QUEBEC CITY, MAY 2018

Variations on the theme of analytic continuation

"Between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain."

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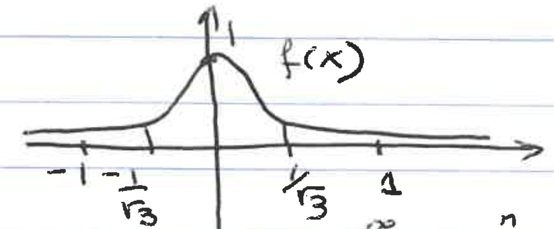
$\boxed{\text{SALES PITCH}}$: DK & E. Lundberg, "Linear Holomorphic PDE and Classical Potential Theory" Surveys & Monographs, AMS, July 2018

I. A warm-up

(i) $f(x) = \frac{1}{1+x^2}$



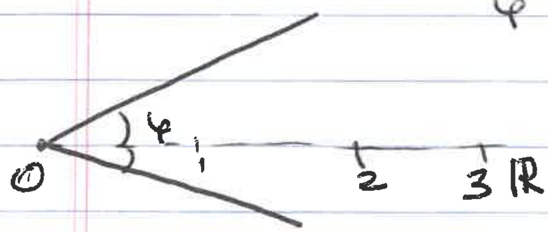
Q. Why does its Taylor series $\sum_0^\infty (-1)^n x^{2n}$ diverge for all $x : |x| \geq 1$? - Ans: ...

(ii) $f(z) := \sum_0^\infty \cos(\sqrt{n}) z^n$, ROC $= 1$.

Q. Does $f(z)$ extend to a larger domain than $\mathbb{D}$

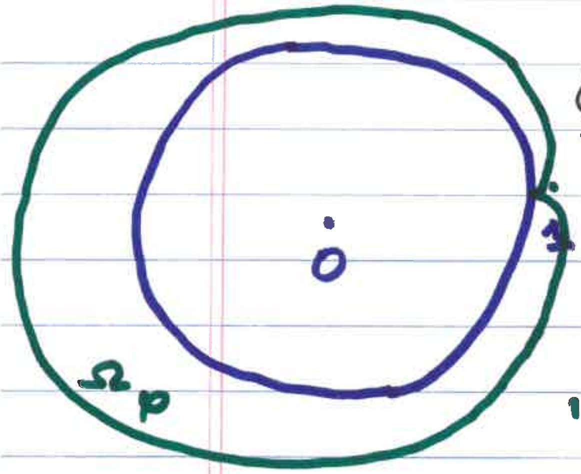
Ans. Yes, $f(z)$ is analytic in $\mathbb{C} \setminus \{1\}$. Why?

(iii) Let $g(z)$ be bounded and analytic in the sector $S_\varphi := \{ z : |\arg z| < \varphi,\ 0<\varphi<\frac{\pi}{2} \}$



and assume that every point on ∂S_φ is a singularity for g, i.e., g doesn't extend beyond S_φ.

Consider $f(z) := \sum_{1}^{\infty} g(n) z^n$, ROC $= 1$.



Q. Can we extend f beyond $\mathbb{T}$?

Ans: Yes, to $\Omega_\varphi := \{ re^{i\theta} :$

$2\pi - \cot\varphi \cdot \log r > \theta > \cot\varphi \cdot \log$

"heart-shaped domain.

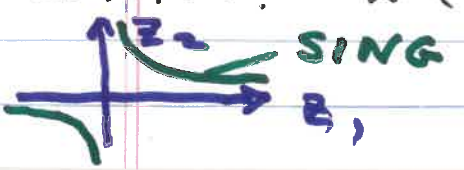
II. (1) ODE vs PDE

$$w^{(n)} + a_{n-1}(z) w^{(n-1)} + \dots + a_0 w(z) = f(z) \qquad (*)$$
$$w(0) = w_0, \dots, \quad w^{(n-1)}(0) = w_{n-1}. \qquad (CP)$$

Thm If $a_0, \dots, a_{n-1}, f$ are analytic in $\Omega \ni \{0\}$, any solution of $(*)$ extends to Ω.

Example $\frac{\partial w}{\partial z_2} = z_1^2 \frac{\partial w}{\partial z_1}$, $w(z_1, 0) = z_1 \qquad (**)$

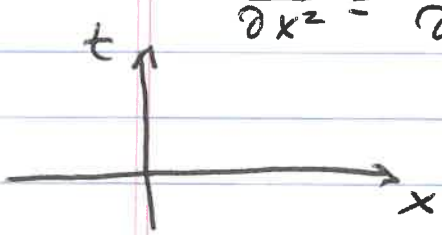
Solution: $w(z_1, z_2) = \frac{z_1}{1 - z_1 z_2}$, singular on $\{ z_1 z_2 = 1 \}$



WHY?

