Factorization in commutative Banach algebras

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Commutative Banach algebras

Throughout, we shall be concerned with commutative Banach algebras = CBAs (always associative and over \( \mathbb{C} \)).

In particular, we shall think about:

- semi-simple CBAs, equivalently (natural) Banach function algebras (= BFAs) on a locally compact space;

- maximal ideals in (unital) uniform algebras - they are closed, unital subalgebras of \((C(X),|\cdot|_X)\), \(X\) compact, that separate the points of \(X\);

- commutative, radical Banach algebras = CRBAs.

We are interested to know if the results are different if we restrict to separable examples.

We shall list a number of properties related to factorization; each implies the next; we are trying to give counter-examples to all reverse implications (in various classes of CBAs).
Some notation

Let $A$ be a BFA on a locally compact space $X$, and take $x \in X$. Then $\varepsilon_x$ is the evaluation functional $\varepsilon_x : f \mapsto f(x)$ and $M_x$ is the corresponding maximal modular ideal; $A$ is **natural** if all characters are evaluation functionals.

Let $A$ be a BFA on $X$. Then $x \in X$ is a **strong boundary point** $= \text{SBP}$ if, for each open neighbourhood $U$ of $x$, there exists $f \in A$ with $f(x) = |f|_X = 1$ and $|f|_{X \setminus U} < 1$ (includes peak points).

For a compact plane set $X$, take $R(X)$ to be the uniform closure of the algebra of rational functions restricted to $X$; it is a natural uniform algebra on $X$. 
Approximate identities

Let $A$ be a CBA. Then an approximate identity (AI) for $A$ is a net $(e_{\nu})$ in $A$ such that $\lim_{\nu} e_{\nu}a = a$ for each $a \in A$; the AI $(e_{\nu})$ is a bounded approximate identity (BAI) if $\sup_{\nu} \|e_{\nu}\| < \infty$, and a contractive approximate identity (CAI) if $\|e_{\nu}\| \leq 1$ for each $\nu$.

(I) Let $A$ be a CBA. Then $A$ has property (I) if $A$ has a BAI.
Approximate identities for uniform algebras

We can characterize when some maximal ideals have a BAI.

**Proposition** Let $A$ be a natural uniform algebra on a compact space $X$, and take $x \in X$. Then the following conditions on $x$ are equivalent:

(a) $x$ is a strong boundary point;

(b) $M_x$ has a BAI;

(c) $M_x$ has a CAI.

A natural uniform algebra on compact $X$ is a **Cole algebra** if all points of $X$ are SBPs. Such algebras not equal to $C(X)$ exist.
Null sequences

Let $E$ be a Banach space. Then a **null sequence** in $E$ is a sequence $(x_n)$ in $E$ such that $\lim_{n \to \infty} \|x_n\| = 0$; the space of null sequences in $E$ is $c_0(E)$, and $c_0(E)$ is itself a Banach space for the norm defined by

$$\|(x_n)\| = \sup\{\|x_n\| : (x_n) \in c_0(E)\}.$$ 

Let $A$ be a CBA. Then **null sequences factor** in $A$ if, for each null sequence $(a_n)$ in $A$, there exist $a \in A$ and a null sequence $(b_n)$ in $A$ such that $a_n = ab_n$ ($n \in \mathbb{N}$).

[Important in automatic continuity theory - see my book.]

(II) Let $A$ be a CBA. Then $A$ has property (II) if all null sequences in $A$ factor.
Cohen’s factorization theorem

The following result is one form of the famous Cohen’s factorization theorem; (much) more general forms are given in my book.

**Theorem** Let $A$ be a CBA with a bounded approximate identity. Then null sequences factor, and so $(I) \Rightarrow (II)$ for $A$.

Reverse implication? **George Willis** (PLMS 1992) gave a separable BFA satisfying $(II)$, but not $(I)$, and this example can be modified to also give a separable CRBA with the same property.

