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Non-Perturbative Localization with Quasiperiodic Potential in Continuous Time

Ilia Binder[1], **Damir Kinzebulatov**[2], **Mircea Voda**[3]

[1] Department of Mathematics, University of Toronto, Toronto, ON, M5S 2E4, Canada.
E-mail: ilia@math.utoronto.ca
[2] Department of Mathematics and Statistics, McGill University, Montreal, QC, H3A 0B9, Canada.
E-mail: damir.kinzebulatov@mail.mcgill.ca
[3] Department of Mathematics, The University of Chicago, Chicago, IL 60637, USA.
E-mail: mircea.voda@math.uchicago.edu

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Abstract: We consider continuous one-dimensional multifrequency Schrödinger operators, with analytic potential, and prove Anderson localization in the regime of positive Lyapunov exponent for almost all phases and almost all Diophantine frequencies.

Contents

1. Introduction

We consider the family of operators on $L^2(\mathbb{R})$ given by

$$[H(\theta, \omega)y](t) = -y''(t) + V(t, \theta + t\omega)y(t), \tag{1.1}$$

where the potential $V : \mathbb{T} \times \mathbb{T}^d \to \mathbb{R}$ is analytic ($\mathbb{T} := \mathbb{R}/\mathbb{Z}$, $d \geq 1$), $\theta \in \mathbb{T}^d$, and ω satisfies a Diophantine condition. More precisely we will work with frequency vectors in a set defined by

$$\mathrm{DC} := \{\omega \in \mathbb{T}^d : \|k \cdot \omega\| \geq c|k|^{-A}, k \in \mathbb{Z}^d \backslash \{0\}\},$$

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for some $A > d$. We used $\|\cdot\|$ to denote the distance to the nearest integer and $|\cdot|$ for the sup-norm on $\mathbb{Z}^d$. We will use $L(\omega, E)$ to denote the Lyapunov exponent associated with our operators (see Sect. 2 for the definition). Our main result is as follows.

Theorem 1.1. *Assume that $L(\omega, E) > 0$ for all $(\omega, E) \in \mathrm{DC} \times [E', E'']$. Then for almost all phases $\theta \in \mathbb{T}^d$ and almost all frequency vectors $\omega \in \mathrm{DC}$ the part of the spectrum of $H(\theta, \omega)$ contained in $[E', E'']$ is pure point with exponentially decaying eigenfunctions.*

Non-perturbative localization results (in the sense that one only requires positivity of the Lyapunov exponent) are well known for discrete Schrödinger operators, dating back to work by Jitomirskaya [Jit99] for the Almost Mathieu operator and by Bourgain and Goldstein [BG00] for general analytic potentials. For continuous Schrödinger operators the only known result, due to Fröhlich, Spencer, and Wittwer [FSW90], deals with potentials of the form

$$K^2(\cos(2\pi t) + \cos(2\pi(\theta + t\omega))) \tag{1.2}$$

with K sufficiently large. At the same time, there was no reason to expect that the discrete results don't carry to the continuous case (indeed, [FSW90] treats both the discrete and the continuous cases). Our motivation for considering this problem stems from the recent work on the inverse spectral theory for continuous quasiperiodic Schrödinger operators started by Damanik and Goldstein [DG14]. Their work is in a perturbative setting (assuming a small coupling constant) and it is natural to try to extend it to a non-perturbative setting. On one hand, one can try to prove the results of [DG14] in the discrete case and then make use of the non-perturbative theory available there (though, it is known that for the inverse spectral problem one should consider Jacobi operators instead of Schrödinger operators). On the other hand, one is motivated to develop the non-perturbative theory in the continuous setting. Our work is a step in this direction.

The fact that Theorem 1.1 is non-vacuous, i.e., there exists a portion of the spectrum where the theorem applies, follows from work on the positivity of the Lyapunov exponent, by Sorets and Spencer [SS91] for the case when the potential is of the form (1.2) and by Bjerklöv [Bje06] for general analytic potentials that assume their minimum value only finitely many times. The result of [Bje06] is perturbative, in the sense that the largeness of the coupling constant depends on the frequency vector. It would be interesting to see whether it is possible to obtain a non-perturbative result on the positivity of the Lyapunov exponent assuming only that V is non-constant. Such results are known in the discrete case (see [SS91] for $d = 1$, and [Bou05b] for $d \geq 1$). We note that such a result in the continuous case may not be completely non-perturbative, because it is well known that the Lyapunov exponent vanishes at high energies (see [Eli92]), and it seems that the transition between energies with positive Lyapunov exponent and energies with zero Lyapunov exponent is of a perturbative nature (see [YZ14, Remark 1.2]).

The proof of Theorem 1.1 follows along the same lines as in the discrete case. The main ingredients are a result on the exponential decay of the finite interval Green's function (see Proposition 4.1) and a result on elimination of resonances (see Proposition 7.2). The basis for these results is a large deviations estimate for the logarithm of the norm of the transfer matrix (see Theorem 3.1), which one obtains immediately from the discrete case. The fact that the large deviations estimate implies the decay of Green's function (on some interval) is trivial in the discrete setting, but not in the continuous setting, as can be seen from the proof of Proposition 4.1. To eliminate the resonances we use the strategy of [BG00] based on semialgebraic sets. The main difference from the discrete case is that to obtain semialgebraic sets it is not enough to approximate the potential by a polynomial. Instead we have to directly approximate the entries of the transfer matrix. This leads to a different set-up for the elimination of resonances (see Remark 7.3) which, unlike the discrete case, requires the use of Cartan's estimate as in [GS08]. Furthermore, the approximating polynomials for the entries of the transfer matrix cannot be obtained via Fourier series as in the discrete case. We use Faber series instead.

The structure of the paper is as follows. The definition and all of the needed properties for the transfer matrix and the Laypunov exponent are presented in Sects. 2 and 3. The exponential decay of Green's function is established in Sect. 4. Sections 5 and 6 contain all the preliminary work needed for the elimination of resonances result from Sect. 7. Finally the proof of Theorem 1.1 is obtained in Sect. 8.

2. Transfer Matrix Formalism

Consider the eigenvalue equation

$$-y''(t) + V(t, \theta + t\omega)y(t) = Ey(t), \quad \theta \in \mathbb{T}^d. \tag{2.1}$$

Let $u_a = u_a(\cdot; \theta, \omega, E)$, $v_a = v_a(\cdot; \theta, \omega, E)$ be the solutions of (2.1) satisfying

$$u_a(a) = 1, u_a'(a) = 0, \quad v_a(a) = 0, v_a'(a) = 1. \tag{2.2}$$

Any solution y of (2.1) satisfies

$$\begin{bmatrix} y(b) \\ y'(b) \end{bmatrix} = M_{[a,b]} \begin{bmatrix} y(a) \\ y'(a) \end{bmatrix}, \quad a \le b$$

where the transfer matrix is defined by

$$M_{[a,b]} = M_{[a,b]}(\theta, \omega, E) := \begin{bmatrix} u_a(b) & v_a(b) \\ u_a'(b) & v_a'(b) \end{bmatrix}.$$

The transfer matrix satisfies

$$M_{[a,b]} = M_{[t,b]} M_{[a,t]}, \quad a \le t \le b \tag{2.3}$$

$$M_{n+[a,b]}(\theta, \omega, E) = M_{[a,b]}(\theta + n\omega, \omega, E), \quad n \in \mathbb{Z}. \tag{2.4}$$

Remark 2.1. The fact that we can only use discrete shifts in (2.4), stems from the fact that we are working with potentials of the form $V(t, \theta + t\omega)$ instead of just $V(\theta + t\omega)$. The reason we consider the more general potentials $V(t, \theta + t\omega)$ is to be able to apply the result of Bjerklöv [Bje06] that guarantees that the statement of our main theorem is not vacuous.

Equation (2.1) can be re-written as

$$Y'(t) = A(t)Y(t), \quad Y(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & 1 \\ V(t, \theta + t\omega) - E & 0 \end{bmatrix}.$$

As a consequence of Grönwall's inequality one has

$$\|Y(b)\| \le \exp\left(\int_a^b \|A(t)\|\, dt\right) \|Y(a)\|.$$

Therefore we have

$$\log\|M_{[a,b]}\| \le (b-a)C(V,|E|). \tag{2.5}$$

Since

$$\det M_{[a,b]} = W(u_a, v_a) = 1,$$

where W stands for the Wronskian, it follows that we also have

$$\log\|M_{[a,b]}^{-1}\| \le (b-a)C(V,|E|). \tag{2.6}$$

Using (2.3), (2.4), (2.5), and (2.6) we see that, as in the discrete case, we have the following "almost invariance" property

$$\left|\log\|M_{[a,b]}(\theta,\omega,E)\| - \log\|M_{[a,b]}(\theta+\omega,\omega,E)\|\right| \le C(V,|E|). \tag{2.7}$$

This property is essential for establishing a large deviations estimate (see Theorem 3.1).

The finite scale Lyapunov exponents are defined by

$$L_{[a,b]}(\omega,E) = \frac{1}{b-a}\int_{\mathbb{T}^d} \log\|M_{[a,b]}(\theta,\omega,E)\|\,d\theta.$$

We let $M_t = M_{[0,t]}$ and $L_t = L_{[0,t]}$. By (2.3) and (2.4), the sequence $(L_n)_{n\ge1}$ is subadditive and so by Fekete's subadditive lemma we can define the Lyapunov exponent

$$L(\omega,E) := \lim_{n\to\infty} L_n(\omega,E) = \inf_{n\ge1} L_n(\omega,E). \tag{2.8}$$

We note that by Kingman's subadditive ergodic theorem we also have

$$L(\omega,E) \overset{\text{a.s.}}{=} \lim_{n\to\infty} \frac{1}{n}\log\|M_n(\theta,\omega,E)\|, \tag{2.9}$$

but we won't make use of this fact.

Let

$$\mathbb{H}_\rho = \{z\in\mathbb{C} : |\operatorname{Im} z| \le \rho\}. \tag{2.10}$$

It is known that there exists $\rho = \rho(V)$ such that V extends to be complex analytic in a neighborhood of $\mathbb{H}_\rho^{d+1}$ and the extension remains periodic in the real direction. In particular, this implies that

$$L_{[a,b]}(\eta,\omega,E) = \frac{1}{b-a}\int_{\mathbb{T}^d} \log\|M_{[a,b]}(\theta+i\eta,\omega,E)\|\,d\theta$$

is well defined for all $E\in\mathbb{C}$ and $\omega\in\mathbb{C}^d$, $\eta\in\mathbb{R}^d$ such that

$$\max(|a|,|b|)\|\operatorname{Im}\omega\| \le \rho/2, \quad \|\eta\| \le \rho/2.$$

As before, we can define

$$L(\eta,\omega,E) = \lim_{n\to\infty} L_n(\eta,\omega,E) = \inf_{n\ge1} L_n(\eta,\omega,E).$$

3. Properties of the Transfer Matrix

All the results of this section are analogues of results obtained in the discrete setting by Goldstein and Schlag [GS01,GS08] (a large deviations estimate and an uniform upper bound can also be found in [BG00]). Since the results we need from [GS01] were already obtained in a multifrequency setting with general Diophantine condition we only discuss their proofs in Appendix A (which includes proofs for Theorem 3.1, Proposition 3.2, and Lemma 3.3). The results we need from [GS08] were obtained in a single frequency setting with a strong Diophantine condition, so we present their proofs in this section. In either case, the proofs are only given in the interest of clarity, as they follow along the same lines as in the originals.

