Some Problems concerning algebras of holomorphic functions

Richard M. Aron
Kent State University

Complex Analysis and Spectral Theory
Université Laval May, 2018
Plan of talk

Problems collected over a long period, with many co-authors: B. Cole, D. Carando, P. Galindo, D. García, T. W. Gamelin, A. Izzo, S. Lassalle, M. Maestre, ...
Plan of talk

Problems collected over a long period, with many co-authors: B. Cole, D. Carando, P. Galindo, D. García, T. W. Gamelin, A. Izzo, S. Lassalle, M. Maestre, ...

1. Basic background
2. A Problem involving $\mathcal{H}_b(X)$
3. Problems involving $\mathcal{H}_\infty(B_X)$
$X$ is a complex Banach space with open unit ball $B_X$. We’ll be interested in three algebras of holomorphic functions $\text{H}_b(X)$, the (Fréchet) algebra of all entire functions $f: X \to \mathbb{C}$ such that $\|f\|_{nB_X} < \infty$, $\forall n$, $\text{H}_\infty(B_X)$, the (Banach) algebra of all holomorphic functions $f: B_X \to \mathbb{C}$ such that $\|f\| = \sup_{x \in B_X} |f(x)| < \infty$. To a lesser extent, we’ll look at $\text{A}_u(B_X)$, the subalgebra of $\text{H}_\infty(B_X)$ consisting of all uniformly continuous holomorphic functions on $B_X$. Let’s call any of these three algebras $A$, for now.
$X$ is a complex Banach space with open unit ball $B_X$. We’ll be interested in three algebras of holomorphic functions

- $\mathcal{H}_b(X)$, the (Fréchet) algebra of all entire functions $f : X \to \mathbb{C}$ such that $\|f\|_{nB_X} < \infty$, $\forall n$,.
1) Basics

$X$ is a complex Banach space with open unit ball $B_X$. We’ll be interested in three algebras of holomorphic functions

- $\mathcal{H}_b(X)$, the (Fréchet) algebra of all entire functions $f : X \to \mathbb{C}$ such that $\|f\|_{nB_X} < \infty$, $\forall n$,

- $\mathcal{H}^\infty(B_X)$, the (Banach) algebra of all holomorphic functions $f : B_X \to \mathbb{C}$ such that $\|f\| = \sup_{x \in B_X} |f(x)| < \infty$. To a lesser extent, we’ll look at
1) Basics

$X$ is a complex Banach space with open unit ball $B_X$. We’ll be interested in three algebras of holomorphic functions

- $\mathcal{H}_b(X)$, the (Fréchet) algebra of all entire functions $f : X \rightarrow \mathbb{C}$ such that $\|f\|_{nB_X} < \infty$, $\forall n$,

- $\mathcal{H}^\infty(B_X)$, the (Banach) algebra of all holomorphic functions $f : B_X \rightarrow \mathbb{C}$ such that $\|f\| = \sup_{x \in B_X} |f(x)| < \infty$. To a lesser extent, we’ll look at

- $\mathcal{A}_u(B_X)$, the subalgebra of $\mathcal{H}^\infty(B_X)$ consisting of all uniformly continuous holomorphic functions on $B_X$. 
X is a complex Banach space with open unit ball $B_X$. We’ll be interested in three algebras of holomorphic functions

- $\mathcal{H}_b(X)$, the (Fréchet) algebra of all entire functions $f : X \to \mathbb{C}$ such that $\|f\|_{nB_X} < \infty$, $\forall n$,

- $\mathcal{H}^\infty(B_X)$, the (Banach) algebra of all holomorphic functions $f : B_X \to \mathbb{C}$ such that $\|f\| = \sup_{x \in B_X} |f(x)| < \infty$. To a lesser extent, we’ll look at

- $\mathcal{A}_u(B_X)$, the subalgebra of $\mathcal{H}^\infty(B_X)$ consisting of all *uniformly* continuous holomorphic functions on $B_X$.