**New Theorem** There is a maximal ideal in a uniform algebra satisfying $(II)$, but not $(I)$.

But our example is not separable - an open point. Proof later.
Factorization of pairs

Let $A$ be a commutative algebra. Then **pairs factor** in $A$ if, for each $a_1, a_2 \in A$, there exist $a, b_1, b_2 \in A$ such that $a_1 = ab_1$ and $a_2 = ab_2$.

(III) Let $A$ be a CBA. Then $A$ has property (III) if all pairs in $A$ factor.

Trivially (II) $\Rightarrow$ (III).

But we cannot yet find any CBA such that pairs factor, but null sequences do not. Ugh.
Factorization

Let \( A \) be an algebra. Then:

\[
A^{[2]} = \{ab : a, b \in A\}, \quad A^2 = \text{lin} A^{[2]}.
\]

**Definition** The algebra \( A \) factors if \( A = A^{[2]} \) and \( A \) factors weakly if \( A = A^2 \).

(IV) Let \( A \) be a CBA. Then \( A \) has property (IV) if \( A \) factors.

Trivially (III) \( \Rightarrow \) (IV). Reverse implication?

Look at \( H^{\infty}(\mathbb{D}) \). It was shown by Ouzomgi (another nephew of Tom Ransford) that there is a maximal ideal \( M_x \) in \( H^{\infty}(\mathbb{D}) \) that factors, but such that pairs do not factor.

We would like a separable BFA and/or a CRBA that factors, but such that null sequences do not. Not known to us yet.
Weak factorization

(V) Let $A$ be a CBA. Then $A$ has property (V) if $A$ factors weakly.

Trivially (IV) $\Rightarrow$ (V).

For a natural BFA on a compact $X$, this means that there are no non-zero point derivations at any point of $X$.

George Willis (PLMS 1992) gave a separable BFA such that every element in $A$ is the sum of two products, and so $A$ factors weakly, but such that $A$ does not factor.

Theorem (Rick Loy) Let $A$ be a separable Banach algebra that factors weakly. Then there exist $m \in \mathbb{N}$ and $M > 0$ such that each $a \in A$, has the form $a = \sum_{i=1}^{m} b_i c_i$, where $\sum_{i=1}^{m} \|b_i\| \|c_i\| \leq M \|a\|$. \qed

In all known examples, we can take $m = 1$ or $m = 2$. 10
Weak factorization for uniform algebras

Embarrassing open point: we have no maximal ideal $M$ in a uniform algebra that factors weakly, but does not factor. Any ideas?

[It is easy to get $M^2 \neq M^2$, but we also want $M^2 = M$.]

**Proposition** Let $X$ be a compact plane set. Then the following conditions on $x \in X$ with respect to the uniform algebra $R(X)$ are equivalent:

(a) $x$ is a peak point;

(b) $M_x$ has a BAI or CAI;

(c) $M_x$ factors;

(d) $M_x$ factors weakly.

So no counter-examples for $R(X)$.  

\[ \square \]
Projective factorization

Let \( A \) be a BA, with projective tensor product \( A \hat{\otimes} A \). There is a unique bounded linear operator \( \pi_A : A \hat{\otimes} A \to A \) with \( \pi_A(a \otimes b) = ab \) for \( a, b \in A \), and then \( \pi_A(A \hat{\otimes} A) \) is a subalgebra of \( A \) and a Banach algebra with respect to the quotient norm from \( (A \hat{\otimes} A, \| \cdot \|_\pi) \).

Let \( A \) be a BA. Then \( A \) factors projectively if the map \( \pi_A : A \hat{\otimes} A \to A \) is a surjection, so that each \( a \in A \) has the form \( \sum_{i=1}^\infty b_ic_i \), where \( b_i, c_i \in A \) and \( \sum_{i=1}^\infty \| b_i \| \| c_i \| < \infty \).

(VI) Let \( A \) be a CBA. Then \( A \) has property (VI) if \( A \) factors projectively.

Trivially (V) \( \Rightarrow \) (VI).

Easy counter-example to the reverse implication: \( A = \ell^1 \), with pointwise product.