Let

$$\mathrm{DC}_t := \{\omega \in \mathbb{T}^d : \|k \cdot \omega\| \geq c|k|^{-A}, k \in \mathbb{Z}^d \backslash \{0\}, |k| \leq t\}.$$

For the purposes of the semialgebraic approximation it is important to keep track of the fact that most finite scale results require only a finite Diophantine condition, as above, and only the positivity of the finite interval Lyapunov exponent (if needed at all).

In what follows we will always assume that the intervals we work on are finite and non-trivial. Also note that all the results are only effective when the size of the interval is large enough. For smaller sizes the constants can be adjusted so that the results hold trivially.

The main ingredient for both the decay of Green's function and the elimination of resonances is the following large deviations estimate.

Theorem 3.1. *Let $I = [a, b]$ and $\varepsilon > 0$. Then for any $\omega \in \mathrm{DC}_{|I|}$, $E \in \mathbb{C}$, and $\eta \in \mathbb{R}^d$, $\|\eta\| \leq \rho(V)$, we have*

$$\mathrm{mes}\{\theta \in \mathbb{T}^d : |\log\|M_I(\theta+i\eta, \omega, E)\| - |I|L_I(\eta, \omega, E)| \geq \varepsilon|I|^{1-\sigma}\} \leq C\exp(-c|I|^{\sigma}),$$

with $c = c(V, d, \mathrm{DC}, |E|, \varepsilon)$, $C = C(V, d, \mathrm{DC}, |E|)$, and $\sigma = \sigma(d, \mathrm{DC}) \in (0, 1)$.

Note. From now on σ will denote the constant from Theorem 3.1.

We will need the following result to relate the Lyapunov exponent with the finite interval Lyapunov exponents, and the finite interval Laypunov exponents with each other.

Proposition 3.2. *Let $I = [a, b]$, $J = [b, c]$, $||I| - |J|| \leq \delta$. If $(\eta, \omega, E) \in \mathbb{R}^d \times \mathrm{DC}_{|I|} \times \mathbb{C}$, is such that $\|\eta\| \leq \rho(V)$, $L_I(\eta, \omega, E) \geq \gamma > 0$, then we have*

$$|L_I(\eta, \omega, E) - L_{I\cup J}(\eta, \omega, E)| \leq \frac{C(\log(1 + |I|))^{1/\sigma}}{|I|} \tag{3.1}$$

with $C = C(V, d, \mathrm{DC}, |E|, \gamma, \delta)$. Furthermore, if $\omega \in \mathrm{DC}$ then

$$|L_I(\eta, \omega, E) - L(\eta, \omega, E)| \leq \frac{C(\log(1 + |I|))^{1/\sigma}}{|I|} \tag{3.2}$$

with $C = C(V, d, \mathrm{DC}, |E|, \gamma)$.

We will use the next estimate to see that positivity of the Lyapunov exponent for some interval also implies positivity for smaller intervals (we do this because we won't be able to apply (3.2) when $\omega \in \mathrm{DC}_{|I|}$). It is possible to adjust the estimate to also give meaningful information when $|I|$ is close to $|J|$, but we are only interested in the case $|I| \gg |J|$.

Lemma 3.3. *Let $I = [a, b]$, $J = [c, d]$, $|I| > |J|$. Then for any $(\eta, \omega, E) \in \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{C}$, $\|\eta\| \leq \rho(V)$ we have*

$$L_J(\eta, \omega, E) \geq L_I(\eta, \omega, E) - C(V, |E|)\left(\frac{|J|+1}{|I|-|J|} + \frac{1}{|J|}\right).$$

Now we start the build-up toward the proof of the uniform upper bound from Proposition 3.11. This result is crucial for the decay of the Green's function and the application of Cartan's estimate. While for the decay of Green's function the simpler estimate from Corollary 3.12 is enough, for Cartan's estimate we also need the estimate to have good stability under (complex) perturbations in (θ, ω, E). See [GS08, Sect. 4] for the discrete counterparts of the results that follow.

One of the ingredients for the proof of Proposition 3.11 will be the fact that the Lyapunov exponent is Lipschitz with respect to η. This follows immediately from the multivariable generalization of the following fact from [GS08].

Lemma 3.4 ([GS08, Lemma 4.1]). *Let $1 > \rho > 0$ and suppose u is subharmonic on*

$$A_\rho := \{z : 1 - \rho < |z| < 1 + \rho\}$$

such that $\sup_{z \in A_\rho} u(z) \leq 1$ and $\int_{\mathbb{T}} u(e(x))\, dx \geq 0$ (we used the notation $e(x) = e^{2\pi i x}$). Then for any r_1, r_2 so that $1 - \rho/2 < r_1, r_2 < 1 + \rho/2$ one has

$$\left| \int_{\mathbb{T}} u(r_1 e(x))\, dx - \int_{\mathbb{T}} u(r_2 e(x))\, dx \right| \leq C_\rho |r_1 - r_2|.$$

The previous Lemma admits the following multivariable extension.

Lemma 3.5. *Let $1 > \rho > 0$ and suppose u is subharmonic in each variable on*

$$A_\rho^d := \{z \in \mathbb{C}^d : 1 - \rho < |z_i| < 1 + \rho, i = 1, \ldots, d\}$$

such that

$$0 \leq u(z) \leq 1, \quad z \in A_\rho^d.$$

Then for any $r, \tilde{r} \in \mathbb{R}^d$ so that $1 - \rho/2 < r_i, \tilde{r}_i < 1 + \rho/2$, $i = 1, \ldots, d$, one has

$$\left| \int_{\mathbb{T}^d} u(r_1 e(x_1), \ldots, r_d e(x_d))\, dx - \int_{\mathbb{T}^d} u(\tilde{r}_1 e(x_1), \ldots, \tilde{r}_d e(x_d))\, dx \right| \leq C_\rho \sum_i |r_i - \tilde{r}_i|.$$

Proof. The proof is by induction on d. The case $d = 1$ holds by Lemma 3.4. We assume the result holds for d and prove it for $d + 1$. Let

$$\begin{aligned} v(z_{d+1}) &= \int_{\mathbb{T}^d} u(r_1 e(x_1), \ldots, r_d e(x_d), z_{d+1})\, dx,\, \tilde{v}(z_{d+1}) \\ &= \int_{\mathbb{T}^d} u(\tilde{r}_1 e(x_1), \ldots, \tilde{r}_d e(x_d), z_{d+1})\, dx. \end{aligned}$$

By the induction assumption

$$|v(z_{d+1}) - \tilde{v}(z_{d+1})| \leq C_\rho \sum_{i=1}^{d} |r_i - \tilde{r}_i|, \quad z_{d+1} \in A_\rho.$$

At the same time we have that v is subharmonic on A_ρ and $0 \le v \le 1$, so we can apply the case $d = 1$ to it to get the desired conclusion. Indeed, we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}} v(r_{d+1}e(x_{d+1}))\,dx_{d+1} - \int_{\mathbb{T}} \tilde{v}(\tilde{r}_{d+1}e(x_{d+1}))\,dx_{d+1}\right| \\
&\le \left|\int_{\mathbb{T}} v(r_{d+1}e(x_{d+1}))\,dx_{d+1} - \int_{\mathbb{T}} v(\tilde{r}_{d+1}e(x_{d+1}))\,dx_{d+1}\right| \\
&\quad + \left|\int_{\mathbb{T}} \left(v(\tilde{r}_{d+1}e(x_{d+1})) - \tilde{v}(\tilde{r}_{d+1}e(x_{d+1}))\right)\,dx_{d+1}\right| \\
&\le C_\rho |r_{d+1} - \tilde{r}_{d+1}| + C_\rho \sum_{i=1}^{d} |r_i - \tilde{r}_i|.
\end{aligned}$$

As an immediate consequence of Lemma 3.5 we have the following result.

Proposition 3.6. *Let* $I = [a, b]$*. If* $(\eta, \omega, E) \in \mathbb{R}^d \times \mathbb{C}^d \times \mathbb{C}$ *are such that*

$$\max(|a|, |b|)\|\operatorname{Im}\omega\| \le \rho(V), \quad \|\eta\| \le \rho(V)$$

then

$$|L_I(\eta, \omega, E) - L_I(\omega, E)| \le C\|\eta\|, C = C(V, d, |E|).$$

The other ingredient needed for Proposition 3.11 is the stability of the logarithm of the norm of the transfer matrix. We have the following "rough" estimate, that will be refined through the use of the Avalanche Principle.

Lemma 3.7. *Let* $I = [a, b]$*. Let* $(\theta_i, \omega_i, E_i) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}$*,* $i = 1, 2$*, such that*

$$|E_2| \le |E_1|, \quad \|\operatorname{Im}\theta_i\| \le \rho(V), \quad \max(|a|, |b|)\|\operatorname{Im}\omega_i\| \le \rho(V).$$

Then we have

$$\begin{aligned}
&|\log\|M_I(\theta_1, \omega_1, E_1)\| - \log\|M_I(\theta_2, \omega_2, E_2)\|| \\
&\quad \le e^{C(V,|E_1|)|I|} \frac{\|\theta_1 - \theta_2\| + \max(|a|, |b|)\|\omega_1 - \omega_2\| + |E_1 - E_2|}{\max_i \|M_I(\theta_i, \omega_i, E_i)\|},
\end{aligned}$$

provided the right-hand side is $\ll 1$*.*

Proof. Let $Y_i(t)$, $i = 1, 2$, be solutions of the equations

$$Y_i'(t) = A_i(t)Y_i(t), \quad A_i(t) = \begin{bmatrix} 0 & 1 \\ V(t, \theta_i + t\omega_i) - E_i & 0 \end{bmatrix}.$$

Using the variations of constants method we have

$$Y_2(b) = M_{[a,b]}Y_2(a) + \int_a^b M_{[a,s]}(A_2(s) - A_1(s))Y_2(s)\,ds$$

where M denotes the transfer matrix corresponding to the equation with $i = 1$. So, if $Y_1(a) = Y_2(a)$, then

$$\begin{aligned}\|Y_2(b) - Y_1(b)\| &\le \int_a^b \|M_{[a,s]}\|\|A_2(s) - A_1(s)\|\|Y_2(s)\|\,ds\\ &\le \|Y_2(a)\|\int_a^b e^{C(V,|E_1|)(s-a)}(C(V)\|\theta_1 - \theta_2\| + C(V)|s|\|\omega_1 - \omega_2\|\\ &\quad +|E_1 - E_2|)\,ds\\ &\le \|Y_2(a)\|e^{C|I|}(\|\theta_1 - \theta_2\| + \max(|a|, |b|)\|\omega_1 - \omega_2\| + |E_1 - E_2|).\end{aligned}$$

This implies

$$\begin{aligned}&\|M_I(\theta_1, \omega_1, E_1) - M_I(\theta_2, \omega_2, E_2)\|\\ &\quad\le e^{C(V,|E_1|)|I|}(\|\theta_1 - \theta_2\| + \max(|a|, |b|)\|\omega_1 - \omega_2\| + |E_1 - E_2|).\end{aligned}$$

The conclusion follows from this and the fact that $|\log x| \lesssim |x-1|$, provided $|x-1| \ll 1$.