(ii) Consider the "standard" initial value problem for the heat equation:

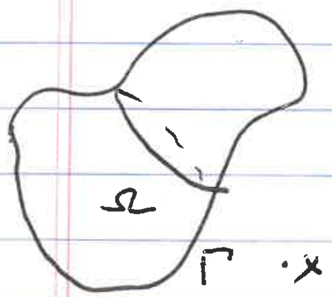
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u(x,0) = f(x), \quad f(x) = e^{-x^4}. \qquad (*)$$



Shocking fact: $\nexists$ solution of $(*)$ that is real-analytic near the origin. **Why?**

III (1) Potential Theory

(i) Herglotz' Question (1914).



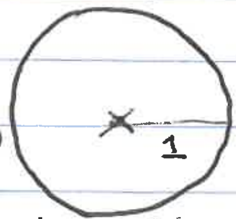
Ω = solid in $\mathbb{R}^3$, $\Gamma = \partial\Omega$ = smooth, algebraic.

Q. How far the gravitational potential

$$u_\Omega(x) = \frac{1}{4\pi}\int_\Omega \frac{dy}{|x-y|}$$

can be continued into Ω?

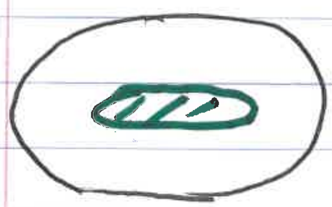
Examples (a) $\Omega = \{|x|<1\}$, $u_\Omega = \frac{c}{|x|}$



(Note: the same holds for ANY polynomial density!)

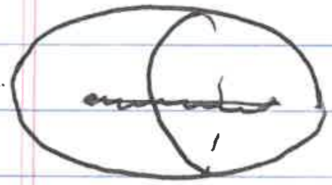
(b) $\Omega := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} < 1,\ a>b \right\}$, an oblate spheroid.

Answer: $\Omega \setminus \{(x_1, x_2, 0) : x_1^2 + x_2^2 \le a^2 - b^2\}$



(c) $\Omega := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} < 1, \quad a > b \right\}$ - prolate spheroid

Answer : $\Omega \setminus \{(x_1, 0, 0) : |x_1| \le \sqrt{a^2 - b^2}\}$

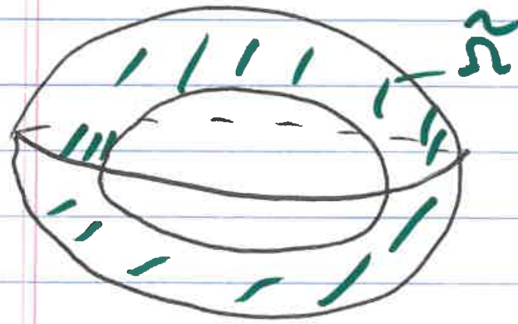


Singularities for (a), (b) are algebraic, while for (c) - logarithmic

(Q) From I. Newton to V. Arnol'd and A. Givental'

Let $\tilde{\Omega}$ be an elliposoidal homoeid, $\tilde{\Omega} := t\Omega \setminus \Omega$,

$t > 1$, $\Omega = \left\{ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} < 1 \right\}$.



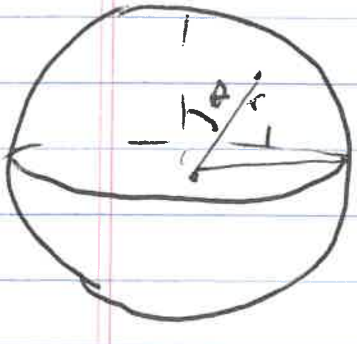
$$u_{p,\tilde{\Omega}} = \frac{1}{4\pi} \int_{\tilde{\Omega}} \frac{p(y)\,dy}{|x-y|},$$

where p is a polynomial of degree m.

Thm

Then, inside Ω (inside the "cavity"), $u_{p,\tilde{\Omega}}$ equals a harmonic polynomial of degree $\le m$. In particular, for $p = \text{const}$, $u_{p,\tilde{\Omega}} \equiv \text{const}$ inside Ω, so $\nabla u_{p,\tilde{\Omega}} = 0$ (Newton's "no gravity in the cavity" thm.)

