle is topologically trivial; hence, $c_{U,\mathbb{T}^2}(D) = 0$.

Therefore, if $H|_U$ is equivalent to a principal divisor, then $c_{U,\mathbb{T}^2}(H|_U) = 0$.

2. We have a surjective map $Q : bX \to b_{\mathbb{T}^2}X$, cf. Section 4.4(1), such that if $D \in \mathrm{Div}\big(b_{\mathbb{T}^2}X\big)$, then $Q^*D \in \mathrm{Div}(bX)$. By definition, divisor $(\iota_0^X)^*H \in \mathrm{Div}_{AP}(X)$ is cylindrical if and only if $Q^*H \in \mathrm{Div}(bX)$ is cylindrical (cf. Definition 4.9).

As in part 1, we define Chern classes $c_{Q^{-1}(U)}(D) \in H^2(Q^{-1}(U),\mathbb{Z})$ of divisors $D \in \mathrm{Div}(bX)$ (note that $Q^{-1}(U) = p_b^{-1}(U_0)$). Therefore, if divisor $Q^*D|_{Q^{-1}(U)}$ is equivalent to a principal divisor, then $c_{Q^{-1}(U)}(Q^*D|_{Q^{-1}(U)}) = 0$. By the functoriality of Chern classes we have

$$c_{Q^{-1}(U)}(Q^*D|_{Q^{-1}(U)}) = Q^*\big(c_{U,\mathbb{T}^2}(H|_U)\big),$$

where $Q^* : H^2(U,\mathbb{Z}) \to H^2(Q^{-1}(U),\mathbb{Z})$ is the map induced by $Q|_U$.

3. Thus, we must prove that if $c_{U,\mathbb{T}^2}(H|_U) \neq 0$, then $Q^*\big(c_{U,\mathbb{T}^2}(H|_U)\big) \neq 0$. Without loss of generality we may assume that U_0 is contractible. Hence, $U \cong U_0 \times \mathbb{T}^2$ is homotopic to $\mathbb{T}^2$ and $Q^{-1}(U) \cong U_0 \times b\mathbb{Z}$ is homotopic to $b\mathbb{Z}$.

Further, for a fixed point $x \in U_0$, the embeddings $\mathbb{T}^2 \cong p_{b,\mathbb{T}^2}^{-1}(x) \hookrightarrow U$ and $b\mathbb{Z} \cong p_b^{-1}(x) \hookrightarrow Q^{-1}(U)$ induce isomorphisms of the corresponding cohomology groups. Identifying under these isomorphisms $c_{U,\mathbb{T}^2}(H|_U)$ with a non-zero element

$\eta \in H^2(\mathbb{T}^2,\mathbb{Z})$ and $Q^*\big(c_{U,\mathbb{T}^2}(H|_U)\big)$ with $q^*(\eta) \in H^2(b\mathbb{Z},\mathbb{Z})$, where $q := Q|_{Q^{-1}(x)} : b\mathbb{Z} \to \mathbb{T}^2$ is the restriction (surjective) homomorphism, we see that it suffices to show that q^* is injective.

To do this, we present $b\mathbb{Z}$ as inverse limit of an inverse family of compact Abelian Lie groups $\mathcal{Z}_\alpha$, $\alpha \in \Lambda$ (a partially ordered set with the infimal element 0), equipped with surjective homomorphisms $q_\alpha^\beta : \mathcal{Z}_\beta \to \mathcal{Z}_\alpha$ for $\alpha \le \beta$ and such that $\mathcal{Z}_0 := \mathbb{T}^2$ and the limit homomorphism $q_0 : b\mathbb{Z} \to \mathcal{Z}_0$ coincides with q. Then if $q^*(\omega) = 0$ for some non-zero $\omega \in H^2(\mathbb{T}^2,\mathbb{Z}) \cong \mathbb{Z}$, there exists an index $\beta > 0$ such that $(q_0^\beta)^*(\omega) = 0 \in H^2(\mathcal{Z}_\beta,\mathbb{Z})$. By definition,

$$\mathcal{Z}_\beta = \mathbb{T}^m \times \bigoplus_{l=1}^{k} \mathbb{Z}/n_l \quad \text{for certain } m\,(\ge 2), k, n_l \in \mathbb{Z}_+, \tag{5.5}$$

where $\mathbb{T}^m := (\mathbb{S}^1)^m$ is the real m-torus.