To refine the previous estimate, let us recall the Avalanche Principle.

Proposition 3.8 ([GS01, Proposition 2.2]). *Let $A_1, \ldots, A_n$ be a sequence of 2×2-matrices. Suppose that*

$$\min_{1\le j\le n} \|A_j\| \ge \mu > n \text{ and}$$
$$\max_{1\le j<n} \left(\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|\right) < \frac{1}{2}\log\mu.$$

Then

$$\left|\log\|A_n \ldots A_1\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\right| < C\frac{n}{\mu}.$$

Proposition 3.9. *Let $I = [a, b]$. Let $(\eta_0, \omega_0, E_0) \in \mathbb{R}^d \times \mathrm{DC}_{|I|} \times \mathbb{C}$, such that $\|\eta_0\| \le \rho(V)$, $L_I(\eta_0, \omega_0, E_0) \ge \gamma > 0$. Let*

$$\ell \ge C(V, d, \mathrm{DC}, |E_0|, \gamma)(\log(1 + |I|))^{1/\sigma}. \tag{3.3}$$

There exists a set

$$\begin{aligned}&\mathcal{B} = \mathcal{B}_{I,\omega_0,E_0,\eta_0,\ell},\ \mathrm{mes}(\mathcal{B}) \le C\exp(-c\ell^\sigma),\\ &C = C(V, d, \mathrm{DC}, |E_0|, \gamma),\ c = c(V, d, \mathrm{DC}, |E_0|)\end{aligned}$$

such that for any $\theta_0 \in \mathbb{T}^d\backslash\mathcal{B}$ and any $(\theta, \omega, E) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}$ satisfying

$$\|\theta - \theta_0 - i\eta_0\| + \max(|a|, |b|)\|\omega - \omega_0\| + |E - E_0| \le \exp(-\ell^2),$$

we have

$$|\log\|M_I(\theta_0 + i\eta_0, \omega_0, E_0)\| - \log\|M_I(\theta, \omega, E)\|| \le \exp(-\gamma\ell/4).$$

Proof. Note that if $\ell \geq |I|^{1/2+}$ then the estimate holds for all θ_0 (provided $|I|$ is large enough) by Lemma 3.7. So it is enough to consider the case $\ell < |I|^{1/2+}$.

Partition I into n intervals J_i (ordered from left to right) of equal length and such that $|J_i| \simeq \ell$. Using the large deviations estimate and (3.1) we can apply the Avalanche Principle with $A_i = M_{J_i}(\theta_0 + i\eta_0, \omega_0, E_0)$, $\mu = \exp(\gamma\ell/2)$, provided θ_0 is outside of the set $\mathcal{B}$ where the large deviations estimate fails on the intervals J_i, $J_i \cup J_{i+1}$. Note that we can apply (3.1) because from Lemma 3.3 it follows that $L_{J_i} \geq 3\gamma/4$ (here we used $\ell < |I|^{1/2+}$). We clearly have

$$\mathrm{mes}(\mathcal{B}) \lesssim nC\exp(-c\ell^\sigma) \leq C\exp(-c\ell^\sigma/2).$$

We use the lower bound (3.3) on ℓ for the above measure estimate and to ensure that $\mu > n$.

From the assumptions on θ, ω, E and Lemma 3.7 it follows that we can also apply the Avalanche Principle with $\tilde{A}_i = M_{J_i}(\theta, \omega, E)$ and the same μ. Subtracting the two Avalanche Principle expansions and applying Lemma 3.7 again we get

$$\begin{aligned}
&|\log\|M_I(\theta_0 + i\eta_0, \omega_0, E_0)\| - \log\|M_I(\theta, \omega, E)\|| \\
&\leq \left|\sum \log\|A_i\| - \log\|\tilde{A}_i\|\right| + \left|\sum \log\|A_{i+1}A_i\| - \log\|\tilde{A}_{i+1}\tilde{A}_i\|\right| + C\frac{n}{\mu} \\
&\lesssim n\exp(C\ell)\exp(-\ell^2) + n\exp(-\gamma\ell/2) \leq \exp(-\gamma\ell/4).
\end{aligned}$$

Corollary 3.10. *Let* $I = [a, b]$. *Let* $(\eta_0, \omega_0, E_0) \in \mathbb{R}^d \times \mathrm{DC}_{|I|} \times \mathbb{C}$, *such that* $\|\eta_0\| \leq \rho(V)$, $L_I(\eta_0, \omega_0, E_0) \geq \gamma > 0$. *Let*

$$\ell \geq C(V, d, \mathrm{DC}, |E_0|, \gamma)(\log(1 + |I|))^{1/\sigma}.$$

Then we have

$$|L_I(\eta_0, \omega_0, E_0) - L_I(\eta_0, \omega, E)| \leq C\exp(-c\ell^\sigma)$$

with $C(V, d, \mathrm{DC}, |E_0|, \gamma)$, $c = c(V, d, \mathrm{DC}, |E_0|)$, *for any* $(\omega, E) \in \mathbb{C}^d \times \mathbb{C}$ *such that*

$$\max(|a|, |b|)\|\omega - \omega_0\| + |E - E_0| \leq \exp(-\ell^2).$$

Proof. The conclusion follows by integrating the estimate from Proposition 3.9.

We are now ready to prove the uniform upper bound.

Proposition 3.11. *Let* $I = [a, b]$. *Let* $(\omega_0, E_0) \in \mathrm{DC}_{|I|} \times \mathbb{C}$ *be such that* $L_I(\omega_0, E_0) \geq \gamma > 0$. *Then*

$$\sup_{\theta\in\mathbb{T}^d} \log\|M_I(\theta + \eta, \omega, E)\| \leq |I|L_I(\omega_0, E_0) + C|I|^{1-\sigma}, C = C(V, d, \mathrm{DC}, |E_0|, \gamma)$$

for any $(\eta, \omega, E) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}$ *such that* $\|\eta\| \leq \rho(V)/(1 + |I|)$ *and*

$$\max(|a|, |b|)\|\omega - \omega_0\| + |E - E_0| \leq \exp(-C(\log(1+|I|))^{2/\sigma}), C = C(V, d, \mathrm{DC}, |E_0|, \gamma). \tag{3.4}$$

Proof. Let $\mathcal{B}_\eta^{(1)} = \mathcal{B}_{\eta,E_0,\omega_0}^{(1)}$ be the set from Proposition 3.9 with $\ell = C(\log(1+|I|))^{1/\sigma}$ and $C = C(V, d, \mathrm{DC}, |E_0|, \gamma)$ large enough so that $\mathrm{mes}(\mathcal{B}_\eta^{(1)}) \le 1/|I|^{4d}$. Let $\mathcal{B}_\eta^{(2)} = \mathcal{B}_{\eta,E_0,\omega_0}^{(2)}$ be the exceptional set from Theorem 3.1. We define

$$\mathcal{B} = \left\{\theta \in \mathbb{C}^d : \|\mathrm{Im}\,\theta\| \le \rho(V), \mathrm{Re}\,\theta \in \mathcal{B}_{\mathrm{Im}\,\theta}^{(1)} \cup \mathcal{B}_{\mathrm{Im}\,\theta}^{(2)}\right\}.$$

We clearly have $\mathrm{mes}(\mathcal{B}) \le 2/|I|^{4d}$ and if $\theta \in \mathbb{C}^d \setminus \mathcal{B}$, $\|\mathrm{Im}\,\theta\| \le \rho(V)/(1+|I|)$, then by Theorem 3.1 and Lemma 3.5

$$\log\|M_I(\theta, \omega_0, E_0)\| \le |I| L_I(\mathrm{Im}\,\theta, \omega_0, E_0) + |I|^{1-\sigma} \le |I| L_I(\omega_0, E_0) + 2|I|^{1-\sigma},$$

and by Proposition 3.9

$$\log\|M_I(\theta, \omega, E)\| \le |I| L_I(\omega_0, E_0) + 3|I|^{1-\sigma} \tag{3.5}$$

for any ω, E satisfying (3.4).

Let $\theta_0 \in \mathbb{T}^d$ arbitrary and η, ω, E satisfying the needed assumptions. Let B_r be the ball centered at $\theta_0 + \eta$ and of radius $r = 1/|I|^2$. Using the submean property of plurisubharmonic functions, (2.5), and (3.5), we have

$$\begin{aligned}\log\|M_I(\theta_0 + \eta, \omega, E)\| &\le \frac{1}{\mathrm{mes}(B_r)} \int_{B_r} \log\|M_I(\theta, \omega, E)\|\, d\theta \\ &\le \frac{1}{\mathrm{mes}(B_r)} \Big(C\,\mathrm{mes}(\mathcal{B}) + (\mathrm{mes}(B_r) - \mathrm{mes}(\mathcal{B}))(|I| L_I(\omega_0, E_0) + 3|I|^{1-\sigma})\Big) \\ &\le |I| L_I(\omega_0, E_0) + C|I|^{1-\sigma}.\end{aligned}$$

This concludes the proof.

Corollary 3.12. *Let $I = [a, b]$. If $(\omega, E) \in \mathrm{DC} \times \mathbb{C}$ are such that $L(\omega, E) \ge \gamma > 0$ then*

$$\sup_{\theta \in \mathbb{T}^d} \log\|M_I(\theta, \omega, E)\| \le |I| L(\omega, E) + C|I|^{1-\sigma}$$
$$\sup_{\theta \in \mathbb{T}^d} \log\|M_I(\theta, \omega, E)^{-1}\| \le |I| L(\omega, E) + C|I|^{1-\sigma}$$

with $C = C(V, d, \mathrm{DC}, |E|, \gamma)$.

Proof. The first estimate follows from Proposition (3.11) and (3.2).

Since $\det M_I = 1$ we have

$$\|M_I^{-1}\|_{\mathrm{HS}} = \|M_I\|_{\mathrm{HS}} \le \sqrt{2}\|M_I\|$$

and the second estimate follows ($\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm).

4. Decay of Green's Function

We consider Green's function on a finite interval $I = [a, b]$ with Dirichlet boundary conditions:

$$G_I(s,t) = G_I(s,t;\theta,\omega,E) = \begin{cases} \dfrac{v_a(s)v_b(t)}{W(v_a,v_b)}, & s \le t \\ \dfrac{v_a(t)v_b(s)}{W(v_a,v_b)}, & t \le s \end{cases}. \tag{4.1}$$

Recall that the functions v satisfy the initial conditions (2.2).