What about a maximal ideal in a uniform algebra? What about a CRBA? Not known to us yet.
Examples of projective factorization

Here $\mathbb{Q}^+ = \{ r \in \mathbb{Q} : r > 0 \}$, $\omega$ is a weight on $\mathbb{R}^+$, and so $\ell^1(\mathbb{Q}^+,\omega)$ is a CBA with respect to convolution multiplication.

At least we have the following:

**Example** Take $A = \ell^1(\mathbb{Q}^+,\omega)$ for a continuous weight $\omega$ on $\mathbb{R}^+$ which may be radical. Then $A$ factors projectively, but pairs do not factor. Surely $A$ does not factor? \[ \square \]
Projective factorization in uniform algebras

Let $A$ be a natural uniform algebra on compact $X$, and take $x, y \in X$. Then $x \sim y$ if $\|\varepsilon_x - \varepsilon_y\| < 2$. This is an equivalence relation on $X$, and the equivalence classes are the Gleason parts (wrt $A$).

These parts form a partition of $X$, and each part is a completely regular and $\sigma$-compact topological space with respect to the Gel’fand topology; by a theorem of Garnett, these are the only topological restrictions on Gleason parts.

Let $A$ be a natural BFA on $K$. A net $(e_\alpha)$ in $A$ is a pointwise approximate identity (PAI) if

$$\lim_{\alpha} e_\alpha(x) = 1 \quad (x \in K);$$

the PAI $(e_\alpha)$ is contractive if $\sup_\alpha \|e_\alpha\| \leq 1$; we obtain a CPAI.

**Proposition (D-Ulger)** A maximal ideal $M_x$ in a uniform algebra has a CPAI iff $\{x\}$ is a one-point part. \qed

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Projective factorization for $R(X)$

**Proposition** Let $X$ be a compact plane set. Then the following conditions on $x \in X$ with respect to the uniform algebra $R(X)$ are equivalent:

(a) $x$ is a peak point;

(b) $\{x\}$ is a one-point Gleason part;

(c) $M_x$ has a CPAI;

(d) $x$ is an isolated point with respect to the Gleason metric;

(e) $M_x$ factors projectively. □

So no counter-examples for $R(X)$.  

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The big disc algebra - 1

Take an irrational number \( \alpha \) with \( 0 < \alpha < 1 \), and consider the ‘open half-plane’ \( H_\alpha \) consisting of points \((m, n) \in \mathbb{Z} \times \mathbb{Z}\) with \( m + n\alpha > 0 \).

Then consider monomials on \( \mathbb{C}^2 \) of the form \( Z^m W^n \), where \((m, n) \in H_\alpha\); here \( Z \) and \( W \) are the coordinate functionals on \( \mathbb{C}^2 \). We take \( \mathcal{A}_{0,\alpha} \) to be the linear span of these monomials and the constant function \( 1 \), and \( \mathcal{A}_\alpha \) to be the uniform closure of this algebra, regarded as a subalgebra of \( C(\mathbb{T}^2) \). Then \( \mathcal{A}_\alpha \) is a separable uniform algebra on its character space \( \Phi_\alpha \) that can be identified with the space \( \mathbb{T}^2 \times [0, 1] \), with the subset \( \mathbb{T}^2 \times \{0\} \) identified to a point, called \( x_0 \); the corresponding maximal ideal in \( \mathcal{A}_\alpha \) at \( x_0 \) is denoted by \( \mathcal{M}_\alpha \). The set \( \{x_0\} \) is a one-point part, but \( x_0 \) is not a peak point. (See the book of Lee Stout for all this.)
**The big disc algebra - 2**

**Proposition** The maximal ideal $\mathfrak{M}_\alpha$ of the big disc algebra factors projectively.

**Proof** For this, we use the fact that it follows from Dirichlet’s theorem on Diophantine approximation that, for each $\varepsilon > 0$, there exist $p, q \in \mathbb{N}$ with

\[
\alpha - \frac{\varepsilon}{q} < \frac{p}{q} < \alpha.
\]

Hence (VI) $\not\Rightarrow$ (I) in the class of separable maximal ideals in uniform algebras.