Since q_0^β is surjective, its restriction $\hat{q}$ to $\mathbb{T}^m$ is surjective as well, and the restriction of $(q_0^\beta)^*(\omega)$ to $\mathbb{T}^m$ is 0 in $H^2(\mathbb{T}^m,\mathbb{Z})$. Thus it suffices to prove that the surjective homomorphism $\hat{q} : \mathbb{T}^m \to \mathbb{T}^2$ induces the injective map of 2-cohomology groups.

Indeed, the kernel of $\hat{q}$ is isomorphic to $\mathbb{T}^{m-2} \oplus \Gamma$, where Γ is a finite Abelian group, and the regular covering $r : M \to \mathbb{T}^2$ of $\mathbb{T}^2$ with the transformation group Γ is also isomorphic to $\mathbb{T}^2$. Moreover, there exists a surjective homomorphism $s : \mathbb{T}^m \to M$ with connected fibres isomorphic to $\mathbb{T}^{m-2}$ such that $\hat{q} = r \circ s$. One easily shows that the exact sequence of groups

$$0 \to \mathbb{T}^{m-2} \cong Ker(s) \to \mathbb{T}^m \xrightarrow{s} M \to 0$$

splits, i.e., there exists a monomorphism $\hat{s} : M \to \mathbb{T}^m$ such that $\mathbb{T}^m = Ker(s) \oplus \hat{s}(M)$. Thus if $\hat{q}^*(\omega) = 0$ for some non-zero $\omega \in H^2(\mathbb{T}^2,\mathbb{Z})$, then the restriction of $q^*(\omega)$ to $\hat{s}(M) \cong M$ is 0 in $H^2(\hat{s}(M),\mathbb{Z})$. This implies that it suffices to prove that $r : M \to \mathbb{T}^2$ induces injection of 2-cohomology groups. But the map $r^* : \mathbb{Z} \cong H^2(\mathbb{T}^2,\mathbb{Z}) \to H^2(M,\mathbb{Z}^2) \cong \mathbb{Z}$ is multiplication by $\#\Gamma$ (= the degree of r); that is, r^* is injective, as required. $\square$

Proof of Proposition 4.22. Recall that X_0 is relatively compact in a Riemann surface $\tilde{X}_0$ such that $\pi_1(X_0) \cong \pi_1(\tilde{X}_0)$. Consider the composite φ of the identity homomorphism $\mathbb{Z} \to \mathbb{Z}$ and the embedding $\mathbb{Z} \hookrightarrow \mathbb{C}$, where $\mathbb{Z}$ is the quotient group of $\pi_1(\tilde{X}_0)$ corresponding to the deck transformation group of the covering $p : X \to X_0$. The holomorphic bundle over $\tilde{X}_0$ associated with homomorphism φ has fibre $\mathbb{C}$ and is given on an acyclic open cover $(U_i)_{i\in I}$ of $\tilde{X}_0$ by an integer-valued additive cocycle $\{c_{ij}\}_{i,j\in I}$. Since $H^1(\widetilde{X},\mathcal{O}) = 0$ and the cover is acyclic, by the Leray theorem this cocycle can be resolved, i.e., there are $c_i \in \mathcal{O}(U_i)$ such that $c_i - c_i = c_{ij}$ on $U_i \cap U_j$. Further, let us consider a refinement $(V_j)_{j\in J}$ of $(U_i)_{i\in I}$ such that each V_j is a relatively compact coordinate chart in some $U_{i(j)}$. Taking a finite cover (W_k) of X_0 by elements V_j and restricting the above cocycle and its resolution to this cover (retaining the same symbols for the restrictions) we get

$$c_k - c_\ell = c_{k\ell} \quad \text{on} \quad W_k \cap W_\ell \tag{5.6}$$

and, moreover, each $c_k \in \mathcal{O}(W_k)$ is bounded. According to the result of [GN] there exists a function $f \in \mathcal{O}(\tilde{X}_0)$ without critical points. Adding to functions c_k

the function cf with a sufficiently large $c \in \mathbb{C}$, we may and will assume that each holomorphic 1-form dc_k is nowhere zero on W_k.