Let’s call any of these three algebras $\mathcal{A}$, for now.
Let $\mathcal{M}(A)$ denote the set of all $\neq 0$ homomorphisms $\varphi : A \to \mathbb{C}$. 
Let $\mathcal{M}(A)$ denote the set of all $\neq 0$ homomorphisms $\varphi : A \to \mathbb{C}$. In the case of Banach algebras, any such $\varphi$ is automatically continuous. However, for $A = \mathcal{H}_b(X)$, the automatic continuity of $\varphi$ is unknown ("Michael problem").
Let $\mathcal{M}(A)$ denote the set of all $\neq 0$ homomorphisms $\varphi : A \to \mathbb{C}$. In the case of Banach algebras, any such $\varphi$ is automatically continuous. However, for $A = \mathcal{H}_b(X)$, the automatic continuity of $\varphi$ is unknown ("Michael problem"). For this talk, we’ll work only with continuous homomorphisms $\varphi$. 
Let $\mathcal{M}(\mathcal{A})$ denote the set of all $\neq 0$ homomorphisms $\varphi : \mathcal{A} \to \mathbb{C}$. In the case of Banach algebras, any such $\varphi$ is automatically continuous. However, for $\mathcal{A} = \mathcal{H}_b(X)$, the automatic continuity of $\varphi$ is unknown ("Michael problem"). For this talk, we’ll work only with continuous homomorphisms $\varphi$. For Banach algebras $\mathcal{A}$, $\mathcal{M}(\mathcal{A}) \subset \overline{B}_{\mathcal{A}^*}$ and is weak-star compact.
Let \( \mathcal{M}(\mathcal{A}) \) denote the set of all \( \neq 0 \) homomorphisms \( \varphi : \mathcal{A} \to \mathbb{C} \). In the case of Banach algebras, any such \( \varphi \) is automatically continuous. However, for \( \mathcal{A} = \mathcal{H}_b(X) \), the automatic continuity of \( \varphi \) is unknown ("Michael problem"). For this talk, we'll work only with continuous homomorphisms \( \varphi \). For Banach algebras \( \mathcal{A} \), \( \mathcal{M}(\mathcal{A}) \subset \mathcal{B}_{\mathcal{A}^*} \) and is weak-star compact.

A few more basics later.
By [A & Berner], every $f \in \mathcal{H}_b(X)$ admits an extension (via a canonical map) to $\tilde{f} \in \mathcal{H}_b(X^{**})$. Moreover, $f \rightarrow \tilde{f}$ is itself a homomorphism, i.e. it’s linear, multiplicative, and continuous.
2) $\mathcal{H}_b(X)$

By [A & Berner], every $f \in \mathcal{H}_b(X)$ admits an extension (via a canonical map) to $\tilde{f} \in \mathcal{H}_b(X^{**})$. Moreover, $f \to \tilde{f}$ is itself a homomorphism, i.e. it’s linear, multiplicative, and continuous. By [Davie & Gamelin], it follows that every $f \in \mathcal{H}^\infty(B_X)$ admits an extension $\tilde{f} \in \mathcal{H}^\infty(B_X^{**})$. The same holds for $\mathcal{A}_u(B_X)$. As above, $f \to \tilde{f}$ is a homomorphism.
By [A & Berner], every $f \in \mathcal{H}_b(X)$ admits an extension (via a canonical map) to $\tilde{f} \in \mathcal{H}_b(X^{**})$. Moreover, $f \rightarrow \tilde{f}$ is itself a homomorphism, i.e. it’s linear, multiplicative, and continuous. By [Davie & Gamelin], it follows that every $f \in \mathcal{H}_\infty(B_X)$ admits an extension $\tilde{f} \in \mathcal{H}_\infty(B_X^{**})$. The same holds for $A_u(B_X)$. As above, $f \rightarrow \tilde{f}$ is a homomorphism.

**Examples**

1. $X = \mathbb{C}$: $\mathcal{M}(\mathcal{H}(\mathbb{C})) = \{\delta_c \mid c \in \mathbb{C}\}$.  
   Also $\mathcal{M}(A_u(B_\mathbb{C})) = \mathcal{M}(A(\mathbb{D})) = \{\delta_c \mid c \in \mathbb{C}, \ |c| \leq 1\}$.

   However, $\mathcal{M}(\mathcal{H}_\infty(\mathbb{D}))$ is very complicated and very interesting.
2) $\mathcal{H}_b(X)$

Examples

2. $X = c_0$: It is known that $\mathcal{M}(\mathcal{H}_b(c_0))$ consists of $\{\delta_b \mid b \in c_0\}$ together with all $\{\tilde{\delta}_{b^{**}} \mid b^{**} \in \ell_\infty\}$, where $\tilde{\delta}_{b^{**}}(f) = \tilde{f}(b^{**})$.

Similarly for $\mathcal{M}(A_u(B_{c_0}))$, except $\|b\|, \|b^{**}\| \leq 1$. 

R. M. Aron, Kent State University
Examples

2. $X = c_0$: It is known that $\mathcal{M}(\mathcal{H}_b(c_0))$ consists of $\{\delta_b \mid b \in c_0\}$ together with all $\{\tilde{\delta}_{b^{**}} \mid b^{**} \in \ell_\infty\}$, where $\tilde{\delta}_{b^{**}}(f) = \tilde{f}(b^{**})$.