We believe that there are null sequences in $\mathfrak{M}_\alpha$ that do not factor, but so far have not proved this.
Dense factorization

Let $A$ be a CBA. Then $A$ factors densely if $A^2$ is dense in $A$.

For a natural BFA on a compact $X$, this means that there are no non-zero, continuous point derivations at any point of $X$.

(VII) Let $A$ be a CBA. Then $A$ has property (VII) if $A$ factors densely.

Trivially (VI) $\Rightarrow$ (VII).

Example Consider $R = C_{*,0}(\mathbb{I})$, the algebra of all continuous functions on $\mathbb{I}$ that vanish at 0, taken with the convolution product. Then $R$ is a CRBA. It is easy to see that it factors densely, but not projectively.
Dense factorization and uniform algebras

**Example** Consider the ‘road-runner set’ $X$, defined by discs $D(x_n, r_n)$. Then $M_0$ in $R(X)$ factors iff $\sum_{i=1}^{\infty} \frac{r_i}{x_i} = \infty$ (**Melnikov**), but factors densely iff $\sum_{i=1}^{\infty} \frac{r_i}{x_i^2} = \infty$ (**Hallstrom**). Thus there are maximal ideals in algebras $R(X)$ that factor densely, but not projectively. \(\square\)

**Side remark 1:** Stu Sidney has an example of a separable uniform algebra on $X$ and $x \in X$ such that $\{x\}$ is a one-point part, but $M_x$ does not factor densely. \(\square\)

**Side remark 2:** There is a (non-separable) uniform algebra such that $M_x$ factors, but $\{x\}$ is not a one-point part. \(\square\)
Extensions of uniform algebras - 1

Let $X$ and $Y$ be compact spaces, and suppose that $\Pi : Y \to X$ is a continuous surjection. Then $\Pi^* : C(X) \to C(Y)$ is defined by the formula

$$\Pi^*(f) = f \circ \Pi \quad (f \in C(X)),$$

so that $\Pi^*$ is an isometric isomorphism of $C(X)$ onto a closed subalgebra of $C(Y)$. A linear contraction $T : C(Y) \to C(X)$ such that $T \circ \Pi^* = I_{C(X)}$ is an averaging operator for $\Pi$. 
Extensions of uniform algebras - 2

Let $X$ be a compact space, take $x_0 \in X$, and let $A$ be a uniform algebra on $X$. Also suppose that $(Y, y_0, B)$ is another uniform algebra. Then $(Y, y_0, B)$ is an extension of $(X, x_0, A)$ with respect to a continuous surjection $\Pi : Y \to X$ and an averaging operator $T : C(Y) \to C(X)$ for $\Pi$ if:

(i) $\Pi^*(A) \subset B$;

(ii) $\Pi^{-1}(\{x_0\}) = \{y_0\}$;

(iii) $T(B) = A$;

(iv) $(Th)(x_0) = h(y_0)$ ($h \in C(Y)$);

(v) $|(Th)(x)| \leq |h|_{\Pi^{-1}(\{x\})}$ ($x \in X$, $h \in C(Y)$).
Extensions of uniform algebras - 3

Basic idea - this goes back to Brian Cole in his thesis; it is in the book of Lee Stout.

Start with a suitable \((X, x_0, A)\), with \(A \neq C(X)\); take an extension; keep on doing it with suitable compatibility conditions built in; index the set of extensions by the ordinals - usually up to \(\omega_1\); act sensibly at limit ordinals. We obtain an enormous, non-separable uniform algebra \((Y, y_0, B)\), and, with care, \(B \neq C(Y)\).

Cole’s original construction made extensions by ‘adding square roots’ to obtain a uniform algebra \((Y, y_0, B)\) with \(B \neq C(Y)\) such that every element in \(M_{y_0}\) is the square of another element in \(M_{y_0}\), and so the end point is a ‘Cole algebra’. (There are also separable examples of Cole algebras.)