We will show that if a large deviations estimate holds on some interval, then we get exponential decay for Green's function on another interval of roughly the same size (this is similar to what happens in the discrete case, see [Bou05a, Proposition7.19]). In fact, due to Poisson's formula we will need this result for the partials of Green's function. Recall that for any solution y of (2.1), on an interval containing I, the Poisson formula reads

$$y(t) = y(b)\partial_s G_I(b,t) - y(a)\partial_s G_I(a,t).$$

Proposition 4.1 *Let $I = [a, b]$. Let $(\omega, E) \in \mathrm{DC} \times \mathbb{C}$ be such that $L(\omega, E) \ge \gamma > 0$. If*

$$\log\|M_I(\theta,\omega,E)\| \ge |I|L(\omega,E) - K,$$

with

$$C(|I|^{1-\sigma}+1) \le K \le |I|^{1-}, C = C(V, d, \mathrm{DC}, |E|, \gamma)$$

(recall that σ is as in Theorem 3.1), then there exists an interval $J = J(\theta, \omega, E)$ such that

$$J \subset I,\ |I| - |J| \le 4K/\gamma$$
$$|G_J(s,t)|, |\partial_s G_J(s,t)| \le \exp(-|s-t|L(\omega,E) + 2K),\ s,t \in J.$$

Proof. By our assumption, at least one of the entries of M_I has to be $\ge \frac{1}{2}\exp(|I|L - K)$. We treat each of the four possibilities separately.
(1) Suppose

$$|v_a(b)| \ge \frac{1}{2}\exp(|I|L - K).$$

In this case we let $J = I$. Using Corollary 3.12 we have

$$|G_I(s,t)| = \left|\frac{v_a(s)v_b(t)}{v_a(b)}\right| \le 2\exp((s-a)L + (b-t)L + C|I|^{1-\sigma} - (b-a)L + K)$$
$$\le \exp(-(t-s)L + 2K)$$

provided $s \le t$ (it is enough to consider this case because $G_I(s,t) = G_I(t,s)$). We used the fact that

$$W(v_a, v_b) = v_a(b) = -v_b(a). \tag{4.2}$$

The bound on $|\partial_s G_I|$ is obtained in the same way, because the bounds from Corollary 3.12 apply to all the entries of the transfer matrix.
(2) Suppose

$$|v_a'(b)| \ge \frac{1}{2}\exp(|I|L - K).$$

For any $t \in (a, b)$, there exists $\tilde{t} \in (t, b)$ such that

$$|v_a'(b) - v_a'(t)| = |v_a''(\tilde{t})(b-t)| = |v_a(\tilde{t})(V(\tilde{t}, \theta + \tilde{t}\omega) - E)(b-t)|. \tag{4.3}$$

Using Corollary 3.12 and choosing t so that $b - t = 2K/\gamma$, it follows that

$$\begin{aligned}|v_a(\tilde{t})| &\geq \frac{1}{C(V, |E|)(b-t)} |v_a'(b) - v_a'(t)| \\ &\geq \frac{1}{2C(b-t)} \exp(|I|L - K) \left[1 - \exp((t-a)L + C|I|^{1-\sigma} - |I|L + K + \log 2)\right] \\ &\geq \frac{1}{2C(b-t)} \exp(|I|L - K) \left[1 - \frac{1}{2}\exp(-(b-t)L + 2K)\right] \\ &\geq \frac{1}{4C(b-t)} \exp(|I|L - K) \geq \exp(|I|L - 3K/2).\end{aligned}$$

The conclusion follows by the reasoning from case (1) applied to $J = [a, \tilde{t}]$.
(3) Suppose

$$|u_a(b)| \geq \frac{1}{2} \exp(|I|L - K).$$

Note that we have

$$W(u_a, v_b) = u_a(b) = v_b'(a).$$

Then, by the reasoning from case (2), there exists $\tilde{t} \in I$, $\tilde{t} - a \leq 2K/\gamma$ such that

$$|v_b(\tilde{t})| \geq \exp(|I|L - 3K/2).$$

Recall from (4.2) that we have $|v_b(\tilde{t})| = |v_{\tilde{t}}(b)|$ and so the conclusion follows by the argument from case (1) applied on $J = [\tilde{t}, b]$.
(4) Suppose

$$|u_a'(b)| \geq \frac{1}{2} \exp(|I|L - K).$$

By the argument from case (2), there exists $\tilde{t}$, $b - \tilde{t} \leq 2K/\gamma$ such that

$$|u_a(\tilde{t})| \geq \exp(|I|L - 3K/2).$$

Following the reasoning from case (3) we get that there exists $\bar{t}$, $\bar{t} - a \leq 2K/\gamma$ such that

$$|v_{\bar{t}}(\tilde{t})| \geq \exp(|I|L - 5K/4).$$

The conclusion follows as in case (1) by taking $J = [\bar{t}, \tilde{t}]$.

Next we illustrate the well-known strategy of iterating Poisson's formula to get the exponential decay of solutions, provided that we have the decay of Green's function.

Lemma 4.2. *Let* $0 < \ell \ll a \ll b$ *and* $m > 0$ *such that* $m\ell \gg 1$. *Suppose that for any* $t \in [a, b]$ *there exists an interval* $J = [c_t, d_t]$, $|J| \leq \ell$, *such that* $t \in J$, *and*

$$|\partial_s G_J(c_t, t)|, |\partial_s G_J(d_t, t)| \leq \exp(-m\ell).$$

Then any solution y *of* (2.1) *satisfies*

$$|y(t)| \leq M \exp(-mt/8), \ t \in [2a, b/2],$$

where $M = \sup_{[a,b]} |y|$.

Proof. For any $t \in [2a, b/2]$ we can iterate Poisson's formula (on intervals J satisfying the assumptions) at least

$$n = \min\left([(t-a)/\ell], [(b-t)/\ell]\right) \geq \frac{t}{4\ell}$$

times to get

$$|y(t)| \leq M2^n \exp(-nm\ell) \leq M \exp(-nm\ell/2) \leq M \exp(-mt/8).$$

5. Cartan Sets

We will use $D(z, r)$ to denote the disk of radius r centered at $z \in \mathbb{C}$.

Definition 5.1. Let $H \gg 1$. For an arbitrary subset $\mathcal{B} \subset D(z_0, 1) \subset \mathbb{C}$ we say that $\mathcal{B} \in \mathrm{Car}_1(H, K)$ if $\mathcal{B} \subset \bigcup_{j=1}^{j_0} D(z_j, r_j)$ with $j_0 \leq K$, and

$$\sum_j r_j < e^{-H}. \tag{5.1}$$

If d is a positive integer greater than one and $\mathcal{B} \subset \prod_{i=1}^d D(z_{i,0}, 1) \subset \mathbb{C}^d$ then we define inductively that $\mathcal{B} \in \mathrm{Car}_d(H, K)$ if for any $J \subset \{1, \ldots, d\}$, $|J| < d$, there exists

$$\mathcal{B}_J \subset \prod_{j \in J} D(z_{j,0}, 1) \subset \mathbb{C}^{|J|},\ \mathcal{B}_J \in \mathrm{Car}_{|J|}(H, K)$$

so that $\mathcal{B}_{J'}(z) \in \mathrm{Car}_{|J'|}(H, K)$ for any $z \in \mathbb{C}^{|J|} \backslash \mathcal{B}_J$, where

$$\mathcal{B}_{J'}(z) = \{w_{J'} : w \in \mathcal{B}, w_J = z\}.$$

We used J' to denote $\{1, \ldots, d\} \backslash J$ and given $z \in \mathbb{C}^d$, z_J denotes the vector $(z_j)_{j \in J}$.

The above definition is a simple extension of [GS08, Definition 2.12], where only the case $|J| = 1$ is considered. The reason behind the definition of Cartan sets is the following result, referred to as the Cartan estimate. The Cartan estimate from [GS08] holds even with this slightly more general definition. The proof is essentially the same, one only needs to use complete induction instead of the regular induction used in [GS08].

Lemma 5.2 ([GS08, Lemma 2.15]). *Let $\varphi(z_1, \ldots, z_d)$ be an analytic function defined in a polydisk $P = \prod_{j=1}^d D(z_{j,0}, 1)$, $z_{j,0} \in \mathbb{C}$. Let $M \geq \sup_{z \in P} \log|\varphi(z)|$, $m \leq \log\left|\varphi(z_0)\right|$, $z_0 = (z_{1,0}, \ldots, z_{d,0})$. Given $H \gg 1$, there exists a set $\mathcal{B} \subset P$, $\mathcal{B} \in \mathrm{Car}_d\left(H^{1/d}, K\right)$, $K = C_d H(M - m)$, such that*

$$\log\left|\varphi(z)\right| > M - C_d H(M - m) \tag{5.2}$$

for any $z \in \prod_{j=1}^d D(z_{j,0}, 1/6) \backslash \mathcal{B}$.

Let us note that the definition of the Cartan sets gives information about their measure.

Lemma 5.3. *For any $\mathcal{B} \subset \prod_{i=1}^d D(z_{i,0}, 1) \subset \mathbb{C}^d$ such that $\mathcal{B} \in \mathrm{Car}_d(H, K)$ we have*

$$\mathrm{mes}_{\mathbb{C}^d}(\mathcal{B}) \lesssim de^{-H} \text{ and } \mathrm{mes}_{\mathbb{R}^d}(\mathcal{B} \cap \mathbb{R}^d) \lesssim de^{-H}.$$

Proof. The case $d = 1$ follows immediately from the definition of Car_1. The case $d > 1$ follows by induction, using Fubini and the definition of Car_d.

We use Cartan's estimate to argue that if the large deviations estimate fails then for fixed phase and frequency the energy must be in a finite union of small intervals, with a good bound on the number of intervals. This is only possible up to some small exceptional sets of phases and frequencies. To be able to apply Cartan's estimate effectively we need to restrict ourselves to the case when the Lyapunov exponent is positive, so that we have the uniform upper bound from Proposition 3.11.

Proposition 5.4. *Let $I = [a, b]$. Let $[E', E''] \subset \mathbb{R}$, $\gamma > 0$ and*

$$\mathcal{P}_I = \{(\omega, E) \in \mathrm{DC}_{|I|} \times [E', E''] : L_I(\omega, E) \geq \gamma\}.$$

Let

$$H \geq C(\log(1 + |I|))^A,\ C = C(V, d, \mathrm{DC}, E', E'', \gamma),\ A = A(\sigma, d).$$

If $|I| \geq C(V, d, \mathrm{DC}, E', E'', \gamma)$ then there exists

$$\Theta_I \subset \mathbb{T}^d,\ \mathrm{mes}(\Theta_I) \leq \exp(-H^{1/(2d+1)}/2)$$

such that if $\theta \in \mathbb{T}^d \setminus \Theta_I$ and $(\omega, E) \in \mathcal{P}_I$ are such that

$$\log\|M_I(\theta, \omega, E)\| \leq |I|L_I(\omega, E) - CH|I|^{1-\sigma},\ C = C(V, d, \mathrm{DC}, E', E'', \gamma)$$

then either $\omega \in \Omega_{I,\theta}$ or $E \in \mathcal{E}_{I,\theta,\omega}$, where $\mathrm{mes}(\Omega_{I,\theta}) \leq \exp(-H^{1/(2d+1)}/2)$ and $\mathcal{E}_{I,\theta,\omega}$ is the union of less than

$$H \exp(C(\log(1 + |I|))^{2/\sigma}),\ C = C(V, d, \mathrm{DC}, E', E'', \gamma)$$

intervals, each of measure less than $\exp(-H^{1/(2d+1)})$.