Similarly for $\mathcal{M}(A_u(B_{c_0}))$, except $\|b\|, \|b^{**}\| \leq 1$.

3. $X = \ell_2$. There are many more non-trivial homomorphisms in $\mathcal{M}(\mathcal{H}_b(\ell_2))$ than merely the evaluation homomorphisms $\delta_x, x \in \ell_2$:
Examples

2. $X = c_0$: It is known that $\mathcal{M}(\mathcal{H}_b(c_0))$ consists of $\{\delta_b \mid b \in c_0\}$ together with all $\{\tilde{\delta}_{b^{**}} \mid b^{**} \in \ell_\infty\}$, where $\tilde{\delta}_{b^{**}}(f) = \tilde{f}(b^{**})$. Similarly for $\mathcal{M}(A_u(B_{c_0}))$, except $\|b\|, \|b^{**}\| \leq 1$.

3. $X = \ell_2$. There are many more non-trivial homomorphisms in $\mathcal{M}(\mathcal{H}_b(\ell_2))$ than merely the evaluation homomorphisms $\delta_x, x \in \ell_2$:

Consider the set $\{\delta_{e_n}\} \subset \mathcal{M}(\mathcal{H}_b(\ell_2))$. It isn’t difficult that this set has an accumulation point $\varphi \in \mathcal{M}(\mathcal{H}_b(\ell_2))$. But $\varphi$ is not a point evaluation homomorphism.
Question  For a fixed $b^{**} \in X^{**}$, with 
\[ \hat{\delta}^{**}_{b} : f \in \mathcal{H}_{b}(X) \rightarrow \tilde{f} \in \mathcal{H}_{b}(X^{**}) \rightarrow \tilde{f}(b^{**}) \], we see that $X^{**}$ can be viewed as a subset of $\mathcal{M}(\mathcal{H}_{b}(X))$. Can we continue this procedure, going from $f \in \mathcal{H}_{b}(X) \rightarrow \tilde{f} \in \mathcal{H}_{b}(X^{**})$, and then from $\tilde{f} \in \mathcal{H}_{b}(X^{**}) \rightarrow \tilde{\tilde{f}} \in \mathcal{H}_{b}(X^{iv})$? In this way, for each fixed $b^{iv} \in X^{iv}$, can we get new homomorphisms 
\[ \hat{\delta}^{**}_{b^{iv}} \in \mathcal{M}(\mathcal{H}_{b}(X)), \quad f \mapsto \tilde{\tilde{f}}(b^{iv}) \]
**Question**  For a fixed $b^{**} \in X^{**}$, with 
\[ \tilde{\delta}^{**}_{b} : f \in \mathcal{H}_{b}(X) \rightarrow \tilde{f} \in \mathcal{H}_{b}(X^{**}) \rightarrow \tilde{f}(b^{**}), \]
we see that $X^{**}$ can be viewed as a subset of $\mathcal{M}(\mathcal{H}_{b}(X))$. Can we continue this procedure, going from $f \in \mathcal{H}_{b}(X) \rightarrow \tilde{f} \in \mathcal{H}_{b}(X^{**})$, and then from $\tilde{f} \in \mathcal{H}_{b}(X^{**}) \rightarrow \tilde{\tilde{f}} \in \mathcal{H}_{b}(X^{iv})$? In this way, for each fixed $b^{iv} \in X^{iv}$, can we get new homomorphisms 
\[ \tilde{\tilde{\delta}}^{iv}_{b} \in \mathcal{M}(\mathcal{H}_{b}(X)), \quad f \mapsto \tilde{\tilde{f}}(b^{iv})? \]
Answer: Sometimes yes, sometimes no. If $X$ is Arens regular, e.g. a $C^{*}$—algebra, or if $X$ is reflexive (trivial), then no.
2) $\mathcal{H}_b(X)$

**Question**  For a fixed $b^{**} \in X^{**}$, with 
$\tilde{\delta}^{**}_b : f \in \mathcal{H}_b(X) \rightarrow \tilde{f} \in \mathcal{H}_b(X^{**}) \rightarrow \tilde{f}(b^{**})$, we see that $X^{**}$ can be viewed as a subset of $\mathcal{M}(\mathcal{H}_b(X))$. Can we continue this procedure, going from $f \in \mathcal{H}_b(X) \rightarrow \tilde{f} \in \mathcal{H}_b(X^{**})$, and then from $\tilde{f} \in \mathcal{H}_b(X^{**}) \rightarrow \tilde{\tilde{f}} \in \mathcal{H}_b(X^{iv})$? In this way, for each fixed $b^{iv} \in X^{iv}$, can we get new homomorphisms 
$\tilde{\tilde{\delta}}^{**}_{b^{iv}} \in \mathcal{M}(\mathcal{H}_b(X))$, $f \mapsto \tilde{\tilde{f}}(b^{iv})$?