By definition the bundle $b_{(\mathbb{C}^*)^2}X$ is defined on the cover (W_k) of X_0 by the 1-cocycle

$$\varphi_{k\ell}(z) := \left(\begin{array}{ll} e^{i\lambda_1 c_{k\ell}} & 0 \\ 0 & e^{i\lambda_2 c_{k\ell}} \end{array}\right), \qquad z \in W_k \cap W_\ell\,.$$

According to (5.6) we have

$$\varphi_{k\ell} = \varphi_k \cdot \varphi_\ell^{-1} \quad \text{on} \quad W_k \cap W_\ell\,,$$

where

$$\varphi_k(z) := \left(\begin{array}{ll} e^{i\lambda_1 c_k(z)} & 0 \\ 0 & e^{i\lambda_2 c_k(z)} \end{array}\right), \qquad z \in W_k\,.$$

Thus $b_{(\mathbb{C}^*)^2}X$ is isomorphic to the trivial bundle $X_0 \times (\mathbb{C}^*)^2$ and this isomorphism is given by the formulas

$$\eta(z, z_1, z_2) = \big(z, z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}\big), \quad z \in W_k, \quad (z_1, z_2) \in (\mathbb{C}^*)^2. \tag{5.7}$$

This formula shows that if λ_1, λ_2 are sufficiently small, then $\eta(b_{\mathbb{T}^2}X)$ belongs to $X_0 \times U$ for a given neighbourhood $U \Subset (\mathbb{C}^*)^2$ of $\mathbb{T}^2$. $\square$

Proof of Proposition 4.23. (1) By results of part II (A) of Section 4.4, the complex manifold X admits a holomorphic injective map into $b_{(\mathbb{C}^*)^2}(X)$. Without loss of generality we identify X with its image in $b_{(\mathbb{C}^*)^2}(X)$. The functions $(\pi \circ \eta)^* f_\alpha$ are holomorphic on subsets $(\pi \circ \eta)^{-1}(U_\alpha) \subset b_{(\mathbb{C}^*)^2}(X)$, hence their restrictions to X are holomorphic on open subsets $X \cap (\pi \circ \eta)^{-1}(U_\alpha) \subset X$. This means that the restrictions of functions $(\pi \circ \eta)^* f_\alpha$ to $b_{\mathbb{T}^2}X$ are holomorphic on subsets $b_{\mathbb{T}^2}X \cap (\pi \circ \eta)^{-1}(U_\alpha) \subset b_{\mathbb{T}^2}X$.

(2) Fix a neighbourhood $U \Subset (\mathbb{C}^*)^2$ such that the projection $q : (\mathbb{C}^*)^2 \to \mathbb{C}^2/\Lambda$ maps U biholomorphically onto its image. Then there exist a finite open cover (B_α) of $\bar{U}$ by open balls and holomorphic functions $f_\alpha \in \mathcal{O}(2B_\alpha)$, where $2B_\alpha$ is the ball with the same center as B_α and of twice the radius of B_α, with norms of gradients bounded away from zero on $2B_\alpha$ such that divisor G on $2B_\alpha$ is defined as the set of zeros of f_α. By definition, E on $(\pi \circ \eta)^{-1}(U)$ is determined by the family of pullbacks $f_\alpha \circ \eta$. In local coordinates (z, z_1, z_2) on $W_k \times B_\alpha$ (considered as a subset of $b_{(\mathbb{C}^*)^2}X$) with W_k as in (5.7) the divisor E is given (for sufficiently small λ_j) by the equation

$$g_\alpha(z, z_1, z_2) := f_\alpha\big(z_1 e^{-i\lambda_1 c_k(z)}, z_2 e^{-i\lambda_2 c_k(z)}\big) = 0. \tag{5.8}$$

Next, for such λ_j the preimage $\eta^{-1}(X_0 \times U) \subset b_{(\mathbb{C}^*)^2}X$ contains the bundle $b_{\mathbb{T}^2}X$ (see (5.7)). Therefore the intersection of $\mathrm{supp}(E)$ with $b_{\mathbb{T}^2}X$ is defined by the above equations with $(z_1, z_2) \in \mathbb{T}^2$. Also for such λ_j, since $\mathrm{supp}(G)$ intersects $\mathbb{T}^2$ transversely in finitely many points, for a fixed $z \in X_0$ the manifold $\mathrm{supp}(E)$ intersects the fibre (torus) over z in $b_{\mathbb{T}^2}X$ transversely in finitely many points as well. Moreover, all points $(z_1, z_2) \in \mathbb{T}^2$ satisfying (5.8) are sufficiently close to the points of the intersection of $\mathrm{supp}(G)$ with U_α and tend to these points uniformly in z as λ_1, λ_2 tend to 0 (this follows from the implicit function theorem).