(3) From Taylor series to series of zonal harmoni



Let $u := \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$ be

an axially symmetric harmonic functio

in the unit ball, $\lim |a_n|^{1/n} = 1$, $P_n(x) =$
$= \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$ are Legendre polynomials.

u must have singularities on the sphere $\{r=1\}$

<u>Q</u>. Where?

<u>Thm</u> (G. Szegö, '54). $u(r,\theta)$ extends harmonically

across $(1, \theta_0, \varphi)$ iff the Taylor series

$f(\zeta) := \sum_0^{\infty} a_n \zeta^n$ extends across $\zeta_0 = e^{i\theta_0}$

<u>Q</u>. Where does it come from?

<u>Extension</u> ('96; Ebenfelt – DK – H.S. Shapiro).

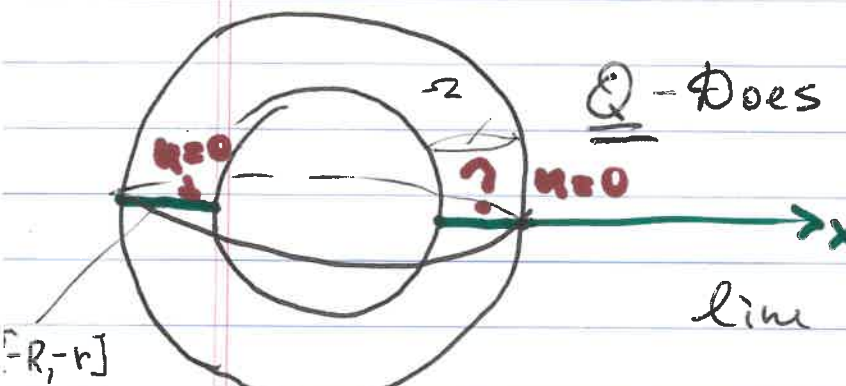
Theorem holds if P_n are replaced by $P_n^{\alpha,\beta}$,

any Jacobi polynomials orthogonal on $(-1,1)$ wrt weight $(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha,\beta > -1$.

($\alpha=\beta=0$ - Legendre, $\alpha=\beta=-\frac{1}{2}$ - Tschebyseff, $\alpha=\beta=\frac{k-3}{2}$, $k>$
- an integer, ultraspherical polynomials.

(IV) Uniqueness Problem for Riesz Potentials

Consider the spherical shell $\Omega := \{x \in \mathbb{R}^3 : r < |x| < R\}$. Let u: $\Delta u = 0$ in Ω, $u\big|_{[-R,-r]} = 0$.



Q – Does u also vanish on $[r, R]$

– Does it matter that the line passes through the center

Exercise dim = 2, Ω = annulus.

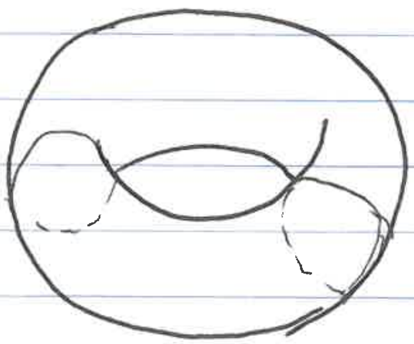
Answers: – "Yes"

– "Yes" if $\frac{R}{r} \le 3$ ("thin" shell)

– "No" if $\frac{R}{r} > 3$ ("thick" shell)

hint from T. Ransford.

Now, what if we ask the Same question for the torus?

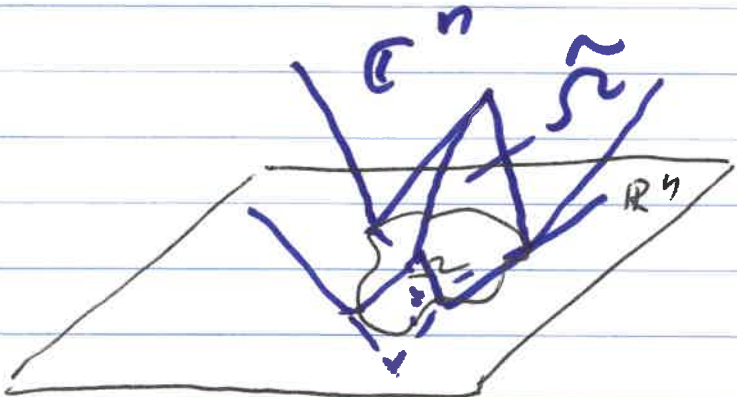


– Ans. "No" – DK-Lundberg.