**Answer:** Sometimes yes, sometimes no. If $X$ is Arens regular, e.g. a $C^*$—algebra, or if $X$ is reflexive (trivial), then no. Namely, to each $b^{iv} \in X^{iv}$, there corresponds $b^{**} \in X^{**}$ such that $\tilde{\tilde{\delta}}^{**}_{b^{iv}} = \tilde{\tilde{\delta}}^{**}_{b^{**}}$. However,
Example

$X = \ell_1$. **Theorem:** There are points $b^{iv} \in \ell^{iv}_1$ such that $\hat{\delta}_{b^{iv}} \neq \hat{\delta}_{b^{**}}$ for any $b^{**} \in \ell^{**}_1$. 

Problems: (1) There are more points in $\ell^{iv}_1$ than there are homomorphisms in $M(H_{b^{}}(\ell_1))$. So, which points $b^{iv}$ of the fourth dual yield new homomorphisms and which do not? (2) Same questions about going to the sixth dual of $\ell_1$. 

R. M. Aron, Kent State University

Problems about Banach and Fréchet algebras of analytic functions
Example

$X = \ell_1$. **Theorem:** There are points $b^{iv} \in \ell_1^{iv}$ such that $\tilde{\delta}_{b^{iv}} \neq \tilde{\delta}_{b^{**}}$ for any $b^{**} \in \ell_1^{**}$.

**Problems:** (1) There are *more points* in $\ell_1^{iv}$ than there are homomorphisms in $\mathcal{M}(\mathcal{H}_b(\ell_1))$. So, which points $b^{iv}$ of the fourth dual yield new homomorphisms and which do not?

(2) Same questions about going to the sixth dual of $\ell_1$. 
Example

$X = \ell_1$. **Theorem**: There are points $b^{iv} \in \ell_1^{iv}$ such that $\tilde{\delta}_{b^{iv}} \neq \tilde{\delta}_{b^{**}}$ for any $b^{**} \in \ell_1^{**}$.

**Problems**: (1) There are more points in $\ell_1^{iv}$ than there are homomorphisms in $M(\mathcal{H}_b(\ell_1))$. So, which points $b^{iv}$ of the fourth dual yield new homomorphisms and which do not?

(2) Same questions about going to the sixth dual of $\ell_1$. 
As before, let $\mathcal{A}$ be one of the following three algebras: $\mathcal{H}_b(X), \mathcal{H}^\infty(B_X), \mathcal{A}_u(B_X)$.

**Observation:** $X^* \subset \mathcal{A}$. Consequently, for any $x^* \in X^*$ and for any (continuous) homomorphism $\varphi \in \mathcal{M}(\mathcal{A})$, $\varphi(x^*) \in \mathbb{C}$ makes sense.
As before, let $A$ be one of the following three algebras: $\mathcal{H}_b(X), \mathcal{H}_\infty(B_X), A_u(B_X)$.

**Observation:** $X^* \subset A$. Consequently, for any $x^* \in X^*$ and for any (continuous) homomorphism $\varphi \in M(A)$, $\varphi(x^*) \in \mathbb{C}$ makes sense. Define $\Pi : A \to ???$ by $\Pi(\varphi) = \varphi|_{X^*}$.

So, what is ????. Answer: It has to be the bidual $X^{**}$. 
As before, let $\mathcal{A}$ be one of the following three algebras: $\mathcal{H}_b(X), \mathcal{H}_\infty(B_X), \mathcal{A}_u(B_X)$.

**Observation:** $X^* \subset \mathcal{A}$. Consequently, for any $x^* \in X^*$ and for any (continuous) homomorphism $\varphi \in \mathcal{M}(\mathcal{A})$, $\varphi(x^*) \in \mathbb{C}$ makes sense. Define $\Pi : \mathcal{A} \to ???$ by $\Pi(\varphi) = \varphi|_{X^*}$.

So, what is ???? Answer: It has to be the bidual $X^{**}$. (Of course, nothing new when $\dim X < \infty$.) For $\mathcal{A} = \mathcal{H}_b(X)$, the range of $\Pi$ is all of $X^{**}$, while in the other two cases, $\mathcal{A} = \mathcal{H}_\infty(B_X)$ or $\mathcal{A}_u(B_X)$, the range is $\overline{B}_X^{**}$. 