By $O_\alpha \subset \mathbb{C}^2$ we denote an open ball with center at 0 in the space of parameters λ_1, λ_2 such that for $\lambda = (\lambda_1, \lambda_2) \in O_\alpha$ the expression on the left hand-side of (5.8) is well defined (i.e., $\big(z_1e^{-i\lambda_1 c_k(z)}, z_2e^{-i\lambda_2 c_k(z)}\big)\in 2B_\alpha$ for such λ and all $z\in W_k$, $(z_1,z_2)\in B_\alpha$). Then we have for $\lambda\in O_\alpha$, $(z,z_1,z_2)\in W_k\times B_\alpha$,

$$
\begin{aligned}
&\frac{\partial g_\alpha}{\partial z}(z,z_1,z_2)\\
&=\frac{\partial f_\alpha}{\partial \xi_1}\big(z_1e^{-i\lambda_1 c_k(z)}, z_2e^{-i\lambda_2 c_k(z)}\big)\cdot z_1e^{-i\lambda_1 c_k(z)}\cdot(-i\lambda_1)\cdot\frac{dc_k}{dz}(z)\\
&+\frac{\partial f_\alpha}{\partial \xi_2}\big(z_1e^{-i\lambda_1 c_k(z)}, z_2e^{-i\lambda_2 c_k(z)}\big)\cdot z_2e^{-i\lambda_2 c_k(z)}\cdot(-i\lambda_2)\cdot\frac{dc_k}{dz}(z)\\
&=-\frac{dc_k}{dz}(z)\bigg(\lambda_1\frac{\partial f_\alpha}{\partial \xi_1}\big(z_1e^{-i\lambda_1 c_k(z)}, z_2e^{-i\lambda_2 c_k(z)}\big)\\
&+\lambda_2\frac{\partial f_\alpha}{\partial \xi_2}\big(z_1e^{-i\lambda_1 c_k(z)}, z_2e^{-i\lambda_2 c_k(z)}\big)\bigg).
\end{aligned}
\tag{5.9}
$$

Suppose that a point $\theta=(\theta_1,\theta_2)\in\mathbb{T}^2$ of the intersection of $\mathrm{supp}(G)$ with $\mathbb{T}^2$ belongs to B_α. The equation

$$\lambda_1\frac{\partial f_\alpha}{\partial \xi_1}(\theta_1,\theta_2)+\lambda_2\frac{\partial f_\alpha}{\partial \xi_2}(\theta_1,\theta_2)=0$$

determines a one-dimensional complex subspace ℓ in the space $\mathbb{C}^2$ of parameters $\lambda=(\lambda_1,\lambda_2)$. Fix a compact subset $K_\theta\subset\partial O_\alpha\setminus\ell$ on the boundary ∂O_α of O_α. Since the numbers of indices α and k in the covers are finite, without loss of generality (decreasing each O_α, if necessary) we may assume that all O_α coincide (denote this set by O). Also, since the number of points of intersection of $\mathrm{supp}(G)$ with $\mathbb{T}^2$ is finite, the above argument shows, that we may choose the sets K_θ so that $K:=\cap K_\theta\neq\emptyset$ and, moreover, K contains points from $\partial O\cap(\mathbb{R}_+)^2$. We set $K_+:=K\cap(\mathbb{R}_+)^2$. Then by the continuity of derivatives of f_α there exist a number $0<t\le 1$ and neighbourhoods N_β in U of points of intersection of $\mathrm{supp}(G)$ with $\mathbb{T}^2$ such that each N_β is a subset of some B_α and for $z\in W_k$ and $(z_1,z_2)\in N_\beta$ and for any $(\lambda_1,\lambda_2)\in t_0K_+$ with $0<t_0\le t$,

$$\left|\frac{\partial g_\alpha}{\partial z}(z,z_1,z_2)\right|\ge ct_0>0,$$

where c is a constant independent of the choice of (z,z_1,z_2) and indices k,α. Here we have used that $\left|\frac{dc_k}{dz}\right|$ is bounded away from zero by a numerical constant by our choice of c_k, see (5.6). As we have noticed before for sufficiently small t_0 the sets of solutions of equations (5.8) belong to unions of open sets $W_k\times N_\beta$.