Proof. Let $r = \exp(-C(\log(1 + |I|))^{2/\sigma})$ and let

$$\begin{aligned} P(\theta_j, r),\ 1 \leq j \lesssim r^{-d} \\ P(\omega_k, r/\max(|a|, |b|)),\ 1 \leq k \lesssim r^{-d} \max(|a|, |b|)^d \\ D(E_l, r),\ 1 \leq l \lesssim |E'' - E'|,\ 1 \leq l \lesssim r^{-1}|E'' - E'| \end{aligned}$$

be covers of $\mathbb{T}^d$, $\mathrm{DC}_{|I|}$, and $[E', E'']$ respectively ($P(\cdot, r)$ denotes a polydisk of radius r). Let

$$\mathcal{I} = \{(j, k, l) : (P(\theta_j, r) \times P(\omega_k, r/\max(|a|, |b|)) \times D(E_l, r)) \cap (\mathbb{T}^d \times \mathcal{P}_I) \neq \emptyset\}.$$

Let $\iota = (j, k, l) \in \mathcal{I}$ and

$$P_\iota(r) = P(\theta_j, r) \times P(\omega_k, r/\max(|a|, |b|)) \times D(E_l, r).$$

By the definition of $\mathcal{I}$ and Theorem 3.1 there exists $(\theta_\iota, \omega_\iota, E_\iota) \in P_\iota(r) \cap (\mathbb{T}^d \times \mathcal{P}_I)$ such that

$$\log\|M_I(\theta_\iota, \omega_\iota, E_\iota)\| \geq |I|L_I(\omega_\iota, E_\iota) - |I|^{1-\sigma}.$$

Let

$$\tilde{P}_\iota(r) = P(\theta_\iota, r) \times P(\omega_\iota, r/\max(|a|, |b|)) \times D(E_\iota, r).$$

Since $L_I(\omega_\iota, E_\iota) \geq \gamma$ we can use Proposition 3.11 to guarantee that

$$\sup\{\log\|M_I(\theta, \omega, E)\| : (\theta, \omega, E) \in \tilde{P}_\iota(12r)\} \leq |I| L_I(\omega_\iota, E_\iota) + C|I|^{1-\sigma}$$

with $C = C(V, d, \mathrm{DC}, E', E'', \gamma)$ (we just need to take C from the definition of r to be large enough).

Next we set things up to apply Lemma 5.2. Let

$$\begin{aligned} f_I &= u_a(b)^2 + u_a'(b)^2 + v_a(b)^2 + v_a'(b)^2 \\ S_\iota(\theta, \omega, E) &= (\theta_\iota + 12r\theta, \omega_\iota + 12r\omega/\max(|a|, |b|), E_\iota + 12rE) \\ \phi_\iota(\theta, \omega, E) &= f_I(S_\iota(\theta, \omega, E)),\ (\theta, \omega, E) \in D(0,1)^{2d+1}. \end{aligned}$$

Note that we have

$$|f_I| \leq \|M_I\|_{\mathrm{HS}}^2 \leq 2\|M_I\|^2$$

and if $(\theta, \omega, E) \in \mathbb{R}^{2d+1}$, then

$$|f_I(\theta, \omega, E)| = \|M_I(\theta, \omega, E)\|_{\mathrm{HS}}^2 \geq \|M_I(\theta, \omega, E)\|^2$$

and therefore

$$\begin{aligned} \log|\phi_\iota(\theta_\iota, \omega_\iota, E_\iota)| &\geq 2(|I| L_I(\omega_\iota, E_\iota) - |I|^{1-\sigma}), \\ \sup_{D(0,1)^{2d+1}} \log|\phi_\iota(\theta, \omega, E)| &\leq 2(|I| L_I(\omega_\iota, E_\iota) + C|I|^{1-\sigma}). \end{aligned}$$

By applying Cartan's estimate to ϕ_ι and by using Corollary 3.10 we get that

$$\log\|M_I(\theta, \omega, E)\| \geq |I| L_I(\omega_\iota, E_\iota) - CH|I|^{1-\sigma} \geq |I| L_I(\omega, E) - 2CH|I|^{1-\sigma}$$

for all

$$(\theta, \omega, E) \in P_\iota(r) \backslash S_\iota(\mathcal{B}_\iota) \subset \tilde{P}_\iota(2r) \backslash S_\iota(\mathcal{B}_\iota)$$

with $\mathcal{B}_\iota \in \mathrm{Car}_{2d+1}(H^{1/(2d+1)}, K)$, $K = CH|I|^{1-\sigma}$.

From the definition of Cartan sets we know that there exists a set $\Theta_\iota \in \mathrm{Car}_d(H^{1/(2d+1)}, K)$ such that if $\theta \notin \Theta_\iota$, then $\mathcal{B}_\iota(\theta) \in \mathrm{Car}_{d+1}(H^{1/(2d+1)}, K)$, with

$$\mathcal{B}_\iota(\theta) = \{(\omega, E) : (\theta, \omega, E) \in \mathcal{B}_\iota\}.$$

Applying the definition again we see that if $(\omega, E) \in \mathcal{B}_\iota(\theta)$ then either

$$\omega \in \Omega_{\iota,\theta},\ \Omega_{\iota,\theta} \in \mathrm{Car}_d(H^{1/(2d+1)}, K)$$

or

$$E \in \mathcal{E}_{\iota,\theta,\omega} := \mathcal{B}_\iota(\theta, \omega),\ \mathcal{E}_{\iota,\theta,\omega} \in \mathrm{Car}_1(H^{1/(2d+1)}, K).$$

Note that by the definition of Car_1 we have that $\mathcal{E}_{\iota,\theta,\omega} \cap \mathbb{R}$ is contained in the union of at most K intervals, each of measure smaller than $\exp(-H^{1/(2d+1)})$.

Let

$$\begin{aligned} j_0 &= j_0(\theta) = \min\{j : \theta \in P(\theta_j, r)\} \\ k_0 &= k_0(\theta, \omega) = \min\{k : (\theta, \omega) \in P(\theta_{j_0}, r) \times P(\omega_k, r/\max(|a|, |b|))\}. \end{aligned}$$

Now the conclusion follows by setting

$$\begin{aligned} \Theta_I &= \cup\{(\theta_j + 12r\Theta_\iota) \cap P(\theta_j, r) : \iota = (j, k, l) \in \mathcal{I}\} \cap \mathbb{R}^d, \\ \Omega_{I,\theta} &= \cup\{(\omega_k + 12r/\max(|a|, |b|)\Omega_{\iota,\theta}) \cap P(\omega_k, r/\max(|a|, |b|)) : \\ &\qquad \iota = (j_0, k, l) \in \mathcal{I}\} \cap \mathbb{R}^d, \\ \mathcal{E}_{I,\theta,\omega} &= \cup\{(E_l + 12r\mathcal{E}_{\iota,\theta,\omega}) \cap D(E_l, r) : \iota = (j_0, k_0, l) \in \mathcal{I}\} \cap \mathbb{R}. \end{aligned}$$

Note that for the measure estimates we use Lemma 5.3.

6. Semialgebraic Sets

Recall that a set $\mathcal{S} \subset \mathbb{R}^n$ is called semialgebraic if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. So, a (closed) semialgebraic set is given by an expression

$$\mathcal{S} = \cup_j \cap_{\ell \in L_j} \{P_\ell s_{j\ell} 0\},$$

where $\{P_1, \dots, P_s\}$ is a collection of polynomials of n variables, $L_j \subset \{1, \dots, s\}$ and $s_{jl} \in \{\geq, \leq, =\}$. If the degrees of the polynomials are bounded by d then we say that the degree of $\mathcal{S}$ is bounded by sd. We refer to [Bou05a, Chapter 9] for more information on semialgebraic sets.

The main result of this section is Lemma 6.7, in which we argue that the set of (θ, ω, E) for which the large deviations estimate fails is contained in a semialgebraic set of controlled size and degree. To this end we will need to approximate the entries of M_I by polynomials of controlled degree. Since V is complex analytic in a neighborhood of $\mathbb{H}_\rho^{d+1}$, $\rho = \rho(V)$ (recall (2.10)), and periodic in the real direction it follows that any entry of M_I is also complex analytic in a neighborhood of

$$\mathbb{H}_{\rho'}^{2d} \times \mathbb{C}, \ \rho' = \frac{\rho}{2(1 + \max(|a|, |b|))}, \tag{6.1}$$

and periodic in the real direction for the phase variables (we chose ρ' such that $\theta + t\omega \in \mathbb{H}_\rho^d$ for $t \in I$). We can use Fourier series and Taylor series to obtain a polynomial approximation in the phase and energy variables, but not in the frequency variables. One could use Taylor series for the frequency variables, but only at the cost of getting different approximating polynomials on different frequency intervals. We avoid this inconvenience by using Faber series.

We recall the basic information we will need about Faber polynomials and Faber series. We refer to [Sue98, Chapters 2, 3] for further information (see also [Mar67, Sects. 3.14–15]). Let $K \subset \mathbb{C}$ be a compact set such that its complement is simply connected (on the Riemann sphere). Let φ_K be the conformal mapping of the complement of K onto the complement of the unit disk, normalized such that $\varphi_K(\infty) = \infty$ and $\varphi_K'(\infty) > 0$. Faber's polynomials, denoted by $\Phi_{K,n}$, $n \geq 0$, are the polynomial parts of the Laurent series expansion of φ_K^n at ∞. It is clear from the definition that $\Phi_{K,n}$ has degree n. Given $R > 1$ we let $\Gamma_{K,R} = \varphi_K^{-1}(\{|z| = R\})$ and we denote by $G_{K,R}$ the bounded domain enclosed by $\Gamma_{K,R}$. It can be seen that

$$\Phi_{K,n}(z) = \frac{1}{2\pi i} \int_{\Gamma_{K,R}} \frac{\varphi_K^n(\zeta)}{\zeta - z} \, d\zeta, \ z \in G_{K,R}. \tag{6.2}$$

If f is an analytic function on $G_{K,R}$, then it can be expanded in a series with respect to the Faber polynomials

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_{K,n}(z), \ z \in G_{K,R}$$

which converges absolutely and locally uniformly in $G_{K,R}$. The Faber coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{|t|=\rho} \frac{f(\varphi_K^{-1}(t))}{t^{n+1}} \, dt$$

for any $\rho \in (1, R)$.

We are now ready to state our abstract approximation result for functions which are analytic on product sets.

Proposition 6.1. *Let $K_1, \ldots, K_m$ be compact sets in $\mathbb{C}$ such that their complements are simply connected (on the Riemann sphere). Let $R > 1$ and let f be an analytic function on a neighborhood of the closure of $G_{K_1,R} \times \cdots \times G_{K_m,R}$. Given $N \geq 0$ and $R' \in (1, R)$, there exists a polynomial P_N of degree at most N such that*

$$\sup\{|f(z) - P_N(z)| : z \in K_1 \times \cdots \times K_m\}$$
$$\leq C(m)\left(\frac{R'}{R}\right)^N \left(\prod_{i=1}^{m} \frac{\ell(\Gamma_{K_i,R'})}{d(K_i, \Gamma_{K_i,R'})}\right) \sup\{|f(z)| : z \in G_{K_1,R} \times \cdots \times G_{K_m,R}\},$$

where $\ell(\Gamma_{K_i,R'})$ denotes the length of $\Gamma_{K_i,R'}$ and $d(K_i, \Gamma_{K_i,R'})$ denotes the distance between K_i and $\Gamma_{K_i,R'}$.