R. M. Aron, Kent State University  
Problems about Banach and Fréchet algebras of analytic functions
As before, let \( A \) be one of the following three algebras: 
\( \mathcal{H}_b(X), \mathcal{H}_\infty(B_X), A_u(B_X) \).

**Observation:** \( X^* \subset A \). Consequently, for any \( x^* \in X^* \) and for any (continuous) homomorphism \( \varphi \in \mathcal{M}(A), \varphi(x^*) \in \mathbb{C} \) makes sense. Define \( \Pi : A \rightarrow X^{**} \) by \( \Pi(\varphi) = \varphi|_{X^*} \).

So, what is \( X^{**} \)? Answer: It has to be the bidual \( X^{**} \).

(Of course, nothing new when \( \dim X < \infty \).) For \( A = \mathcal{H}_b(X) \), the range of \( \Pi \) is all of \( X^{**} \), while in the other two cases, \( A = \mathcal{H}_\infty(B_X) \) or \( A_u(B_X) \), the range is \( B_X^{**} \).

**Definition**

Let \( z^{**} \) be in the range of \( \Pi \). The fiber over \( z^{**} \) is just \( \Pi^{-1}(z^{**}) \).
1) Basics

Definition

The *cluster set* of a function $f \in \mathcal{H}^\infty(B_X)$ at the point $z^{**} \in \overline{B}_{X^{**}}$ is the set of all limits of values of $f$ along nets in $B_X$ that converge weak-star to $z^{**}$. 

Corona Theorem (L. Carleson - 1962) The collection $\delta(D)$ of point evaluations at points of the open unit disc is dense in the space of all homomorphisms $\mathcal{M}(\mathcal{H}^\infty(D))$ on $\mathcal{H}^\infty(D)$. 

R. M. Aron, Kent State University

Problems about Banach and Fréchet algebras of analytic functions
Definition
The \textit{cluster set} of a function $f \in \mathcal{H}^\infty(B_X)$ at the point $z^{**} \in \overline{B_X}^{**}$ is the set of all limits of values of $f$ along nets in $B_X$ that converge weak-star to $z^{**}$.

Let’s restrict to $\mathcal{A} = \mathcal{H}^\infty(D)$. Recall that $\delta(D) \equiv \{\delta_c \mid c \in D\} \subset \mathcal{M}($\mathcal{H}^\infty(D)$).
1) Basics

Definition
The cluster set of a function $f \in \mathcal{H}^\infty(B_X)$ at the point $z^{**} \in \overline{B}_X^{**}$ is the set of all limits of values of $f$ along nets in $B_X$ that converge weak-star to $z^{**}$.

Let’s restrict to $A = \mathcal{H}^\infty(D)$. Recall that
$$\delta(D) \equiv \{\delta_c \mid c \in D\} \subset \mathcal{M}(\mathcal{H}^\infty(D)).$$

Corona Theorem (L. Carleson - 1962) The collection $\delta(D)$ of point evaluations at points of the open unit disc is dense in the space of all homomorphisms $\mathcal{M}(\mathcal{H}^\infty(D))$ on $\mathcal{H}^\infty(D)$. 
Carleson’s theorem (312) appeared one year after a somewhat overlooked paper by I. J. Schark (10). In it, among other things I. J. Schark proved

Cluster Value Theorem (I. J. Schark - 1961)

Fix \( f \in H_\infty(D) \) and \( c \in D \). Then the following sets are equal:

\[
\{ w \in \mathbb{C} \mid \exists (z_n) \subset D, z_n \to c \text{ and } f(z_n) \to w \} ;
\]

\[
\{ \varphi(f) \mid \varphi \in M(H_\infty(D)) \mid \Pi(\varphi) = c \}.
\]

Remarks 0. Schark’s result is trivial if \( |c| < 1 \).

1. Carleson’s theorem \( \rightarrow \) I. J. Schark’s theorem, but \( \leftarrow \) is false.

2. The analogous result to Carleson’s theorem for higher dimensions, e.g. \( C^2 \) with the Euclidean or max norms, is unknown. Put briefly, for \( \text{dim} X = 1 \), there are no known counterexamples; for \( \text{dim} X \geq 2 \), there are no known positive results. On the other hand,

3. There is no known situation in which I. J. Schark’s theorem is false.
Carleson’s theorem (312) appeared one year after a somewhat overlooked paper by I. J. Schark (10). In it, among other things I. J. Schark proved

