Nonlinear maps commuting with the λ-Aluthge transform under certain operations

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(The talk is based on joint work with F. Chabbabi)

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I. Introduction

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For $T \in \mathcal{B}(H)$, we denote the module of T by $|T| = (T^*T)^{1/2}$ and we shall always write, without further mention, $T = V|T|$ to be the unique polar decomposition of T, where V is a partial isometry satisfying $\mathcal{N}(V) = \mathcal{N}(T)$, where $\mathcal{N}(T)$ is the kernel of T.

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The Aluthge transform introduced by Aluthge, as

$$\Delta(T) = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(H),$$

to extend some properties of hyponormal operators.

Later, Okubo introduced a more general notion called $\lambda-$Aluthge transform which has also been studied in detail.
For $\lambda \in [0, 1]$, the $\lambda-$Aluthge transform is defined by,

$$\Delta_\lambda(T) = |T|^\lambda V |T|^{1-\lambda}, \quad T \in \mathcal{B}(H).$$

Notice that $\Delta_0(T) = V|T| = T$, and $\Delta_1(T) = |T|V$ which is known as Duggal's transform.

The interest of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example,

$$\sigma_*(\Delta_\lambda(T)) = \sigma_*(T), \text{ for every } \quad T \in \mathcal{B}(H), \tag{1}$$

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where σ_* runs over a large family of spectra.
Another important property is that $Lat(T)$, the lattice of T-invariant subspaces of H, is nontrivial if and only if $Lat(\Delta_\lambda(T))$ is nontrivial.

Next, we shall introduce some operator classes which will be used in the talk.

Definition

Let $T \in \mathcal{B}(H)$.
(i) T is normal if $T^*T = TT^*$,
(ii) T is quasi-normal if $T^*TT = TT^*T$,
(iii) T is subnormal if T has a normal extension,
(iv) for $p > 0$, T is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$.

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The following inclusion relations are well known and they are proper.

{Normal} $\subset$ {quasi-normal} $\subset$ {subnormal} $\subset$ {hyponormal} $\subset$ {semi-hyponormal} $\subset$ {normaloid}.

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Aluthge transform has been defined in the paper discussed on hyponormal operators by A. Aluthge, as follows:

Theorem(Aluthge 1990)

For $p \in]0, 1]$, let $T \in \mathcal{B}(H)$ be a p-hyponormal operator. Then the following assertions hold:
(i) $\Delta(T)$ is $(p + \frac{1}{2})$-hyponormal if $0 < p \leq \frac{1}{2}$;
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In particular, if T is semi-hyponormal then $\Delta(T)$ is hyponormal.

II. Main results

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We write here this general result in the context of $\mathcal{B}(H)$.

Theorem (BMN 2016)

Let H and K be two complex Hilbert spaces, with $\dim(H) \geq 2$. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective and linear map. Then Φ satisfies $\Delta_\lambda \circ \Phi = \Phi \circ \Delta_\lambda$ if and only if there exist a unitary operator $U : H \to K$ and a constant $\alpha \neq 0$ such that
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We improve the above result of BMN, replacing linearity by additivity for the map Φ. Also, we give new proof, based on maps that preserve the set of nilpotent operators. More precisely, we show that:

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Theorem A (F. Chabbabi, M.M. 2017)

If H, K are two Hilbert spaces of infinite dimensional and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective additive map and $\lambda \in]0, 1[$. Then $\Delta_\lambda \circ \Phi = \Phi \circ \Delta_\lambda$ if and only if there exist a unitary or anti-unitary operator $U : H \to K$ and $0 \neq \alpha \in \mathbb{C}$, such that

$\Phi(T) = \alpha UTU^* \quad$ for every $T \in \mathcal{B}(H)$.

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In particular, T is nilpotent of order $d+1$ if and only if $\Delta_\lambda(T)$ is nilpotent of order d.

As a direct consequence of this theorem, we obtain the following :

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Corollary

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective additive map. If Φ commutes with the λ-Aluthge transform for some $\lambda \in]0, 1]$, then we have

$$T^d = 0 \iff (\Phi(T))^d = 0.$$

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That is Φ preserves strongly nilpotent operators in both directions. Now, to conclude, we use the form of additive maps that preserve the nilpotent operators.