Proof. If we take the Faber series expansion of f with respect to one of its variables, it is clear that the coefficients will be analytic with respect to the other variables. Since the Faber series converges absolutely we obtain through iteration the following expansion for f on $G_{K_1,R} \times \cdots \times G_{K_m,R}$

$$f(z_1, \cdots, z_m) = \sum_n a_n \Phi_{K_1,n_1}(z_1) \cdots \Phi_{K_m,n_m}(z_m),$$

with the coefficients given by

$$a_n = \frac{1}{(2\pi i)^m} \int_{|t_m|=R} \cdots \int_{|t_1|=R} \frac{f(\varphi_{K_1}^{-1}(t_1), \cdots, \varphi_{K_m}^{-1}(t_m))}{t_1^{n_1+1} \cdots t_m^{n_m+1}} \, dt_1 \cdots dt_m.$$

Note that we have

$$|a_n| \leq \frac{1}{(2\pi)^m R^{|n|}} \sup\{|f(z)| : z \in G_{K_1,R} \times \cdots \times G_{K_m,R}\},$$

where $|n| = n_1 + \cdots + n_m$. Also, from (6.2) (with R' instead of R) it follows that

$$\sup\{|\Phi_{K_i,n_i}(z)| : z \in K_i\} \leq \frac{(R')^{n_i} \ell(\Gamma_{K_i,R'})}{2\pi d(K_i, \Gamma_{K_i,R'})}.$$

Therefore the conclusion holds by taking

$$P_N(z_1, \ldots, z_m) = \sum_{|n| \leq N} a_n \Phi_{K_1,n_1}(z_1) \cdots \Phi_{K_m,n_m}(z_m).$$

Remark 6.2. The previous proposition is a more explicit version of the direct statement of the so called Bernstein-Walsh theorem. Such results are also known for functions which are not necessarily defined on a product set, see [Sic81,Lev06], but their statements are not explicit enough for our purposes.

Lemma 6.3. *Let* $I = [a, b]$*,* $T \geq 0$*,* $[E', E''] \subset \mathbb{R}$*. There exists a polynomial* $P_I(\theta, \omega, E)$ *of degree less than*

$$C[(1 + \max(|a|, |b|))(1 + |I|)(1 + T)]^2, \; C = C(V, d, E', E'') \tag{6.3}$$

such that

$$\left| \log\|M_I(\theta + t\omega, \omega, E)\| - \frac{1}{2} \log |P_I(\theta + t\omega, \omega, E)| \right| \lesssim 1,$$

for any $(\theta, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d \times [E', E'']$ *and* $|t| \leq T$.

Proof. Let $f(\theta, \omega, E)$ denote one of the entries of $M_I(\theta, \omega, E)$. We already noted that f is analytic on $\mathbb{H}^{2d}_{\rho'} \times \mathbb{C}$ (see (6.1)). Let $K_i = [-L_i, L_i]$, where

$$L_i = \begin{cases} 1 + T, & i = 1, \ldots, d \\ 1, & i = d + 1, \ldots, 2d \\ \max(|E'|, |E''|), & i = 2d + 1 \end{cases}.$$

We want to apply Proposition 6.1 to approximate f by a polynomial on $\prod_i K_i$. The mappings needed for Proposition 6.1 are scaled versions of the Zhukowsky transform:

$$\varphi_{K_i}(z) = \frac{z}{L_i} + \sqrt{\left(\frac{z}{L_i}\right)^2 - 1}, \quad \varphi_{K_i}^{-1}(w) = \frac{L_i}{2}\left(w + \frac{1}{w}\right).$$

If we let $R = 1 + \varepsilon$, with $\varepsilon \ll \rho'/(\max_i L_i)$, then $\prod_i G_{K_i,R} \subset \mathbb{H}^{2d}_{\rho'} \times \mathbb{C}$. We choose $R' = (1 + R)/2$. It is elementary to see that

$$\ell(\Gamma_{K_i,R'}) \leq \pi L_i \left(R' + \frac{1}{R'}\right), \quad d(\Gamma_{K_i,R'}, K_i) = \frac{L_i}{2}\left(R' + \frac{1}{R'} - 2\right).$$

By Proposition 6.1, for any $N \geq 0$, there exists a polynomial P_N of degree less than N such that on $\prod_i K_i$ we have

$$\begin{aligned} |f - P_N| &\leq C(d)\left(\frac{R'}{R}\right)^N \left(\frac{(R')^2 + 1}{(R' - 1)^2}\right)^{2d+1} \|f\|_\infty \\ &\leq C \exp(-cN\varepsilon) \exp(-2(2d + 1) \log \varepsilon) \exp(C|I|), \end{aligned}$$

provided $\varepsilon \ll 1$. Clearly, if we choose N as in (6.3) we get that $|f - P_N| \leq \exp(-cN\varepsilon/2)$.

By approximating each entry of M_I in this way we obtain a 2×2 matrix with polynomial entries with the desired degree bounds and such that

$$\|M_I(\theta + t\omega, \omega, E) - \tilde{M}_I(\theta + t\omega, \omega, E)\| \ll 1$$

for any $(\theta, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d \times [E', E'']$ and $|t| \leq T$. Therefore, the conclusion holds with $P_I = \|\tilde{M}_I\|^2_{\mathrm{HS}}$.

We will also need a way to approximate the Lyapunov exponent $L_I(\omega, E)$. We use the same strategy of averaging over the phase shifts of $\log\|M_I\|$ as in [BG00, Lemma9.1], but we give a different proof. We base our proof on the following fact.

Lemma 6.4. ([Bou05a, Corolarry 9.7]) *Let $\mathcal{S} \subset [0,1]^d$ be semialgebraic of degree B and* $\operatorname{mes} \mathcal{S} < \eta$*. Let N be an integer such that*

$$\log B \ll \log N < \log \frac{1}{\eta}.$$

Then, for any $\theta_0 \in \mathbb{T}^d$, $\omega \in \mathrm{DC}_N$

$$\#\{n = 1, \dots, N : \theta_0 + n\omega \in \mathcal{S} (\bmod 1)\} < N^{1-\delta}$$

for some $\delta = \delta(\mathrm{DC})$.

In order to apply Lemma 6.4 we will need a semialgebraic approximation of the set where the large deviations estimate fails, but only in the phase variable.

Lemma 6.5. *Let $I = [a, b]$, $\omega \in \mathbb{T}^d$, $E \in [E', E'']$, and*

$$\mathcal{B}_I(H) = \mathcal{B}_I(H, \omega, E) := \{\theta \in \mathbb{T}^d : |\log\|M_I(\theta, \omega, E)\| - |I|L_I(\omega, E)| \geq H\}.$$

There exists a semialgebraic set $\mathcal{S}_I(H) = \mathcal{S}_I(H, \omega, E)$ of degree less than

$$C(1 + \max(|a|, |b|))(1 + |I|),\ C = C(V, d, E', E'')$$

such that

$$\mathcal{B}_I(H) \subset \mathcal{S}_I(H) \subset \mathcal{B}_I(H/2),$$

provided $H \gg 1$.

Proof. Let P_I be the polynomial from Lemma 6.3 with $T = 1$. Then

$$\left|\log\|M_I(\theta, \omega, E)\| - \frac{1}{2} \log |P_I(\theta, \omega, E)|\right| \leq C_0$$

and the conclusion follows by taking

$$\mathcal{S}_I(H) = \left\{\theta : \left|\frac{1}{2} \log |P_I(\theta, \omega, E)| - |I|L_I(\omega, E)\right| \geq H - C_0\right\}.$$

Now we can prove the estimate that will let us approximate L_I.

Lemma 6.6. *Let $I = [a, b]$, $\omega \in \mathrm{DC}_{|I|}$, $E \in [E', E'']$ such that $L_I(\omega, E) \geq \gamma > 0$. If $|I| \geq C(V, d, \mathrm{DC}, E', E'')$ then*

$$\left|\frac{1}{N} \sum_{n=1}^{N} \log\|M_I(\theta + n\omega, \omega, E)\| - |I|L_I(\omega, E)\right| \leq 2|I|^{1-\sigma}$$

for any integer N such that

$$C(V, d, \mathrm{DC}, E', E'') \log \max(|a|, |b|, |I|) \leq \log N < c(V, d, \mathrm{DC}, E', E'')|I|^{1-\sigma}.$$

Proof. Let

$$\mathcal{B} = \{\theta \in \mathbb{T}^d : |\log\|M_I(\theta, \omega, E)\| - |I|L_I(\omega, E)| \geq |I|^{1-\sigma}\}.$$

By Lemma 6.5 and Theorem 3.1, there exists a semi-algebraic set $\mathcal{S}$ such that

$$\mathcal{B} \subset \mathcal{S}, \quad \deg \mathcal{S} \leq C \max(|a|, |b|)|I|, \quad \operatorname{mes} \mathcal{S} \leq \exp(-c|I|^{1-\sigma}),$$

provided $|I| \geq C(V, d, \mathrm{DC}, E', E'')$.

For any $\theta \in \mathbb{T}^d \setminus \mathcal{S}$ we have

$$|I|L_I(\omega, E) - |I|^{1-\sigma} \leq \log\|M_I(\theta, \omega, E)\| \leq |I|L_I(\omega, E) + |I|^{1-\sigma},$$

whereas for $\theta \in \mathcal{S}$ we have

$$0 \leq \log\|M_I(\theta, \omega, E)\| \leq C|I|.$$

From the above and Lemma 6.4 we get

$$\begin{aligned} 2|I|^{1-\sigma} &\geq \frac{N - N^{1-\delta}}{N}|I|^{1-\sigma} + \frac{N^{1-\delta}}{N}C|I| \\ &\geq \frac{1}{N}\sum_{n=1}^{N} \log\|M_I(\theta + n\omega, \omega, E)\| - |I|L_I(\omega, E) \\ &\geq \frac{N - N^{1-\delta}}{N}(-|I|^{1-\sigma}) + \frac{N^{1-\delta}}{N}(-|I|L_I(\omega, E)) \geq -2|I|^{1-\sigma} \end{aligned}$$

provided

$$C(V, d, \mathrm{DC}, E', E'') \log \max(|a|, |b|, |I|) \leq \log N < c(V, d, \mathrm{DC}, E', E'')|I|^{1-\sigma}.$$

This concludes the proof.