**Cluster Value Theorem** (I. J. Schark - 1961) Fix \( f \in \mathcal{H}^\infty(\mathbb{D}) \) and \( c \in \overline{\mathbb{D}}. \) Then the following sets are equal:

\[
\{ w \in \mathbb{C} \mid \exists (z_n) \subset \mathbb{D}, z_n \to c \text{ and } f(z_n) \to w \};
\]

\[
\{ \varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(\mathbb{D})) \mid \Pi(\varphi) = c \}.
\]
Carleson’s theorem (312) appeared one year after a somewhat overlooked paper by I. J. Schark (10). In it, among other things I. J. Schark proved

**Cluster Value Theorem** (I. J. Schark - 1961) Fix $f \in \mathcal{H}^\infty(D)$ and $c \in \overline{D}$. Then the following sets are equal:

$\{ w \in \mathbb{C} | \exists (z_n) \subset D, z_n \to c \text{ and } f(z_n) \to w \}$;

$\{ \varphi(f) | \varphi \in \mathcal{M}(\mathcal{H}^\infty(D)) \mid \Pi(\varphi) = c \}$.

**Remarks 0.** Schark’s result is trivial if $|c| < 1$.

1. Carleson’s theorem $\Rightarrow$ I. J. Schark’s theorem, but $\Leftarrow$ is false.
1) Basics

Carleson’s theorem (312) appeared one year after a somewhat overlooked paper by I. J. Schark (10). In it, among other things I. J. Schark proved

**Cluster Value Theorem** (I. J. Schark - 1961) Fix $f \in \mathcal{H}^\infty(D)$ and $c \in \overline{D}$. Then the following sets are equal:

$\{ w \in \mathbb{C} \mid \exists (z_n) \subset D, z_n \to c \text{ and } f(z_n) \to w \}$

$\{ \varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(D)) \mid \Pi(\varphi) = c \}$

**Remarks 0.** Schark’s result is trivial if $|c| < 1$.

1. Carleson’s theorem $\Rightarrow$ I. J. Schark’s theorem, but $\Leftarrow$ is false.
2. The analogous result to Carleson’s theorem for higher dimensions, e.g. $\mathbb{C}^2$ with the Euclidean or max norms, is unknown. Put briefly, for dim $X = 1$, there are no known counterexamples; for dim $X \geq 2$, there are no known positive results. On the other hand,
3. There is no known situation in which I. J. Schark’s theorem is false.
First, we’re interested in a cluster value theorem, à la I. J. Schark. To start, for a given complex Banach space $X$, observe that

$$\delta(B_X) \equiv \{\delta_c \mid c \in B_X\} \subset \mathcal{M}(\mathcal{H}^\infty(B_X)).$$

Also, as before, endow $\mathcal{M}(\mathcal{H}^\infty(B_X))$ with the weak-star topology, considering it as a subspace of $(\mathcal{H}^\infty(B_X)^*, \text{weak-star})$. Harder Problem: Is the Cluster Value Theorem still true? Namely, for a fixed $f \in \mathcal{H}^\infty(B_X)$ and a fixed point $z^{**} \in B_X^{**}$, are the following two sets equal?

$$\{w \in \mathbb{C} \mid \exists \text{ net } (z^{\alpha})_{\alpha} \in B_X, z^{\alpha} \to z^{**} \text{ weak-\ast} \land f(z^{\alpha}) \to w\};$$

$$\{\phi(f) \mid \phi \in \mathcal{M}(\mathcal{H}^\infty(B_X)), \Pi(\phi) = z^{**}\}.$$
First, we’re interested in a cluster value theorem, à la I. J. Schark. To start, for a given complex Banach space $X$, observe that
$\delta(B_X) \equiv \{\delta_c \mid c \in B_X\} \subset \mathcal{M}(\mathcal{H}^\infty(B_X))$. Also, as before, endow $\mathcal{M}(\mathcal{H}^\infty(B_X))$ with the weak-star topology, considering it as a subspace of $(\mathcal{H}^\infty(B_X)^*, \text{weak-star})$.

**Harder Problem:** Is the Cluster Value Theorem still true? Namely, for a fixed $f \in \mathcal{H}^\infty(B_X)$ and a fixed point $z^{**} \in \overline{B}_X^{**}$, are the following two sets equal?

\[
\{w \in \mathbb{C} \mid \exists \text{ net } (z_\alpha)_\alpha \in B_X, \ z_\alpha \to z^{**} \ \text{weak} - * \ & f(z_\alpha) \to w\};
\]

\[
\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)), \ \Pi(\varphi) = z^{**}\}.
\]
Remark Unlike the case $\dim X < \infty$, the fiber over any, even an \textit{interior} point of $B_{X^{**}}$ is rich. In particular, $\beta \mathbb{N} \subset \Pi^{-1}(0)$. Even in this case, the \textit{easier} (?) problem is open in general:
(3) Problems involving \( \mathcal{H}^\infty(B_X) \)