Now, we give a complete form of the bijective (not necessarily linear) maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H, K are Hilbert spaces of dimension greater than 2, that satisfy

$$\Delta_\lambda(\Phi(A) \star \Phi(B)) = \Phi(\Delta_\lambda(A \star B)) \text{ for all } A, B \in \mathcal{B}(H),$$

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4. $A \star B = A^n B A^m$ with $n, m \in \mathbb{N}$, such that $n + m \geq 1$, the (n, m)-Jordan triple product.

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5. $A \star B = A + \omega B, \quad 0 \neq \omega \in \mathbb{C}.$

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$$\Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \text{ for every } A, B \in \mathcal{B}(H),$$

if and only if, there exists a unitary or anti-unitary operator $U : H \to K$ such that

$$\Phi(T) = UTU^* \qquad \text{for every } T \in \mathcal{B}(H).$$

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For ω-addition, we have :

Theorem (F. Chabbabi and M.M.)

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map and ω be a non-zero complex number. Then Φ satisfies
$\Delta_\lambda\Big(\Phi(S) + \omega\Phi(T)\Big) = \Phi\Big(\Delta_\lambda(S + \omega T)\Big), \quad \text{for all} \quad S, T \in \mathcal{B}(H),$
if and only if there exist a unitary or anti-unitary operator $U : H \to K$ and $0 \neq \alpha \in \mathbb{C}$, such that Φ has the following form :

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Remark

If $\omega \in \mathbb{R}$ then Φ is linear or anti-linear; both cases can hold. On the other hand, if ω is not real, then Φ is necesserly linear and U is a unitary operator.

Now, for n and m non-negative integers with $n + m \geq 1$, consider the operation $\mathcal{B}(H) \times \mathcal{B}(H) \ni (A, B) \mapsto A \star B = A^n B A^m$, $(n, m)-$Jordan-Triple product.

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Theorem (F. Chabbabi and M.M.)

Let H and K be two complex Hilbert spaces with $\dim H \geq 2$. Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then Φ satisfies $\Delta_\lambda(\Phi(A)^n \Phi(B) \Phi(A)^m) = \Phi(\Delta_\lambda(A^n B A^m))$ for all $A, B \in \mathcal{B}(H)$, if and only if there exist a unitary or anti-unitary operator $U : H \to K$ and a constant $\alpha \in \mathbb{C}$, with $\alpha^{n+m} = 1$, such that

$$\Phi(T) = \alpha U T U^* \qquad \text{for every } T \in \mathcal{B}(H).$$

Remark

Even if the hypothesis on the map Φ is purely algebraic, the conclusion gives automatically the continuity of the map.
Also, the linearity of Φ is not assumed, we get it automatically.

Proof of the last theorem for $n + m \geq 1$

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$$(x \otimes y)u =< u, y > x \text{ for } u \in H.$$

It is easy to show that every rank one operator has the previous form and that $x \otimes y$ is an orthogonal projection, if and only if $x = y$ and $\|x\| = 1$.

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Lemma1

Let $x, y \in H$ be nonzero vectors. We have

$$\Delta_\lambda(x \otimes y) = \frac{< x, y >}{\|y\|^2}(y \otimes y) \text{ for every } \lambda \in]0, 1[.$$

Lemma 2

Let $T \in \mathcal{B}(H)$ and $n \geq 2$ be a fixed integer number. Suppose that

$$< T^n x, x >= (< Tx, x >)^n \quad \text{for all unit vectors } x \in H.$$

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In the next we will denote by

$$\mathcal{U}_k(H) = \{U \in \mathcal{B}(H); \quad U \text{ unitary and } \ U^k = I\}.$$

Lemma 3

Let $T \in \mathcal{B}(H)$ and $k \geq 2$ be an integer, suppose that T and T^* are one-to-one. For every $\lambda \in]0, 1]$, we have the following

$$\Delta_\lambda(T^k) = T \iff T \in \mathcal{U}_{k-1}(H).$$

For a Hilbert space H and an integer number $k \in \mathbb{N}^*$, we denote by

$$\mathcal{Q}_k(H) = \left\{T \in \mathcal{B}(H); \text{ such that } T \text{ quasi-normal and } T^{k+1} = T\right\}.$$

The following lemma gives a discretion of $\mathcal{Q}_k(H)$.

