Boundary behaviour of optimal polynomial approximants

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This talk is based on some recent and upcoming papers with various combinations of co-authors, including Dmitry Khavinson, Conni Liaw,Myrto Manolaki, Daniel Seco, and Brian Simanek.

Outline

Introduction to optimal polynomial approximants

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Side trip to Jentzsch's Theorem

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Definition of Dirichlet spaces D_α

For $-\infty < \alpha < \infty$, the space D_α consists of all analytic functions $f\colon \mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\} \to \mathbb{C}$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

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Given two functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in D_α, we also have the associated inner product

$$\langle f, g\rangle_\alpha = \sum_{k=0}^{\infty} (k+1)^\alpha a_k \overline{b_k}.$$

Examples

- $\alpha = -1$ corresponds to *the Bergman space* A^2, consisting of functions with
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- $\alpha = 1$: *the (classical) Dirichlet space* D of functions whose derivatives have finite area integral:
$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

Optimal polynomial approximants: opa!

Definition

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In other words, p_n is an optimal polynomial of order n to $1/f$ if

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where Pol_n denotes the space of polynomials of degree at most n.
Note: $p_n f$ is the orthogonal projection of 1 onto the subspace $f \cdot \text{Pol}_n$. Therefore, optimal approximants p_n always exist and are unique for any nonzero function f, and any degree $n \geq 0$.

Initial motivation for studying opa - Cyclicity

A function $f \in D_\alpha$ is said to be *cyclic* in D_α if the subspace generated by polynomials multiples of f,

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The optimal polynomial approximants are the "best" such p_n.

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Now notice that in all the D_α spaces,

$$\langle 1, \varphi_k f \rangle_\alpha = \overline{\varphi_k(0) f(0)}.$$

Formula for the Optimal Approximants

We thus have the following formula:

Proposition

Let $\alpha \in \mathbb{R}$ and $f \in D_\alpha$. For integers $k \geq 0$, let φ_k be the orthonormal polynomials for the weighted space $D_{\alpha,f}$. Let p_n be the optimal approximants to $1/f$. Then

$$p_n(z) = \overline{f(0)} \sum_{k=0}^{n} \overline{\varphi_k(0)} \varphi_k(z).$$

Connection with reproducing kernels

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Therefore, $K_n(z, 0) = p_n(z) f(z)$. This means that results about the behaviour of optimal approximants are actually results about the behaviour of weighted reproducing kernels!

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- Does $p_n(e^{i\theta}) \to 1/f(e^{i\theta})$?
- Can there be more than one limit point?
- Can it happen that the set $\{p_n(e^{i\theta}) : n = 0, 1, 2, \ldots\}$ is dense in $\mathbb{C}$?

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Their zeros are the roots of unity (except for $z = 1$), which are dense in the unit circle $\mathbb{T}$.
It turns out that this phenomenon is quite general and was discovered by Robert Jentzsch in his Ph.D. thesis in Berlin in 1914.

Jentzsch Theorem

Theorem (Jentzsch, 1914)

Given any power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with radius of convergence 1, *every point on the unit circle is a limit point of the zeros of the partial sums of the series.*

Historical Notes

Robert Jentzsch was a talented mathematician born in Königsberg in 1890. He was a student of Georg Frobenius in Berlin. He was also a poet. He was killed in battle in 1918.

There is a nice historical article "Jentzsch, Mathematician and Poet" by P. Duren, A.K. Herbig, and D. Khavinson in the Mathematical Intelligencer in 2008.

Jentzsch's Theorem for Optimal Polynomial Approximants

Theorem (BKLSS, 2016)

Let $\alpha \in \mathbb{R}$ and let $f \in D_\alpha$ be cyclic, such that $1/f$ has a singularity on the unit circle, and $f(0) \neq 0$. Then every point on the unit circle is a limit point of the zeros of the optimal approximants of $1/f$.

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Moral of the story: Strange things happen for optimal polynomial approximants *outside* the unit disk. *Inside* the unit disk, the optimal approximants converge to $1/f$, if f is cyclic. So what happens on the circle?

Theorem (C.B., M. Manolaki, D. Seco, 2018)

Let f be a monic polynomial of degree d with distinct zeros $z_1, z_2, \ldots z_d$ that lie on or outside the unit disk. For each n, let p_n be the optimal approximant of $1/f$ in D_α. Write $\omega_k := (k+1)^\alpha$ and denote by $d_{k,n}$ the Taylor coefficients of $p_n f - 1$. Then there exists a vector $A_n = (A_{1,n}, ..., A_{d,n})^t$ independent of k, such that for $k = 0, ..., n+d$, we have

$$d_{k,n} = \frac{1}{\omega_k} \sum_{i=1}^{d} A_{i,n} \overline{z_i^k}.$$

Theorem (continued...)

Moreover A_n is the only solution to the linear system

$$E_{Z,n}A_n = (-1, -1, \dots, -1)^t,$$

where the matrix $E_{Z,n}$ is invertible and has coefficients

$$E_{Z,n,l,m} = \sum_{k=0}^{n+d} \frac{\overline{z_m}^k z_l^k}{\omega_k}.$$

Idea of the proof

The proof is based on the idea that since $p_n f$ is the projection of 1 onto $f \cdot \mathrm{Pol}_n$, $p_n f - 1$ is orthogonal to $z^t f$ for $t = 0, \ldots, n$.

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Lemma

Let a sequence $\{e_k\}_{k\in\mathbb{N}}$ *satisfy the recurrence relation (for* $k \geq d$*)*

$$\sum_{r=0}^{d} e_{k-r}\hat{q}(r) = 0,$$

where q *is a polynomial* q *of degree* d *satisfying* $q(0) = 1$*, with simple zeros* $\{w_1, ..., w_d\} \subset \mathbb{C}$*. Then, there exist some constants* $H_1, ..., H_d \in \mathbb{C}$ *such that, for all* k*,* e_k *is given by*

$$e_k = \sum_{i=1}^{d} H_i w_i^{-k}.$$

Example

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$$\lim_{n\to\infty} A_n(z) = 0 - \frac{z}{1-z} + \frac{1}{1-z} = 1.$$

Bounded

The previous theorem can be leveraged to prove the following.

Theorem (C.B., M. Manolaki, D. Seco, 2018)

Let f be a polynomial of degree d with simple zeros that all lie outside the open unit disk, and let $\alpha \leq 1$. Let p_n be the n^{th} optimal polynomial approximant of $1/f$ in D_α. Let $z_0 \in \mathbb{T}$, not a zero of f. Then

$$(p_n f - 1)(z_0) \to 0 \quad as \quad n \to \infty.$$

Further Remarks

- It turns out that this most recent result is the *first* step in showing that there are functions f for which the set $p_n(e^{i\theta})$ is *dense* in $\mathbb{C}$! This is a preliminary result (C.B., M. Manolaki, D. Seco, 2018).

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- Can we describe the functions f for which this kind of universality happens? For which the opposite happens? Are there cases in between?

Thank you!