Entire functions mapping given countable dense real sets onto each other bijectively

(preliminary report)

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ORDER

An order is a way of giving meaning to an expression of the form

$$x < y.$$

Examples: On people,

Height, weight, age and income are orders.

Nationality, religion, color and gender are not. orders.

ORDER ISOMORPHISMS

Every well-ordered set is order-isomorphic to a unique ordinal. Note: $\mathbb{Q}$ not well-ordered.

Definition. An ordered set is **dense**, if between every two elements, there is a third. Note: $\mathbb{Q}$ is dense.

Cantor 1895
If A and B are countable dense ordered sets without first elements, then there is an order isomorphism

$$f : A \to B.$$

Corollary
If A and B are countable dense subsets of $\mathbb{R}$,
then there is a homeomorphism

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(A) = B.$$

Franklin 1925
If A and B are countable dense subsets of $\mathbb{R}$,
then there is a bianalytic mapping

$$f : \mathbb{R} \to \mathbb{R}, \quad \text{with} \quad f(A) = B.$$

Franklin 1925 (again)

For A and B countable dense subsets of $\mathbb{R}$, there exists a bianalytic map $f : \mathbb{R} \to \mathbb{R}$, such that:

$f(\mathbb{R}) \subset \mathbb{R}$;

f restricts to a bijection of A onto B (hence, $f(\mathbb{R}) = \mathbb{R}$).

Morayne 1987

If A and B are countable dense subsets of $\mathbb{C}^n$ (respectively $\mathbb{R}^n$), $n > 1$, there is a measure preserving biholomorphic mapping of $\mathbb{C}^n$ (respectively bianalytic mapping of $\mathbb{R}^n$) which maps A to B.

Rosay-Rudin 1988

Same result for $\mathbb{C}^n$ only.

Remarks

Franklin's proof invokes the statement that the uniform limit of analytic functions is analytic, which is false.

For $\mathbb{C}^1$, Morayne, Rosay-Rudin results are false.
For $n = 1$, Morayne conclusion $\Rightarrow$ Franklin,
but Morayne proof fails for $n = 1$.

Theorem. For A and B countable dense subsets of $\mathbb{R}$, there exists an entire function f such that:
$f(\mathbb{R}) \subset \mathbb{R}$;
f restricts to a bijection of A onto B (hence, $f(\mathbb{R}) = \mathbb{R}$).

Proof. $A = \{\alpha_1, \alpha_2, \ldots\}$; $B = \{\beta_1, \beta_2, \ldots\}$.

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \left(z + \sum_{j=1}^{n} \lambda_j h_j(z) \right) = z + \sum_{j=1}^{\infty} \lambda_j h_j(z),$$

$$h_1 = 1; \quad \text{and} \quad h_n(z) = e^{-z^2} \prod_{k=1}^{n-1} (z - \alpha_k), \quad \text{for} \quad n = 2, 3, \ldots,$$

λ_j's small $\Rightarrow$ f entire and $f'(x) > 0, \forall x \in \mathbb{R}$,
λ_j's real $\Rightarrow$ $f(\mathbb{R}) \subset \mathbb{R}$.

$$h_n(z) = 0, \quad \text{iff} \quad z = \alpha_k,\ k = 2, \ldots, n-1.$$

Choose λ_n so $f_n(\alpha_n) = \beta_n$.

□

I OVERSIMPLIFIED

Choose enumerations $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots, \}$. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are rearrangements of $\{a_n\}$ and $\{b_n\}$ chosen recursively.

First, choose $\alpha_1, \lambda_1, \beta_1, \beta_2 \neq \beta_1$, so $f_1(\alpha_1) = \beta_1$.