The common denominator:

analytic continuation of solutions of holomorphic PDE in $\mathbb{C}^n$ and propagation of singularitie through $\mathbb{C}^n$ to reach $\mathbb{R}^n$.

Uniqueness Problem



All solutions of all PDE

$(\Delta^m + \text{lower terms})u = f$

extend to $\hat{\Omega} := \mathbb{C}^n \setminus \bigcup_{x \in \mathbb{R}^n \setminus \Omega} \Gamma_x$, $\Gamma_x = \{\sum_1^n (z_j - x_j)^2 = 0\}$,

so-called "cell of harmonicity".

(Example for $\Omega = \{|x| < 1\}$, $\hat{\Omega} = \{z : (\|z\|^4 - |\sum_1^n z_j^2|^2)^{1/2} + \|z\|^2 < 1\}$ – the Lie ball.)

Thus, our question of uniqueness transforms to a pure geometric question whether $\hat{\Omega}$ $\cap \{ z_j = c_j,\ j = 2, .., n\}$, i.e. with a given complex line, is connected, or not.

Return to Herglotz' question:

Seeking singularities of u_Ω:

$$\Delta u_\Omega = \chi_\Omega.$$

Consider M (modified Schwarz potential) solving the (CP)

$$\Delta M = 1 \text{ near } \Gamma = \partial\Omega \qquad (*)$$

$$M|_\Gamma = \nabla M|_\Gamma = 0.$$

The function $u := \begin{cases} u_\Omega, & \text{outside} \\ u_\Omega - M, & \text{inside} \end{cases}$

gives the desired continuation. For the weight P, a polynomial, or an entire function, we only need to modify (*) replacing 1 by P.

Thus, "philosophically", we are interested in a question whether the singularities of solutions of a (CP) are dictated by the initial variety Γ and the operator (Δ here) not by a particular initial data P (as long as it has none in $\mathbb{C}$").

Examples: (A glimpse of the deep and beautiful theory launched by J. Leray).

(1) Let $\Gamma = \{ w = z^3 \}$

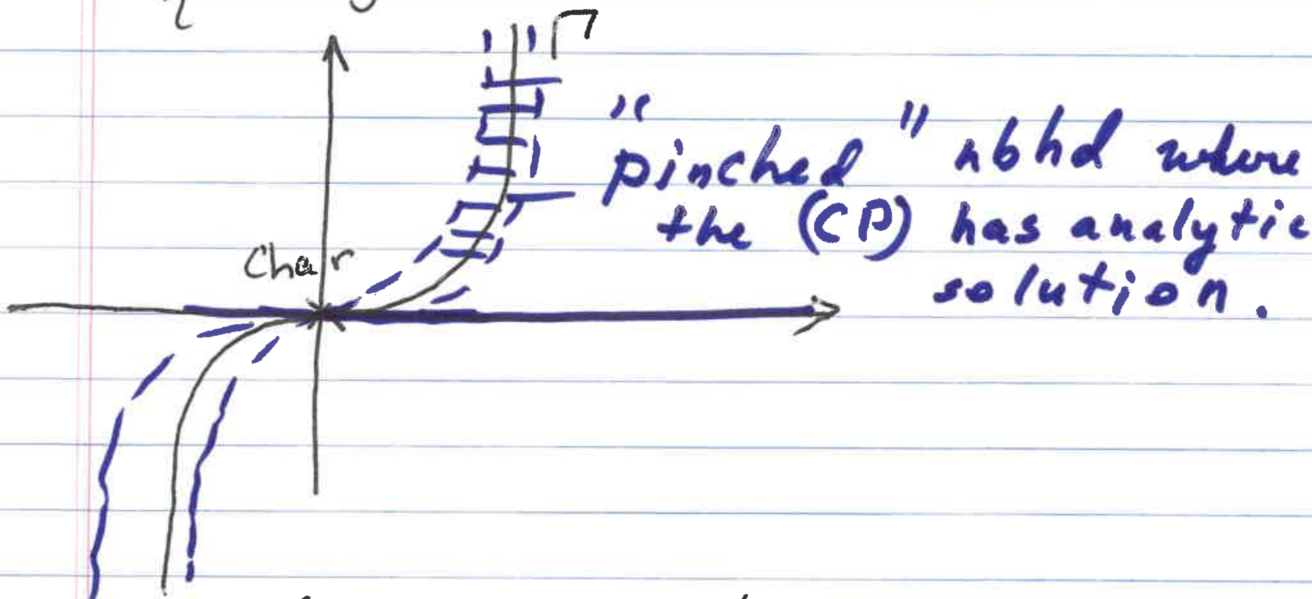
$$(CP) \quad \begin{cases} \partial_{zw} = 0, \\ u = zw \text{ on } \Gamma, \ \frac{\partial u}{\partial w} = z \text{ on } \Gamma, \ \frac{\partial u}{\partial z} = w \text{ on } \Gamma \end{cases}$$

$(0,0)$ is the "bad" point on Γ (characteristic) where Cauchy-Kovalevskaya theorem doesn't work.