**Remark** Unlike the case \( \dim X < \infty \), the fiber over *any, even an interior* point of \( B_{X^{**}} \) is rich. In particular, \( \beta\mathbb{N} \subset \Pi^{-1}(0) \). Even in this case, the easier (?) problem is open in general:

**Easier Problem:** For a fixed \( f \in \mathcal{H}^\infty(B_X) \), are the following two sets equal?

\[
\left\{ w \in \mathbb{C} \mid \exists \text{ net} \left( z_{\alpha} \right)_{\alpha} \in B_X, z_{\alpha} \to 0 \text{ weakly} \quad \& \quad f(z_{\alpha}) \to w \right\};
\]
\[
\left\{ \varphi(f) \mid \varphi \in M(\mathcal{H}^\infty(B_X)), \Pi(\varphi) = 0 \right\}.
\]
Problems involving $\mathcal{H}^\infty(B_X)$

Remark Unlike the case $\dim X < \infty$, the fiber over any, even an interior point of $B_{X^{**}}$ is rich. In particular, $\beta\mathbb{N} \subset \Pi^{-1}(0)$. Even in this case, the easier (?) problem is open in general:

Easier Problem: For a fixed $f \in \mathcal{H}^\infty(B_X)$, are the following two sets equal?

$$\{ w \in \mathbb{C} \mid \exists \text{ net } (z_\alpha)_\alpha \in B_X, \ z_\alpha \to 0 \text{ weakly } & f(z_\alpha) \to w \};$$

$$\{ \varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)), \ \Pi(\varphi) = 0 \}.$$
Yes, even to the “harder” question, if $X = c_0$. 
Yes, even to the "harder" question, if $X = c_0$.

**Theorem.** Fix $f \in \mathcal{H}^\infty(B_{c_0})$ and $z^{**} \in \overline{B}_{\ell\infty}$. Then the two sets

$$\{ w \in \mathbb{C} \mid \exists \text{ net } (z_\alpha)_\alpha \in B_{c_0}, \ z_\alpha \to z^{**} \text{ weak } - * \ \& \ f(z_\alpha) \to w \}$$

and

$$\{ \varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0})), \ \Pi(\varphi) = z^{**} \}$$

are equal.
It is unknown if a similar result holds for the apparently simpler case of $X = \ell_2$. 
It is unknown if a similar result holds for the apparently simpler case of $X = \ell_2$.

One basic idea for proof of harder problem, $X = c_0$. Notation: For $g \in \mathcal{H}^\infty(B_{c_0})$ and $n \in \mathbb{N}$, define $g_n \in \mathcal{H}^\infty(B_{c_0})$ by $g_n(x_1, \ldots, x_n, x_{n+1}, \ldots) \equiv g(0, \ldots, 0, x_{n+1}, \ldots)$.

**Lemma**

Fix $\varphi \in \mathcal{M}(\mathcal{H}(\infty(B_{c_0})))$ so that $\Pi(\varphi) = 0$. For any $g \in \mathcal{H}^\infty(B_{c_0})$ and any $n \in \mathbb{N}$, $\varphi(g) = \varphi(g_n)$. 

---

R. M. Aron, Kent State University

Problems about Banach and Fréchet algebras of analytic functions
It is unknown if a similar result holds for the apparently simpler case of $X = \ell_2$.

One basic idea for proof of harder problem, $X = c_0$. Notation: For $g \in \mathcal{H}^\infty(B_{c_0})$ and $n \in \mathbb{N}$, define $g_n \in \mathcal{H}^\infty(B_{c_0})$ by $g_n(x_1, \ldots, x_n, x_{n+1}, \ldots) \equiv g(0, \ldots, 0, x_{n+1}, \ldots)$.

**Lemma**
Fix $\varphi \in \mathcal{M}(\mathcal{H}(\mathcal{H}^\infty(B_{c_0})))$ so that $\Pi(\varphi) = 0$. For any $g \in \mathcal{H}^\infty(B_{c_0})$ and any $n \in \mathbb{N}$, $\varphi(g) = \varphi(g_n)$.

**Remark** The lemma is false if $c_0$ is replaced by $\ell_2$ (and so we’re stuck).
Fibers Recall: For a complex Banach space $X$, $\Pi : \mathcal{M}(\mathcal{H}^\infty(B_X)) \to \overline{B}_X^\ast\ast$, $\Pi(\varphi) \equiv \varphi|_{X^\ast}$.