Suppose we have respectively distinct

$$\alpha_1, \dots, \alpha_{2n-1}; \quad \lambda_1, \dots, \lambda_{2n-1}; \quad \beta_1, \dots, \beta_{2n}$$

$$\alpha_{2k-1} = (\text{first } a_i) \in A \setminus \{\alpha_j : j < 2k-1\}, \quad k = 1, \dots, n$$

$$\beta_{2k} = (\text{first } b_i) \in B \setminus \{\beta_j : j < 2k\}, \quad k = 1, \dots, n$$

$$f(\alpha_j) = \beta_j, \quad j = 1, \dots, 2n-1$$

Choose
$\alpha_{2n}, \lambda_{2n},$
$\beta_{2n+1}, \alpha_{2n+1}, \lambda_{2n+1},$
$\beta_{2(n+1)}$
with

$$f_{2n}(\alpha_{2n}) = \beta_{2n} \qquad f_{2n+1}(\alpha_{2n+1}) = \beta_{2n+1}$$

α_1	λ_1	β_1
$-$	$-$	β_2
$-$	$-$	$-$
.	.	.
.	.	.
.	.	.
α_{2n-1}	λ_{2n-1}	β_{2n-1}
$[\alpha_{2n}$	$\lambda_{2n}]$	β_{2n}
α_{2n+1}	$[\lambda_{2n+1}$	$\beta_{2n+1}]$
$-$	$-$	$\beta_{2(n+1)}$

How to find $[\alpha_{2n}, \lambda_{2n}]$ such that

$$\beta_{2n} = f_{2n}(\alpha_{2n}) = \alpha_{2n} + \sum_{j=1}^{2n-1} \lambda_j h_j(\alpha_{2n}) + \lambda_{2n} h_{2n}(\alpha_{2n}) =$$

$$f_{2n-1}(\alpha_{2n}) + \lambda_{2n} h_{2n}(\alpha_{2n}).$$

Put

$$g(x, y) = f_{2n-1}(x) + y h_{2n}(x).$$

Fix y_n small. Show $g(\cdot, y_n) : \mathbb{R} \to \mathbb{R}$ surjective. So, $\exists \alpha$ with $g(\alpha, y_n) = \beta_{2n}$. Implicit function theorem implies, there is $(\alpha_{2n}, \lambda_{2n})$ near (α, y_n), with $g(\alpha_{2n}, \lambda_{2n}) = \beta_{2n}$ and $\alpha_{2n} \in A$.

Theorem (again) $A, B \subset \mathbb{R}$ countable dense. There is f entire which restricts to bijection $A \to B$.

Can impose further conditions on f.

1. Interpolation: $f(a_n) = b_n$, $n \in \mathbb{Z}$, for increasing sequences without limit points.

2. Growth: $M(f, r) > \rho(r)$, given $\rho(r) > 0$ continuous,

3. Universality: translates of f are dense in space of entire functions.

Lemma. Suppose E approximation subset of $\mathbb{C}$ disjoint from $\mathbb{R}$; $\{c_n\}$ sequence of distinct complex numbers tending to ∞ disjoint from E and $\mathbb{R}$; $\{d_n\}$ arbitrary sequence in $\mathbb{C}$, ; $\{a_n\}$ and $\{b_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, strictly increasing sequences of real numbers tending to ∞, as $n \to \infty$ and ϵ positive continuous functions on $\mathbb{C}$. Then, for every function $g \in A(E)$, there exists an entire function Φ, such that $|\Phi - g| < \epsilon$ on E; $M(\Phi, r) > 1/\epsilon(r)$; $\Phi(c_n) = d_n$, $n = 1, 2, \ldots$; Φ maps $\mathbb{R}$ bijectively onto $\mathbb{R}$; $\Phi' > 0$ on $\mathbb{R}$ and $\Phi(a_n) = b_n$, $n = 0, \pm 1, \pm 2, \ldots$.

Lemma. Same hypotheses, there exists entire function H, which tends to 0 as $z \to \infty$ on E, whose zeros are precisely the points a_n, $n = 0, \pm 1, \pm 2, \ldots$ and the points c_n, $n = 1, 2, \ldots$, and such that both $|H|$ and $|H'|$ dominated by ϵ on $\mathbb{R}$.

Replace

$$f(z) = z + \sum_{j=1}^{\infty} \lambda_j h_j(z)$$

by

$$f(z) = \Phi(z) + H(z) \sum_{j=1}^{\infty} \lambda_j h_j(z).$$

Most populous confederacy of Iroquois:

Wendat = Wyandot = Huron

In Canada, they are mainly here in Québec city.

TIAWENHK

MERCI