Thus we have proved that $\mathrm{supp}(H):=\mathrm{supp}(E)\cap b_{\mathbb{T}^2}X$ can be covered by finitely many sets in $b_{(\mathbb{C}^*)^2}(X)$ of the form $W_k\times N_\beta$ (in suitable local coordinates on $b_{(\mathbb{C}^*)^2}(X)$) and there exist functions $g_{k\beta}\in\mathcal{O}(W_k\times N_\beta)$ such that $\mathrm{supp}(H)\cap(W_k\times N_\beta)$ is the set of zeros of $g_{k\beta}$ and the modulus of the derivative of $g_{k\beta}$ with respect to $z\in W_k$ is bounded away from zero on $W_k\times N_\beta$. This shows that $H\in\mathrm{Div}\,(b_{\mathbb{T}^2}X)$ is smooth.

(3) We use the results of part 1 of the proof of Proposition 4.21. It suffices to show that there exists a contractible coordinate chart $U_0\Subset X_0$ such that the Chern class $c_{U,\mathbb{T}^2}(H|_U)\neq 0$ with $U:=p_{b,\mathbb{T}^2}^{-1}(U_0)$.

Since divisor G on $(\mathbb{C}^*)^2$ is not equivalent to a principal divisor, its Chern class $c(G) \neq 0$ in $H^2((\mathbb{C}^*)^2, \mathbb{Z})$ (see, e.g., [GH]). For any contractible coordinate chart $U_0 \Subset X_0$ the divisor G_{U_0} on $U_0 \times (\mathbb{C}^*)^2$ is defined as the pullback of G with respect to the projection $\pi : U_0 \times (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$. In particular, its Chern class $c(G_{U_0})$ equals $\pi^*\big(c(G)\big)$ and is non-zero because π induces isomorphisms of cohomology groups. Since for such U_0 the open set $U' := p_{b,(\mathbb{C}^*)^2}^{-1}(U_0) \subset b_{(\mathbb{C}^*)^2}X$ is biholomorphic by means of η to $U_0 \times (\mathbb{C}^*)^2$, from our construction we obtain that the restriction $E|_{U'}$ of divisor E $(:= (\pi \circ \eta)^* G)$ to U' is the same as $\eta^* G_{U_0}$. Thus the Chern class $c(E|_{U'}) \neq 0$ in $H^2(U', \mathbb{Z})$. Finally, $U := p_{b,\mathbb{T}^2}^{-1}(U_0) \subset b_{\mathbb{T}^2}X$ is a deformation retract of U' and therefore because $H|_U$ coincides with $(E|_{U'})|_U$ the Chern class $c_{U,\mathbb{T}^2}(H|_U) \neq 0$ (in fact, it coincides with $c(E|_{U'})$ under the identification of $H^2(U', \mathbb{Z})$ with $H^2(U, \mathbb{Z})$). □

Proof of Proposition 4.24. We construct a smooth divisor G on $(\mathbb{C}^*)^2$ that has a non-zero Chern class and whose support intersects the real torus $\mathbb{T}^2$ transversely.

Let $\Lambda := (2\pi\mathbb{Z} + 2\pi i\mathbb{Z})^2$, $\Gamma := (2\pi i\mathbb{Z})^2 \subset \mathbb{C}^2$. Then $\mathbb{C}^2/\Lambda$ (with respect to the action of Λ on $\mathbb{C}^2$ by translations) is a complex two-dimensional torus and $\mathbb{C}^2/\Gamma$ is the product of two infinite cylinders. Let $c : \mathbb{C}^2 \to \mathbb{C}^2/\Gamma$ be the (holomorphic) quotient map. Then there exists a biholomorphic map $q_1 : (\mathbb{C}^*)^2 \to \mathbb{C}^2/\Gamma$ defined by the formula

$$q_1(\zeta_1, \zeta_2) := c\big((\log \zeta_1, \log \zeta_2)\big), \quad (\zeta_1, \zeta_2) \in (\mathbb{C}^*)^2;$$

here $\log : \mathbb{C}^* \to \mathbb{C}$ is the multi-valued logarithmic function.

Further, denoting by $q_2 : \mathbb{C}^2/\Gamma \to (\mathbb{C}^2/\Gamma)/(2\pi\mathbb{Z})^2 = \mathbb{C}^2/\Lambda$ the corresponding holomorphic quotient map, we obtain that the regular covering $q : (\mathbb{C}^*)^2 \to \mathbb{C}^2/\Lambda$ with the deck transformation group $\mathbb{Z}^2$ can be obtained as the composite $q_2 \circ q_1$.