Lemma

Assume that $T \in \mathcal{Q}_k(H)$, then T^k is an orthogonal projection on $\mathcal{R}(T)$.

In rest of the talk, H and K are tow Hilbert spaces with $\dim(H) \geq 2$, and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a bijective map satisfying

$$(*) \quad \Delta_\lambda(\Phi(A)^n\Phi(B)\Phi(A)^m) = \Phi(\Delta_\lambda(A^nBA^m)) \text{ for all } A, B \in \mathcal{B}(H)$$

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$$\Delta_\lambda(T^k) = T \iff T \in \mathcal{U}_{k-1}(H).$$

Corollary 5

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map satisfying $(*)$ with $n \neq 0$ and $m \neq 0$, then $\Phi(0) = 0, \quad \text{and} \;\; \Phi(I) \in \mathcal{U}_{n+m}(H)$, and

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In the rest of the talk, without loss of generality, we can assume $\Phi(I) = I$. As a direct consequence of the preceding lemmas, we have the following corollary.

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Let Φ be a bijective map satisfying $(*)$. Then

1. Φ commutes with Δ_λ, in particular it sends the quasi-normal operator to quasi-normal operators.
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1. Φ commutes with Δ_λ, in particular it sends the quasi-normal operator to quasi-normal operators.
2. Φ preserves the set of orthogonal projections, and the set of rank one projections in both directions.
3. $\Phi(P+Q) = \Phi(P) + \Phi(Q)$ for all orthogonal projections P, Q such that $P \perp Q$.

In the following lemma we introduce a function $h : \mathbb{C} \to \mathbb{C}$, which will be used later to prove the linearity of Φ.

Theorem

There exists a bijective function $h : \mathbb{C} \to \mathbb{C}$ such that

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(iv) for all unit vectors $x \in H$, $y \in K$ such that $\Phi(x \otimes x) = y \otimes y$, we have

$$< \Phi(A)y, y >= h(< Ax, x >) \quad \text{for every} \quad A \in \mathcal{B}(H),$$

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$$< \Phi(A)y, y >= h(< Ax, x >) \quad \text{for every} \quad A \in \mathcal{B}(H),$$

(v) Let $P = x \otimes x$, $P' = x' \otimes x'$ be two rank one orthogonal projections such that $P \perp P'$. Then

$$\Phi(\alpha P + \beta P') = h(\alpha)\Phi(P) + h(\beta)\Phi(P'), \quad \text{for every } \alpha, \beta \in \mathbb{C}.$$

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where $W(A) = \{< Ax, x >; \ \ \|x\| = 1\}$ is the numerical range of A.
By (iv) of last theorem, we have

$$h(\mathcal{E}) \subset h(W(A)) = W(\Phi(A)),$$

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By (iv) of last theorem, we have

$$h(\mathcal{E}) \subset h(W(A)) = W(\Phi(A)),$$

Now , $W(\Phi(A))$ is bounded and thus h is bounded on the bounded subset. Which implies that h is continuous, since it is automorphism. We derive that h is a continuous automorphism over the complex field $\mathbb{C}$. It follows that h is the identity or the complex conjugation map.