Examples of Joel Feinstein show that the final uniform algebra can have a variety of other interesting properties.
Extensions of uniform algebras - 4

Main example This is how we construct a maximal ideal $M_x$ in a uniform algebra such that null sequences in $M$ factor, but $x$ is not a SBP, and so $M_x$ does not have a BAI. The main technicality is to find an extension $(Y, y_0, B)$ of a given uniform algebra $(X, x_0, A)$ such that a given null sequence $(f_n)$ in $M_{x_0}$ factors in $M_{y_0}$.

We construct a compact subspace $Y$ of the space $X \times \mathbb{C}^\mathbb{N} \times \mathbb{C}$ that satisfies certain conditions, namely a point $(x, (z_n), w)$ is such that:

(i) $z_n w = f_n(x)$ ($n \in \mathbb{N}$);

(ii) $|w| = k_x := \max\{|f_n(x)|^{1/2} : n \in \mathbb{N}\}$;

(iii) $|z_n|^2 \leq |f_n(x)|$ ($n \in \mathbb{N}$).
Extensions of uniform algebras - 5

The continuous projection is

\[ \Pi : (x, (z_n), w) \mapsto x, \quad Y \to X. \]

Also we have \( p_n : (x, (z_n), w) \mapsto z_n, \quad Y \to \mathbb{C} \), and
\( q : (x, (z_n), w) \mapsto w, \quad Y \to \mathbb{C} \).

When \( k_x = 0 \), the ‘fibre’ above \( x \) is a singleton, otherwise it is a lot of circles.

Then take \( B \) to be smallest closed subalgebra of \((C(Y), |\cdot|_Y)\) containing \( \Pi^*(A) \) and all of the functions \( p_n \) and \( q \).

The map \( T : C(Y) \to C(X) \) is given by

\[
(Th)(x) = \frac{1}{2\pi} \int_0^{2\pi} h \left( x, \frac{f_n(x)}{k_x} e^{-i\theta}, k_x e^{i\theta} \right) \, d\theta
\]

for \( h \in C(Y) \).

\( \square \)
Our conclusion after some more technicalities is the following:

**Theorem** There are a natural uniform algebra $A$ on a compact space $X$ and a point $x \in X$ such that all null sequences in $M_x$ factor, but $M_x$ does not have a BAI, equivalently, $x$ is not a SBP for $A$, and, further, such that each element in $M_x$ is the square of another element in $M_x$ and $\{y\}$ is a one-point part with respect to $A$ for each $y \in X$.

Again note that our example is not separable.
Esterle’s classification of CRBAs

The 3rd conference on Banach algebras was held at Calstate, Long Beach, 13-31 July, 1981; the conference proceedings were published as Lecture Notes in Mathematics, 975. The most impressive paper in these proceedings is Jean Esterle’s ‘Classification of CRBAs’; see my book, §4.9.

Esterle’s classification has nine classes, each smaller than the one before. In each case, a class is distinct from its predecessor, save that maybe his Classes III and IV coincide, and maybe his Classes V and VI coincide.

Class V is defined to consist of the CRBAs $R$ such that there is $a \neq 0$ in $R$ with $a \in \overline{a^2R}$; with CH, this condition is equivalent to ‘for each infinite compact space $K$ there is a discontinuous homomorphism from $C(K)$ into $R^\#$.’
Two classes

Let $A$ be a CBA, and consider the following two properties:

(A) $\lim_{n \to \infty} a^n \cdot A \neq \{0\}$ for some $a \in A$;

(B) $\lim_{n \to \infty} a_1 \cdots a_n \cdot A \neq \{0\}$ for some $(a_n)$ in $A$.

These two conditions specify classes III and IV, respectively, of Esterle in the case of CRBAs. In the case of integral domains $A$, we have:

(A) there exists an element $a \in A^\bullet$ that can be factored successively as $a = ba_1$, $a_1 = ba_2$, $a_2 = ba_3$, ... for some $b$ and $(a_n)$ in $A$;

(B) there exists an element $a \in A^\bullet$ that can be factored successively as $a = b_1a_1$, $a_1 = b_2a_2$, $a_2 = b_3a_3$, ... for some sequences $(a_n)$ and $(b_n)$ in $A$.