We start by constructing a smooth divisor V on $\mathbb{C}^2/\Lambda$ having a non-zero Chern class. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be standard complex coordinates on $\mathbb{C}^2$. They produce local coordinates on $\mathbb{C}^2/\Lambda$ denoted analogously. It follows that

$$\omega_0 = dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2 = 2(dx_1 \wedge dx_2 + dy_1 \wedge dy_2)$$

is a d-closed $(1,1)$-form on $\mathbb{C}^2/\Lambda$ having a non-zero de Rham cohomology class $[\omega_0] \in H^2_{DR}(\mathbb{C}^2/\Lambda)$. We also consider a positive d-closed $(1,1)$-form

$$\eta := ki(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = 2k(dx_1 \wedge dy_1 + dx_2 \wedge dy_2), \quad k \in \mathbb{N}.$$

Since ω_0 and η have integer coefficients, they represent integral cohomology classes, i.e., $[\omega_0]$ and $[\eta]$ belong to the image of $H^2(\mathbb{C}^2/\Lambda, \mathbb{Z})$ in $H^2_{DR}(\mathbb{C}^2/\Lambda)$ under the de Rham isomorphism (see, e.g., [GH, Ch. 2.6]). By taking k sufficiently large one obtains that $\omega := \omega_0 + \eta$ is a positive d-closed $(1,1)$-form representing an integral cohomology class. By the Lefschetz $(1,1)$-theorem (observe that $\mathbb{C}^2/\Lambda$ is projective) the cohomology class $[\omega]$ is the Chern class of a positive line bundle L_ω on $\mathbb{C}^2/\Lambda$. Increasing k, if necessary, we can embed (using the Kodaira theorem) $\mathbb{C}^2/\Lambda$ into a projective space $\mathbb{CP}^N$ by means of holomorphic sections of the bundle L_ω so that L_ω is the pullback of the hyperplane bundle on $\mathbb{CP}^N$. By the Bertini theorem the preimage in $\mathbb{C}^2/\Lambda$ of a generic hyperplane $H \subset \mathbb{CP}^N$ determines a smooth divisor V in $\mathbb{C}^2/\Lambda$ with Chern class $[\omega]$. We claim that

Lemma 5.5. Under a suitable choice of H the support of the constructed divisor V intersects the image $q(\mathbb{T}^2) \subset \mathbb{C}^2/\Lambda$ transversely in finitely many points.

Proof. Clearly, q embeds $\mathbb{T}^2$ into $\mathbb{C}^2/\Lambda$ so that $q(\mathbb{T}^2)$ is a (real) analytic submanifold of $\mathbb{C}^2/\Lambda$. It is also totally real meaning that $T_x \cap iT_x = 0$ and $T_x + iT_x = T_x^{\mathbb{C}}$ at each $x \in q(\mathbb{T}^2)$, where T_x is the tangent space to $q(\mathbb{T}^2)$ at x and $T_x^{\mathbb{C}}$ is the minimal complex subspace of the tangent space to $\mathbb{C}^2/\Lambda$ at x containing T_x (in our case it coincides with this tangent space). This follows from the fact that $\mathbb{T}^2 \subset (\mathbb{C}^*)^2$ is totally real and q is biholomorphic in a neighbourhood of $\mathbb{T}^2$. Without loss of generality we will identify $\mathbb{C}^2/\Lambda$ with its image in $\mathbb{CP}^N$.

Let us choose a generic hyperplane $H \subset \mathbb{CP}^N$ transversely intersecting $\mathbb{C}^2/\Lambda$ and intersecting $q(\mathbb{T}^2)$ transversely at least at one point. To do that we pick $x \in q(\mathbb{T}^2)$ and decompose the complex tangent space $T_x(\mathbb{CP}^N)$ of $\mathbb{CP}^N$ at x as $T_x^{\mathbb{C}} \oplus L$, where L is a complex subspace of codimension 2 of $T_x(\mathbb{CP}^N)$. Further, in $T_x^{\mathbb{C}}$ choose a one-dimensional complex subspace L' which intersects the real part T_x of $T_x^{\mathbb{C}}$ by 0. Then $L + L'$ is a complex hyperplane in $T_x(\mathbb{CP}^N)$ transversely intersecting T_x. Let $H' \subset \mathbb{CP}^N$ be the hyperplane whose tangent space at x coincides with $L + L'$. Then H' intersects $q(\mathbb{T}^2)$ transversely at x. Further, by the Bertini theorem we can perturb H' to get a hyperplane H that also transversely intersects $q(\mathbb{T}^2)$ at least at one point and transversely intersects $\mathbb{C}^2/\Lambda$.