Lemma 6.7. *Let $I = [a, b]$, $[E', E''] \subset \mathbb{R}$, $\gamma > 0$. If $|I| \geq C(V, d, \mathrm{DC}, E', E'', \gamma)$ and*

$$\begin{aligned} &T = (\max(|a|, |b|, |I|))^C, \ C = C(V, d, \mathrm{DC}, E', E'') \\ \mathcal{B}_I(H, \gamma) := \{(\theta, \omega, E) \in \mathbb{T}^d \times \mathrm{DC}_T \times [E', E''] : \quad &\log\|M_I(\theta, \omega, E)\| \leq |I|L_I(\omega, E) - H, \\ &L_I(\omega, E) \geq \gamma\} \end{aligned}$$

then there exists a semi-algebraic set $\mathcal{S}_I = \mathcal{S}_I(H, \gamma)$ of degree less than $T^{C(d)}$ such that

$$\mathcal{B}_I(H, \gamma) \subset \mathcal{S}_I(H, \gamma) \subset \mathcal{B}_I(H/2, \gamma/2),$$

provided

$$H \geq C|I|^{1-\sigma}, \ C = C(V, d, \mathrm{DC}, E', E'').$$

Proof. Let P_I be the polynomial from Lemma 6.3. Then

$$\left|\log\|M_I(\theta+t\omega,\omega,E)\| - \frac{1}{2}\log|P_I(\theta+t\omega,\omega,E)|\right| \le C_0,\ |t|\le T$$

and the degree of P_I is less than T^C. If furthermore, the power in the definition of T is large enough so that we can apply Lemma 6.6 with $N=[T]$, then the conclusion follows by taking $\mathcal{S}_I(H,\gamma)$ to be the set of $(\theta,\omega,E)\in\mathbb{T}^d\times \mathrm{DC}_T\times[E',E'']$ such that

$$\frac{1}{2}\log|P_I(\theta,\omega,E)| \le \frac{1}{N}\sum_{n=1}^{N}\frac{1}{2}\log|P_I(\theta+n\omega,\omega,E)| + 2C_0 + C|I|^{1-\sigma} - H$$

$$\gamma|I| \le \frac{1}{N}\sum_{n=1}^{N}\frac{1}{2}\log|P_I(\theta+n\omega,\omega,E)| + C_0 + C|I|^{1-\sigma}.$$

7. Elimination of Resonances

As in the discrete case, the elimination of resonances is based on the following result.

Lemma 7.1 ([Bou05a, Lemma 9.9]). *Let $\mathcal{S}\subset[0,1]^{2n}$ be a semialgebraic set of degree B and $\mathrm{mes}_{2n}\,\mathcal{S}<\eta$, $\log B\ll\log\frac{1}{\eta}$. We denote $(\theta,\omega)\in[0,1]^n\times[0,1]^n$ the product variable. Fix $\varepsilon>\eta^{\frac{1}{2n}}$. Then there is a decomposition*

$$\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$$

$\mathcal{S}_1$ satisfying

$$\mathrm{mes}_n(\mathrm{Proj}_\omega\,\mathcal{S}_1) < B^C\varepsilon$$

and $\mathcal{S}_2$ satisfying the transversality property

$$\mathrm{mes}_n(\mathcal{S}_2\cap L) < B^C\varepsilon^{-1}\eta^{\frac{1}{2n}}$$

for any n-dimensional hyperplane L s.t. $\max_{0\le j\le n-1}|\mathrm{Proj}_L(e_j)|<\frac{1}{100}\varepsilon$ (we denote by $e_0,\dots,e_{n-1}$ the ω-coordinate vectors).

Our elimination result is as follows.

Proposition 7.2. *Assume that $L(\omega,E)\ge\gamma>0$ for all $(\omega,E)\in\mathrm{DC}\times[E',E'']$. Let $I=[a,b]$, $J=[a',b']$. If*

$$|J|\simeq|I|^A\ge\max(|a|,|b|,|a'|,|b'|),\ |I|,A\ge C(V,d,\mathrm{DC},E',E'',\gamma),$$

there exists a set

$$\Theta_J,\ \mathrm{mes}\,\Theta_J\le\exp(-c|J|^\alpha),\ c=c(V,d,\mathrm{DC},E',E''),\alpha=\alpha(d,\mathrm{DC}),$$

and for each $\theta\in\mathbb{T}^d\backslash\Theta_J$ and $B>0$ there exists a set

$$\Omega_{I,J,\theta,B},\ \mathrm{mes}\,\Omega_{I,J,\theta,B}\le|J|^{C-B},\ C=C(V,d,\mathrm{DC},E',E''),$$

such that the following holds. For any $\theta\in\mathbb{T}^d\backslash\Theta_J$, $\omega\in\mathrm{DC}\backslash\Omega_{I,J,\theta,B}$, and $E\in[E',E'']$ we have that if

$$\log\|M_J(\theta,\omega,E)\|\le|J|L(\omega,E)-|J|^{1-\sigma/2},$$

then

$$\log\|M_I(\theta + n\omega, \omega, E)\| > |I|L(\omega, E) - |I|^{1-\sigma}$$

for any $|J|^B \le |n| \le \exp(c|I|^\sigma)$*,* $c = c(V, d, \mathrm{DC}, E', E'', A, \gamma)$ *(recall that* σ *is as in Theorem* 3.1*).*

Proof. Let Θ_J be the set from Proposition 5.4 with $H = |J|^{\sigma/2}/C$. The measure estimate on Θ_J holds with $\alpha = \sigma/(4d + 2)$.

Fix $\theta_0 \in \mathbb{T}^d \backslash \Theta_J$. Consider the set $\mathcal{B}$ of $(\theta, \omega, E) \in \mathbb{T}^d \times \mathrm{DC} \times [E', E'']$ such that

$$\begin{aligned} L(\omega, E) &\ge \gamma, \\ \log\|M_J(\theta_0, \omega, E)\| &\le |J|L(\omega, E) - |J|^{1-\sigma/2}, \\ \log\|M_I(\theta, \omega, E)\| &\le |I|L(\omega, E) - |I|^{1-\sigma}. \end{aligned}$$

By Proposition 3.2 we have that $\mathcal{B} \subset \mathcal{B}'$ where $\mathcal{B}'$ is defined by

$$\begin{aligned} &(\theta, \omega, E) \in \mathbb{T}^d \times \mathrm{DC}_{|J|^C} \times [E', E''] \\ &L_J(\omega, E) \ge \gamma/2, \\ &\log\|M_J(\theta_0, \omega, E)\| \le |J|L_J(\omega, E) - |J|^{1-\sigma/2}/2, \\ &\log\|M_I(\theta, \omega, E)\| \le |I|L_I(\omega, E) - |I|^{1-\sigma}/2. \end{aligned}$$

By Lemma 6.7 we know that $\mathcal{B}' \subset \mathcal{S} \subset \mathcal{B}''$ with $\mathcal{S}$ semialgebraic of degree less than $|J|^C$ and $\mathcal{B}''$ defined by

$$\begin{aligned} &(\theta, \omega, E) \in \mathbb{T}^d \times \mathrm{DC}_{|J|^C} \times [E', E''] \\ &L_J(\omega, E) \ge \gamma/4, \\ &\log\|M_J(\theta_0, \omega, E)\| \le |J|L_J(\omega, E) - |J|^{1-\sigma/2}/4, \\ &\log\|M_I(\theta, \omega, E)\| \le |I|L_I(\omega, E) - |I|^{1-\sigma}/4. \end{aligned}$$

To get the conclusion we want $(\{\theta_0 + n\omega\}, \omega, E) \notin \mathcal{B}$ for ω outside an exceptional set and all $E \in [E', E'']$. It is enough to argue that $(\{\theta_0 + n\omega\}, \omega) \notin \mathcal{S}' := \mathrm{Proj}_{(\theta,\omega)} \mathcal{S}$ for ω outside an exceptional set. We achieve this by invoking Lemma 7.1. By the Tarski-Seidenberg principle (see [Bou05a, Proposition 9.2]) the set $\mathcal{S}'$ is known to be semialgebraic of degree less than $|J|^C$. We need to estimate $\mathrm{mes}(\mathcal{S}')$.

We have

$$\mathrm{mes}(\mathcal{S}') \le \mathrm{mes}(\mathrm{Proj}_{(x,\omega)} \mathcal{B}'').$$

Let Ω_{J,θ_0}, $\mathrm{mes}(\Omega_{J,\theta_0}) \le \exp(-c|J|^\alpha)$ be the set from Proposition 5.4. Consider the set

$$\Theta'' = \mathrm{Proj}_\theta\{(\theta, \omega, E) \in \mathcal{B}'' : \omega \notin \Omega_{J,\theta_0}\}.$$

If $(\theta, \omega, E) \in \mathcal{B}''$ and $\omega \notin \Omega_{J,\theta_0}$ then by Proposition 5.4 we have that $E \in \mathcal{E}_{J,\theta_0,\omega}$ which is the union of less than $\exp(C(\log|J|)^{2/\sigma})$ intervals each having measure less than $\exp(-c|J|^\alpha)$. If A is large enough so that $|J|^\alpha \ge |I|^2$ then Theorem 3.1 and Lemma 3.7 imply that

$$\mathrm{mes}(\Theta'') \le \exp(C(\log|J|)^{2/\sigma}) \exp(-c|I|^\sigma) \le \exp(-c(A)|I|^\sigma).$$

We conclude that

$$\mathrm{mes}(\mathcal{S}') \le \mathrm{mes}(\mathrm{Proj}_{(x,\omega)} \mathcal{B}'') \le \mathrm{mes}(\Omega_{J,\theta_0}) + \mathrm{mes}(\Theta'') \le \exp(-c|I|^\sigma).$$

Let

$$\mathcal{S}' = \mathcal{S}_1' \cup \mathcal{S}_2'$$

be the decomposition afforded by Lemma 7.1 with $\varepsilon = 200/|J|^B$. The set of $\{\theta_0 + n\omega\}$ with $\omega \in [0,1]^d$ is contained in a union of hyperplanes $L_{n,\alpha}$, $\alpha \leq |n|^d$. The hyperplanes $L_{n,\alpha}$ are parallel to the hyperplane $(n\omega, \omega)$, $\omega \in \mathbb{R}^d$, and therefore

$$|\operatorname{Proj}_{L_{n,\alpha}} e_j| \leq \frac{1}{|n|} < \frac{\varepsilon}{100} \text{ for all } \alpha, e_j, |n| \geq |J|^B$$

(e_j are as in Lemma 7.1). The conclusion follows by letting

$$\Omega_{I,J,\theta_0,B} = \{\omega : (\omega, \{\theta_0 + n\omega\}) \in \mathcal{S}' \text{ for some } |J|^B \leq |n| \leq \exp(c|I|^\sigma)\}.$$

Note that by Lemma 7.1 we have

$$\begin{aligned} \operatorname{mes}(\Omega_{I,J,\theta_0,B}) &\leq \operatorname{mes}(\operatorname{Proj}_\omega \mathcal{S}_1') + \sum_{\alpha,n} \operatorname{mes}(\mathcal{S}_2' \cap L_{n,\alpha}) \\ &\lesssim \frac{|J|^C}{|J|^B} + \sum_n |n|^d |J|^{C+B} \exp(-c|I|^\sigma) \leq \frac{|J|^C}{|J|^B} \end{aligned}$$

provided the constant c in the upper bound of $|n|$ is small enough.

Remark 7.3. We assume the notation from the proof of the previous Proposition. Following the discrete case strategy (see [Bou05a, Chapter 10]) we could set things up so that the set $\mathcal{B}$ used for elimination is determined by

$$\begin{aligned} v_{a'}(b'; \theta_0, \omega, E) &= 0, \\ \log\|M_I(\theta, \omega, E)\| &\leq |I| L(\omega, E) - |I|^{1-\sigma}, \end{aligned} \tag{7.1}$$

where $v_{a'}(b'; \theta_0, \omega, E)$ plays the same role as the finite volume Dirichlet determinant did in the discrete case. The benefit of this set-up is that the first equation restricts E to a finite set of values (the eigenvalues in the interval $[E', E'']$), which lets us project onto (θ, ω) and get a small set (of course, we would also need an estimate for the number of eigenvalues in $[E', E'']$). The problem is that when we pass to the semialgebraic approximation the first equality becomes an inequality and the previous reasoning breaks. In the discrete case one approximates the potential by a polynomial $\tilde{V}$ and as a result one gets a new operator $\tilde{H}$ to which one can apply the reasoning that leads to (7.1) for the semialgebraic approximation in the same way as for H. This doesn't work in the continuous case because we are forced to approximate the solutions, rather than the potential.