Clearly (A) $\Rightarrow$ (B) and our (IV) $\Rightarrow$ (B).
Another uniform algebra - 1

We cannot distinguish (A) and (B) for CRBAs, but we can do this for uniform algebras.

Let $\Pi$ be the open half-plane

$$\Pi = \{z = x + iy \in \mathbb{C} : x > 0\},$$

and let $A = A^b(\Pi)$, the (non-separable) uniform algebra of all bounded, continuous functions on $\Pi$ that are analytic on $\Pi$.

For

$$f = \sum \{\alpha_r \delta_r : r \in \mathbb{R}^+\} \in \ell^1(\mathbb{R}^+),$$

denote its Laplace transform by $\mathcal{L}f$, so that

$$(\mathcal{L}f)(z) = \sum \{\alpha_r e^{-zr} : r \in \mathbb{R}^+\} \quad (z \in \Pi).$$

Denote by $M$ and $B$, respectively, the closures in $(A, |\cdot|_\Pi)$ of $\{\mathcal{L}f : f \in \ell^1(\mathbb{Q}^+)\}$ and $\{\mathcal{L}f : f \in \ell^1(\mathbb{Q}^+)\}$.

Thus $B$ is a separable, unital, closed subalgebra of $A$, and hence a uniform algebra on $\Pi$, and $M$ is a maximal ideal in $B$. 28
Another uniform algebra - 2

**Lemma 1** The maximal ideal $M$ factors projectively.

Let $I$ consist of the functions $F \in B$ such that $|F(z)| = O(e^{-ax})$ as $z \to \infty$ in $\Pi$ for some $a > 0$.

**Lemma 2** Suppose that $F \in I$. Then

$$\bigcap_{n=1}^{\infty} F^n B = \{0\}.$$

**Proposition** Let $F \in M \setminus I$. Then there exists $z \in \Pi$ such that $F(z) = 0$.

**Proof** This is an extension of a classical theorem of H. Bohr using Nevanlinna’s theorem.
Another uniform algebra - 3

**Theorem** The maximal ideal $M$ of the separable, unital uniform algebra $B$ factors projectively, and so satisfies (B), but $M$ does not satisfy (A), and hence null sequences do not factor in $M$.

**Proof** To show that $M$ does not satisfy (A), it suffices to show that $\bigcap F^n M = \{0\}$ for each $F \in M$. If $F \in I$, this follows from Lemma 2. If $F \in M \setminus I$, by the proposition there exists $z \in \Pi$ such that $F(z) = 0$, and so each $G \in \bigcap F^n M$ is analytic on a neighbourhood of $z$ and has a zero of infinite order at $z$, and so $G \equiv 0$, giving the result in this case. \qed
Transfer to the disc - 1

We can transfer the above algebras $B$ and $M$ from the half-plane $\Pi$ to the unit disc $\mathbb{D}$. Indeed, take $r \in \mathbb{R}^{+\bullet}$, and define

$$f_r(z) = \exp \left( r \left( \frac{z + 1}{z - 1} \right) \right) \quad (z \in \mathbb{D} \setminus \{1\}).$$

The functions $f_r$ belong to $H^\infty(\mathbb{D})$. So $B$ is the unital subalgebra of $H^\infty(\mathbb{D})$ generated by the functions $f_r$ for $r \in \mathbb{Q}^+$.

Restrict $H^\infty(\mathbb{D})$ to the fibre above 1, and call the character space of the restriction algebra $\mathcal{M}_1$, as in Hoffman. The common zero set of the functions $f_r$ is called $Z$; it is non-empty, the union of Gleason parts for $H^\infty(\mathbb{D})$, and disjoint from the Shilov boundary.

Consider the compact space $K$ formed by identifying the points of $\mathcal{M}_1$ that are not separated by $B$. 31
Transfer to the disc - 2

**Theorem** The algebra $B$ is a natural, separable uniform algebra on $K$, and the point corresponding to $Z$ gives the maximal ideal $M$, and so is a one-point part off the Shilov boundary. Thus $M$ does not have a BAI, but it does factor projectively.

Does it factor? Does it factor weakly? Do pairs factor?