The solution u of (CP):

$$u = \frac{z^4}{4} + \frac{3w^{4/3}}{4}$$ is ramified

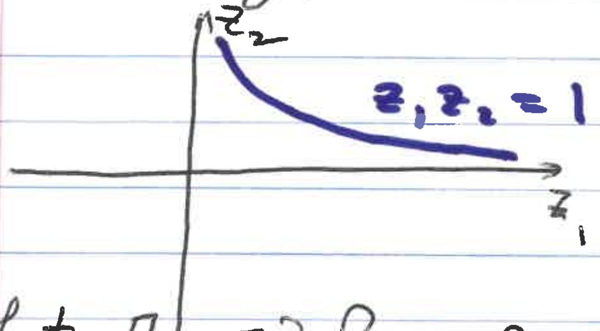
around $\{w=0\}$ tangent to Γ at $(0,0)$.



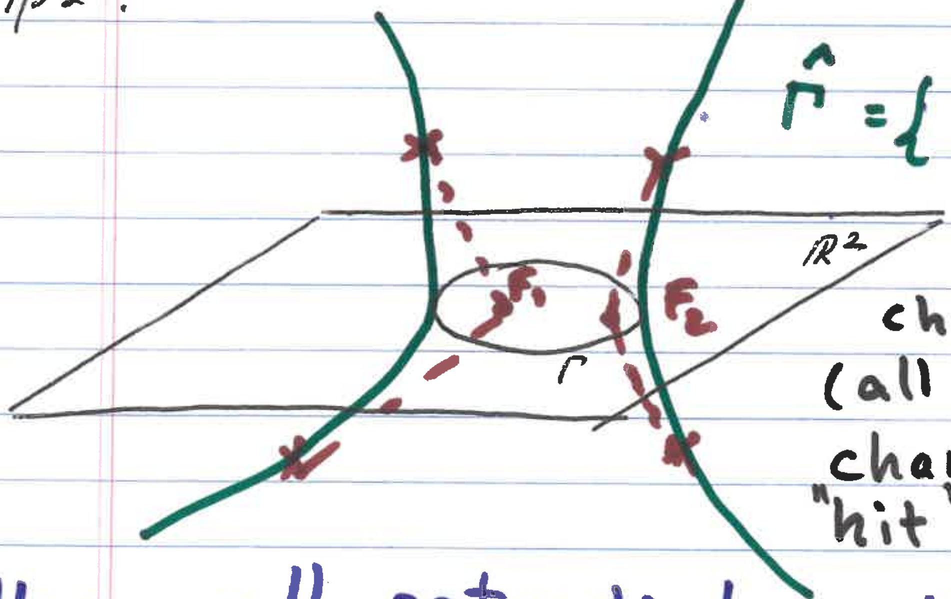
(2) The "heat equation" problem: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(x,0) = f(x)$. Ans: The initial surface $\{t=0\}$ is everywhere characteristic. The C-K thm never gets off the ground.

(3) Recall $\frac{\partial w}{\partial z_2} = z_1^2 \frac{\partial w}{\partial z_1}$, $w(z_1, 0) = f(z_1)$, f - a polynomial, or entire.

$$w = f\left(\frac{z_1}{1 - z_1 z_2}\right)$$ in general develops singularities on the (same!) surface $\{z_1 z_2 = 1\}$ that is tangent to the initial plane $\{z_2 = 0\}$ at ∞, the only "characteristic" point on the initial variety.



(4) Let $\Gamma = \partial\Omega$, $\Omega := \{\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$ be an ellipse.



$\hat{\Gamma} = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\} \subset \mathbb{C}^2$

$\hat{\Gamma}$ has four characteristic points (all in $\mathbb{C}^2$). Leray's characteristic tangen "hit" $\mathbb{R}^2$ at foci

Hence, all potentials with entire densities continue to $\mathbb{R}^2 \setminus \{\text{foci}\}$.