Next, consider a projection π in $\mathbb{CP}^N$ along H onto a one-dimensional projective subspace $\ell \subset \mathbb{CP}^N$ transversely intersecting H at a single point. In fact, π is a meromorphic map of $\mathbb{CP}^N$ onto $\ell \cong \mathbb{CP}^1$ and so it is defined outside a projective subspace of $\mathbb{CP}^N$ of (complex) codimension two. Perturbing H, if necessary, we may assume that the latter subspace does not intersect $q(\mathbb{T}^2)$ so that π is well defined on $q(\mathbb{T}^2)$. By our construction of H the image $\pi(q(\mathbb{T}^2))$ contains interior points (because in a neighbourhood of a point of transversal intersection of H and $q(\mathbb{T}^2)$, the map $\pi|_{q(\mathbb{T}^2)}$ is diffeomorphic). Therefore by Sard's theorem for almost each interior point $z \in \pi(q(\mathbb{T}^2))$ (with respect to the measure on $\mathbb{CP}^1$ determined by the Fubini-Study volume form) the preimage $\pi^{-1}(z)|_{q(\mathbb{T}^2)}$ consists of finitely many non-critical points of $\pi|_{q(\mathbb{T}^2)}$. But this means that the complex hyperplane $\pi^{-1}(z)$ in $\mathbb{CP}^N$ intersects $q(\mathbb{T}^2)$ transversely. Finally, we can perturb $\pi^{-1}(z)$ (using the Bertini theorem) so that the perturbed hyperplane intersects $q(\mathbb{T}^2)$ transversely in finitely many points and also intersects transversely $\mathbb{C}^2/\Lambda$ (determining the required divisor V). □

Now, we define a smooth divisor G on $(\mathbb{C}^*)^2$ as the pullback by map q of the divisor V. Let us show that the Chern class $[q^*\omega] = [q^*\omega] \in H^2_{DR}(\mathbb{C}^2/\Lambda)$ of G is non-zero. First, $[q^*\eta] = 0$ by the definition of η (because each term of $q^*\eta$ is a d-closed 2-form on $\mathbb{C}^*$ which homotopic to $\mathbb{S}^1$). Thus we must check that $[q^*\omega_0] \neq 0$. Since q_1 is a biholomorphism, it suffices to check that $[q_2^*\omega_0] \in H^2_{DR}(\mathbb{C}^2/\Gamma)$ is non-zero. Using coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ on $\mathbb{C}^2/\Gamma$ induced from the standard coordinates on $\mathbb{C}^2$ we easily obtain

$$[q_2^*\omega_0] = 2[dx_1 \wedge dx_2 + dy_1 \wedge dy_2] = 2[dy_1 \wedge dy_2].$$

The latter cohomology class is non-zero; to see this one integrates $dy_1 \wedge dy_2$, e.g., over an embedded real torus $\{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2/\Gamma : x_1, x_2 = \text{const}\}$ getting a non-zero value which, as follows from the Stokes theorem, contradicts to the assumption that $dy_1 \wedge dy_2$ is exact.

Lemma 5.5 implies that the support of G intersects $\mathbb{T}^2 \subset (\mathbb{C}^*)^2$ transversely in finitely many points. □

References

[Bo] H. Bohr, Almost Periodic Functions. *Chelsea*, 1951.

[Bre] G. E. Bredon, Sheaf theory. Second edition. Graduate Texts in Mathematics 170 *Springer-Verlag*, 1997.

[Br1] A. Brudnyi, Projections in the space H^∞ and the Corona Theorem for coverings of bordered Riemann surfaces. *Ark. Mat.* **42** (2004), 31-59.

[Br2] A. Brudnyi, On holomorphic functions of slow growth on coverings of strongly pseudoconvex manifolds. *J. Funct. Anal.* **249** (2007), 354–371.

[Br3] A. Brudnyi, Hartogs-type theorems on coverings of Stein manifolds. *Internat. J. Math.* **17** (2006), 339-349.