8. Proof of the Main Result

We are now ready to prove Theorem 1.1.

Let $N_0 = N_0(V, d, E', E'', \gamma)$ and $C_0 = C_0(V, d, E', E'', \gamma)$ be large enough and define $N_k = (N_{k-1})^{C_0}$, $k \geq 1$. Let Θ_k, $\Omega_{k,\theta}$ be the sets from Proposition 7.2 with $J_k = [-N_{k+1}, N_{k+1}]$, $I_k = [-N_k, N_k]$ and B such that $|J_k|^B \in [N_{k+2}/4, N_{k+2}/2]$, Note that we have

$$\operatorname{mes}(\Theta_k) \leq \exp(-cN_{k+1}), \;\; \operatorname{mes}(\Omega_{k,\theta}) \leq N_{k+2}^{-1/2}.$$

Let

$$\Theta = \bigcap_{\underline{k}=0}^{\infty} \bigcup_{k \geq \underline{k}} \Theta_k.$$

Given $\theta \in \mathbb{T}^d \backslash \Theta$ there exists k_0 such that $\theta \in \mathbb{T}^d \backslash \Theta_k$, $k \geq k_0$. Let

$$\Omega_\theta = \bigcap_{\underline{k}=k_0}^{\infty} \bigcup_{k \geq \underline{k}} \Omega_{k,\theta}.$$

By Borel-Cantelli we clearly have that $\operatorname{mes}(\Theta) = \operatorname{mes}(\Omega_\theta) = 0$.

Let $\theta \in \mathbb{T}^d \backslash \Theta$, $\omega \in \mathrm{DC} \backslash \Omega_\theta$. It is well known that the energies with polynomially bounded solutions are dense in the spectrum (see [Sim82, Corolarry C.5.5]). So, given $E \in [E', E'']$ so that there exists $y \not\equiv 0$ satisfying $H(\theta, \omega)y = Ey$ and

$$|y(t)| \leq (1 + |t|)^C, \tag{8.1}$$

it is enough to show that y decays exponentially.

If

$$\log\|M_{J_k}(\theta, \omega, E)\| > |J_k| L(\omega, E) - |J_k|^{1-\sigma/2}$$

for infinitely many k, then Proposition 4.1 together with Poisson's formula and (8.1) imply that $y \equiv 0$. Therefore, for k large enough we must have

$$\log\|M_{J_k}(\theta, \omega, E)\| \leq |J_k| L(\omega, E) - |J_k|^{1-\sigma/2}$$

and by Proposition 7.2

$$\log\|M_{I_k}(\theta + n\omega, \omega, E)\| > |I_k| L(\omega, E) - |I_k|^{1-\sigma}, \;\; N_{k+1}/2 \leq |n| \leq 2N_{k+2}.$$

Using Proposition 4.1 we can iterate Poisson's formula as in Lemma 4.2 and get that

$$|y(t)| \leq (1 + |t|)^C \exp(-c|t| L(\omega, E)) \leq \exp(-c|t| L(\omega, E)/2), \;\; |t| \in [N_{k+1}, N_{k+2}].$$

This concludes the proof.

A. Appendix

Before we prove the large deviations estimate we need to recall the following result from [GS01].

Theorem A.1 ([GS01, Theorem 8.5]). *Let d be a positive number. Suppose $u : D(0, 2)^d \to [-1, 1]$ is subharmonic in each variable. Given $r \in (0, 1)$ there exists a polydisk*

$$\Pi = D(x_1^{(0)}, r) \times \cdots \times D(x_d^{(0)}, r) \subset \mathbb{C}^d$$

with $x_1^{(0)}, \ldots, x_d^{(0)} \in [-1, 1]$ and a Cartan set $\mathcal{B} \in \mathrm{Car}_d(H)$, $H = \exp(-r^{-\beta})$ so that

$$|u(z) - u(z')| \lesssim r^\beta \text{ for all } z, z' \in \Pi \backslash \mathcal{B}. \tag{A.1}$$

The constant $\beta > 0$ depends only on the dimension d.

We will also need the following fact about the discrepancy of the sequence of shifts of a Diophantine vector. Let $R = \prod_i [a_i, b_i] \subset [0, 1]^d$. It is known (see [Hla73]) that for $\omega \in \mathrm{DC}_N$ we have

$$\#\{n : n\omega \in R, 1 \le n \le N\} = N\,\mathrm{Vol}(R) + C(d, DC)O(N^{1-1/A}\log^2 N). \tag{A.2}$$

Proof of Theorem 3.1. Let

$$u(\theta) = \frac{\log\|M_I(\theta_0 + \rho\theta + i\eta, \omega, E)\|}{C|I|}, \quad v(\theta) = \frac{\log\|M_I(\theta + i\eta, \omega, E)\|}{C|I|},$$

where $\theta_0 = (1/2, \ldots, 1/2) \in \mathbb{T}^d$. We choose $\rho = \rho(V)$ such that u is defined on $D(0, 2)^d$ and $C = C(V, |E|)$ such that

$$|u(\theta)| \le 1, \theta \in D(0, 2)^d$$

and

$$|v(\theta)| \le 1, |v(\theta) - v(\theta + \omega)| \le \frac{1}{|I|}, \theta \in \mathbb{T}^d \tag{A.3}$$

(recall that we have (2.5) and (2.7)). Applying Theorem A.1 to u with $r \in (0, 1)$ we get that there exists $R = x^{(0)} + [-\rho r, \rho r]^d \subset [0, 1]^d$ such that

$$|v(\theta) - v(\theta')| \lesssim r^\beta, \theta, \theta' \in R\backslash\mathcal{B}', \tag{A.4}$$

with $\mathrm{mes}(\mathcal{B}') \lesssim d\rho^d \exp(-r^{-\beta})$. Note that in terms of the notation of Theorem A.1 we have $R = \Pi \cap \mathbb{R}^d$, $\mathcal{B}' = \mathcal{B} \cap \mathbb{R}^d$, and the measure estimate for $\mathcal{B}'$ follows from Lemma Lemma 5.3.
It follows from (A.2) that for any $\theta \in \mathbb{T}^d$ there exists

$$k \le k_0 := [C(V, d, \mathrm{DC})r^{-2dA}] \text{ such that } \theta + k\omega \in R$$

(the factor of 2 in the exponent of r can be replaced by $1+\varepsilon$). Therefore, as a consequence of (A.3) and (A.4) we have

$$|v(\theta) - v(\theta')| < C(V, d, \mathrm{DC})\left(r^\beta + \frac{r^{-2dA}}{|I|}\right), \theta, \theta' \in \mathbb{T}^d\backslash\tilde{\mathcal{B}}$$

with

$$\tilde{\mathcal{B}} := \cup_{k=0}^{k_0}(\mathcal{B}' + k\omega), \quad \mathrm{mes}(\tilde{\mathcal{B}}) \le C(V, d, \mathrm{DC})r^{-2dA}\exp(-r^{-\beta}).$$

Taking

$$|I|^{-\frac{1}{2dA+\beta}} \le r \le c(V, d, \mathrm{DC}) \tag{A.5}$$

we have

$$|v(\theta) - v(\theta')| < Cr^\beta, \theta, \theta' \in \mathbb{T}^d\backslash\tilde{\mathcal{B}}, \quad \mathrm{mes}(\tilde{\mathcal{B}}) \le \exp(-r^{-\beta}/2).$$

It is now straightforward to see that

$$\mathrm{mes}\{\theta : |\log\|M_I(\theta + i\eta, \omega, E)\| - |I|L_I(\eta, \omega, E)| > C|I|r^\beta\} \le \exp(-r^{-\beta}/2), C = C(V, d, \mathrm{DC}, |E|).$$

The conclusion follows immediately by choosing r so that

$$C|I|r^\beta = \varepsilon|I|^{1-\sigma}.$$

Note that due to (A.5) we need to take $\sigma < \beta/(2dA + \beta)$. □

We use the following simple result to pass from continuous time Lyapunov exponents to discrete time Lyapunov exponents.

Lemma A.2. *Let* $I = [a, b]$. *Then for any* $\omega \in \mathbb{T}^d$, $E \in \mathbb{C}$, $\eta \in \mathbb{R}^d$, $\|\eta\| \le \rho(V)$, $n \in \mathbb{Z}$, $n \ge 1$ *we have*

$$|L_I(\eta, \omega, E) - L_n(\eta, \omega, E)| \le \frac{C(V, |E|)}{|I|}(|n - |I|| + 2).$$

Proof. By (2.3) and the bounds on the transfer matrix and its inverse we have

$$|\log\|M_I\| - \log\|M_J\|| \le C\,|I \cup J \backslash (I \cap J)|\,, C = C(V, |E|)$$

for any other closed finite interval J. The conclusion follows by applying this fact with $J = [[a], [a] + n]$ and the definition of the finite scale Lyapunov exponents.

Proof of Proposition 3.2. With the same proof as that of [GS01, Lemma 10.1] we have

$$0 \le L_n(\eta, \omega, E) - L(\eta, \omega, E) \le C(V, d, \mathrm{DC}, |E|)\frac{(\log n)^{1/\sigma}}{n}. \tag{A.6}$$

Note that in fact the proof of [GS01, Lemma10.1] only points out the adjustments that need to be made to the proof of [GS01, Lemma 4.2]; up to these adjustments the proof of [GS01, Lemma 4.2] works as is for our setting too. Furthermore, from the proof of [GS01, Lemma4.2] we also have that

$$|L_n(\eta, \omega, E) - L_{2n}(\eta, \omega, E)| \le C(V, d, \mathrm{DC}, |E|)\frac{(\log n)^{1/\sigma}}{n}. \tag{A.7}$$

The proof of (A.7) only relies on the large deviations estimate at scale $\ell \simeq (\log n)^{1/\sigma}$ and therefore it is enough to have $\omega \in \mathrm{DC}_\ell \supset \mathrm{DC}_n$. Furthermore, one only needs that $L_\ell(\eta, \omega, E) \gtrsim \gamma$ and to have this it is enough to assume $L_n(\eta, \omega, E) \ge \gamma$ (due to Lemma 3.3).

The conclusions follow from (A.7) and (A.6) together with Lemma A.2. □

Proof of Lemma 3.3. Let $m = [|I|] + 2$, $n = [|J|] + 1$. We have $m = kn + r$ and by subadditivity

$$mL_m \le knL_n + rL_r.$$

It follows that

$$L_n \ge L_m + \frac{r(L_m - L_r)}{kn} \ge L_m - \frac{rC(V, |E|)}{kn} \ge L_m - \frac{nC(V, |E|)}{m - n}$$
$$\ge L_m - C(V, |E|)\frac{|J| + 1}{|I| - |J|}.$$

The conclusion follows from Lemma A.2. □

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