Suppose $X = \ell_2$. If $\|z\| = \|w\| = 1$, then $\Pi^{-1}(z) \preceq \Pi^{-1}(w)$. The same result holds if $\|z\|$ and $\|w\|$ are both $< 1$. What if $1 = \|z\| > \|w\|$?
(3) Problems involving $\mathcal{H}^\infty(B_X)$

Fibers  Recall: For a complex Banach space $X$, 
$\Pi : \mathcal{M}(\mathcal{H}^\infty(B_X)) \to B_{X^{**}}$, $\Pi(\phi) \equiv \phi|_{X^*}$.

Fix $X$ and two points $z^{**}$ and $w^{**}$ in $B_{X^{**}}$.
Fibers  Recall: For a complex Banach space $X$, 
$\Pi : \mathcal{M}(\mathcal{H}\infty(B_X)) \to B_{X}^{**}$, $\Pi(\varphi) \equiv \varphi|_{X^*}$.

Fix $X$ and two points $z^{**}$ and $w^{**}$ in $B_{X}^{**}$.  
**Problem** What is the relation between the two fibers $\Pi^{-1}(z^{**})$ and $\Pi^{-1}(w^{**})$?
(3) Problems involving $\mathcal{H}^\infty(B_X)$

**Fibers**
Recall: For a complex Banach space $X$, 
$\Pi : M(\mathcal{H}^\infty(B_X)) \to \overline{B}_X^\ast\ast$, $\Pi(\varphi) \equiv \varphi|_{X^\ast}$.

Fix $X$ and two points $z^{\ast\ast}$ and $w^{\ast\ast}$ in $\overline{B}_X^\ast\ast$.

**Problem**
What is the relation between the two fibers $\Pi^{-1}(z^{\ast\ast})$ and $\Pi^{-1}(w^{\ast\ast})$?

Suppose $X = \ell_2$. If $\|z\| = \|w\| = 1$, then $\Pi^{-1}(z) \simeq \Pi^{-1}(w)$. The same result holds if $\|z\|$ and $\|w\|$ are both $< 1$. What if $1 = \|z\| > \|w\|$?
Suppose $X = c_0$. Then $\|z\|, \|w\| < 1 \Rightarrow \Pi^{-1}(z) \subseteq \Pi^{-1}(w)$. But for $\|z\| = \|w\| = 1$, the situation is murky.
Suppose $X = c_0$. Then $\|z\|, \|w\| < 1 \Rightarrow \Pi^{-1}(z) \preceq \Pi^{-1}(w)$. But for $\|z\| = \|w\| = 1$, the situation is murky.

For the special cases $\mathcal{H}^\infty(D)$ and $\mathcal{H}^\infty(D^2)$, what is known is that $\Pi^{-1}(1) \preceq \Pi^{-1}(a, b)$, if one of $|a|, |b| = 1$ and the other is $< 1$. 
Suppose $X = c_0$. Then $\|z\|, \|w\| < 1 \Rightarrow \Pi^{-1}(z) \preceq \Pi^{-1}(w)$. But for $\|z\| = \|w\| = 1$, the situation is murky.

For the special cases $\mathcal{H}^\infty(D)$ and $\mathcal{H}^\infty(D^2)$, what is known is that $\Pi^{-1}(1) \preceq \Pi^{-1}(a, b)$, if one of $|a|, |b| = 1$ and the other is $< 1$. Also, $\Pi^{-1}(1)$ and $\Pi^{-1}(1, 1)$ are not homeomorphic.

(But the argument really uses dimension 1.)
Suppose $X = c_0$. Then $\|z\|, \|w\| < 1 \Rightarrow \Pi^{-1}(z) \subset \Pi^{-1}(w)$. But for $\|z\| = \|w\| = 1$, the situation is murky.

For the special cases $\mathcal{H}_\infty(D)$ and $\mathcal{H}_\infty(D^2)$, what is known is that $\Pi^{-1}(1) \subset \Pi^{-1}(a, b)$, if one of $|a|, |b| = 1$ and the other is $< 1$. Also, $\Pi^{-1}(1)$ and $\Pi^{-1}(1, 1)$ are not homeomorphic. (But the argument really uses dimension 1.)

Remark Even if $\dim X < \infty$ (so $B_X = B_{X^{**}}$) and even if $\|z\|, \|w\| < 1$, the problem, of whether $\pi^{-1}(z)$ and $\pi^{-1}(w)$ are (somehow) the “same” is apparently unknown in general.

The problem is that, in general, it isn’t known if $\Pi^{-1}(z) = \{\delta_z\}$ if $\dim X < \infty$ and $\|z\| < 1$. 

R. M. Aron, Kent State University  Problems about Banach and Fréchet algebras of analytic functions