[Br4] A. Brudnyi. Representation of holomorphic functions on coverings of pseudoconvex domains in Stein manifolds via integral formulas on these domains. *J. Funct. Anal.* **231** (2006), 418-437.

[BrK1] A. Brudnyi, D. Kinzebulatov, On uniform subalgebras of L^∞ on the unit circle generated by almost periodic functions. *St. Petersburg J. Math.* **19** (2008), 495-518.

[BrK2] A. Brudnyi, D. Kinzebulatov, Algebras of holomorphic functions on coverings of complex manifolds. Cartan theorems. *Preprint.*

[BrK3] A. Brudnyi, D. Kinzebulatov, Holomorphic almost periodic functions on coverings of Stein manifolds. *C. R. Math. Rep. Acad. Sci. Canada.* **32** (2010), 65-80.

[Bu1] L. Bungart, On analytic fibre bundles I. Holomorphic fibre bundles with infinite dimensional fibres. *Topology* **7** (1968), 55-68.

[Bu2] L. Bungart, Holomorphic functions with values in locally convex spaces and applications to integral formulas. *Proc. Amer. Math. Soc.* **111** (1964), 317-344, 1964; Errata, ibid. **113** (1964).

[Fav1] S. Favorov, Zeros of holomorphic almost periodic functions. *J. d'Analyse Math.* **84** (2001), 51-66.

[Fav2] S. Favorov, Almost periodic divisors, holomorphic functions, and holomorphic mappings. *Bull. Sci. Math.* **127** (2003), 859-883.

[FR] S. Favorov and A. Rashkovskii, Holomorphic almost periodic functions. *Acta Appl. Math.* **65** (2003), 217-235.

[FRR] S. Favorov, A. Rashkovskii and L. Ronkin, Almost periodic divisors in a strip. *J. d'Analyse Math.* **74** (1998), 325-345.

[F2R] S. Favorov, A. Rashkovskii and L. Ronkin, Almost periodic currents and holomorphic chains. *C. R. Acad. Sci. Paris* **327** (1998), 302-307.

[For] O. Forster, Lectures on Riemann Surfaces. *Springer-Verlag*, 1981.

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry. *Wiley*, 1994.

[G] A. Grothendieck, Products tensoriels toplogiques et espaces nucleaires. *Mem. Amer. Math. Soc.* 16, 1955.

[GR] R. Gunning and H. Rossi, Analytic Functions of Several Complex Variables. *Prentice-Hall*, 1965.

[GN] R. Gunning and R. Narasimhan, Immersion of open Riemann surfaces. Math. Annalen **174** (1967), 103-108.

[JT] B. Jessen and H. Tornehave, Mean motions and zeros of almost periodic functions. *Acta Math.* **77** (1945), 137-279.

[L] J. Leiterer, Holomorphic vector bundles and the Oka-Grauert principle. Several complex variables. IV. Algebraic aspects of complex analysis. *Encyclopedia Math. Sci.*, 10, 1990.

[Ln] B. Levitan, Almost Periodic Functions (in Russian). *Moscow*, 1953.

[Lv] B. Levin, Distribution of Zeros of Entire Functions. *AMS*, 1980.

[Lin] V. Lin, Liouville coverings of complex spaces, and amenable groups. (Russian) *Mat. Sb.* **132(2)** (1987), 202-224; translation in *Math. USSR-Sb.* **60** (1988), 197-216.

[Ma] W. Maak, Eine neue Definition der fastperiodischen Funktionen. *Abh. Math. Sem. Univ. Hamburg* **11** (1936), 240-244.

[N] J. von Neumann, Almost periodic functions in a group. I. *Trans. Amer. Math. Soc.* **36** (1934), 445-492.

[Rn] L. Ronkin, Jessen's theorem for holomorphic almost periodic functions in tube domains. *Sibirskii Mat. Zh.* **28** (1987), 199-204.

[S] B.V. Shabat, Introduction to Complex Analysis: Functions of Several Variables. *AMS*, 1992.

[St] K. Stein, Überlagungen holomorph-vollstandiger komplexe Räume. *Arch. Math.* **7** (1956), 354-361.

[T] H. Tornehave, On the zeros of entire almost periodic function. The Harald Bohr Centenary (Copenhagen 1987). *Math. Fys. Medd. Danske* **42** (1989), 125-142.

[W] A. Weil, L'integration dans les groupes topologiques et ses applications. *Hermann, Paris*, 1965.

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