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Let $y \in K$ and $x \in H$ be unit such that $y \otimes y = \Phi(x \otimes x)$. Let $\alpha \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$ be arbitrary. Using, Step 1, we get

$$\begin{aligned} < \Phi(A+B)y, y > &= h(< (A+B)x, x >) \\ &= h(< Ax, x > + < Bx, x >) \\ &= h(< Ax, x >) + h(< Bx, x >) \\ &= < \Phi(A)y, y > + < \Phi(B)y, y > \\ &= < (\Phi(A) + \Phi(B))y, y >, \end{aligned}$$

and

$$\begin{aligned} < \Phi(\alpha A)y, y > &= h(< \alpha Ax, x >) \\ &= h(\alpha)h(< Ax, x >) \\ &= h(\alpha) < \Phi(A)y, y > . \end{aligned}$$

Therefore, we conclude that for all unit vectors $y \in K$,

$$< \Phi(A+B)y, y > = < (\Phi(A) + \Phi(B))y, y > \quad \text{and}$$

$$< \Phi(\alpha A)y, y > = h(\alpha) < \Phi(A)y, y >,$$

It follows that

$$\Phi(A+B) = \Phi(A) + \Phi(B) \text{ and } \Phi(\alpha A) = h(\alpha)\Phi(A), \ \ \forall A, B \in \mathcal{B}(H).$$

Therefore Φ is **linear or anti-linear** since h is the identity or the complex conjugation.

Step 4. There exists a unitary or anti-unitary operator $U \in \mathcal{B}(H, K)$, such that $\Phi(T) = UTU^*$ for every $T \in \mathcal{B}(H)$.

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Since Φ commute with λ-Aluthge transform, unital and linear or anti-linear, in particular Φ is additive, by Theorem A , we have

$$\Phi(T) = UTU^* \quad \text{for all} \;\; T \in \mathcal{B}(H)$$

for some unitary or anti-unitary operator $U \in \mathcal{B}(H, K)$.
The proof of theorem is complete.

Spectral radius via Aluthge transform

For $T \in \mathcal{B}(H)$, the spectrum of T is denoted by $\sigma(T)$ and its spectral radius by $r(T)$.
Yamazaki established the following interesting formula for the spectral radius

Theorem (Yamazaki)

If $T \in \mathcal{B}(H)$, then

$$\lim_{n\to\infty} \|\Delta_\lambda^n(T)\| = r(T),$$

where $\Delta_\lambda^{(n)}$ is the n-th iterate of Δ_λ, i.e.; $\Delta_\lambda^{(n+1)}(T) = \Delta_\lambda(\Delta_\lambda^{(n)}(T))$, $\Delta_\lambda^0(T) = T$.

In this section, we use Rota's Theorem, in order to obtain new formulas of spectral radius via λ-Aluthge transformation.

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Theorem (F.Chabbabi, M.M)

If $T \in \mathcal{B}(H)$, then for every $n \geq 0$,

$$\begin{aligned} r(T) &= \inf\{\|\Delta_\lambda^{(n)}(XTX^{-1})\|,\ X \in \mathcal{B}(H) \ \text{ invertible } \} \\ &= \inf\{\|\Delta_\lambda^{(n)}(e^A T e^{-A})\|,\ A \in \mathcal{B}(H) \ \text{ self adjoint } \}. \end{aligned}$$

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An operator T is said to be normaloid if $r(T) = \|T\|$.

As an immediate consequence of the theorem, we obtain the following corollary which is a characterization of normaloid operators via λ-Aluthge transformation.

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Corollary

If $T \in \mathcal{B}(H)$, then the following assertions are equivalent :
(i) T is normaloid;
(ii) $\|T\| \leq \|\Delta_\lambda(XTX^{-1})\|$, for all invertible $X \in \mathcal{B}(H)$;
(iii) $\|T\| \leq \|\Delta_\lambda^{(n)}(XTX^{-1})\|$, for all invertible $X \in \mathcal{B}(H)$ and for all natural number n.

Theorem (F.Chabbabi, M.M)

Let $T \in \mathcal{B}(H)$. Then

$$r(T) = \lim_{k} \|\Delta_{\lambda}(T^{k})\|^{1/k}$$

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As direct consequence of Theorem, we get :

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If $T \in \mathcal{B}(H)$, then the following assertions are equivalent :
(i) T is normaloid;
(ii) $\|T\|^k = \|\Delta_\lambda(T^k)\|$, for all natural number k;
(iii) $\|T\|^k = \|\Delta_\lambda^{(n)}(T^k)\|$, for every natural number k, n.

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