Conformal Graph Directed Markov Systems: Recent Advances

by

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on the occasion of

Thomas Ransford's 60th birthday

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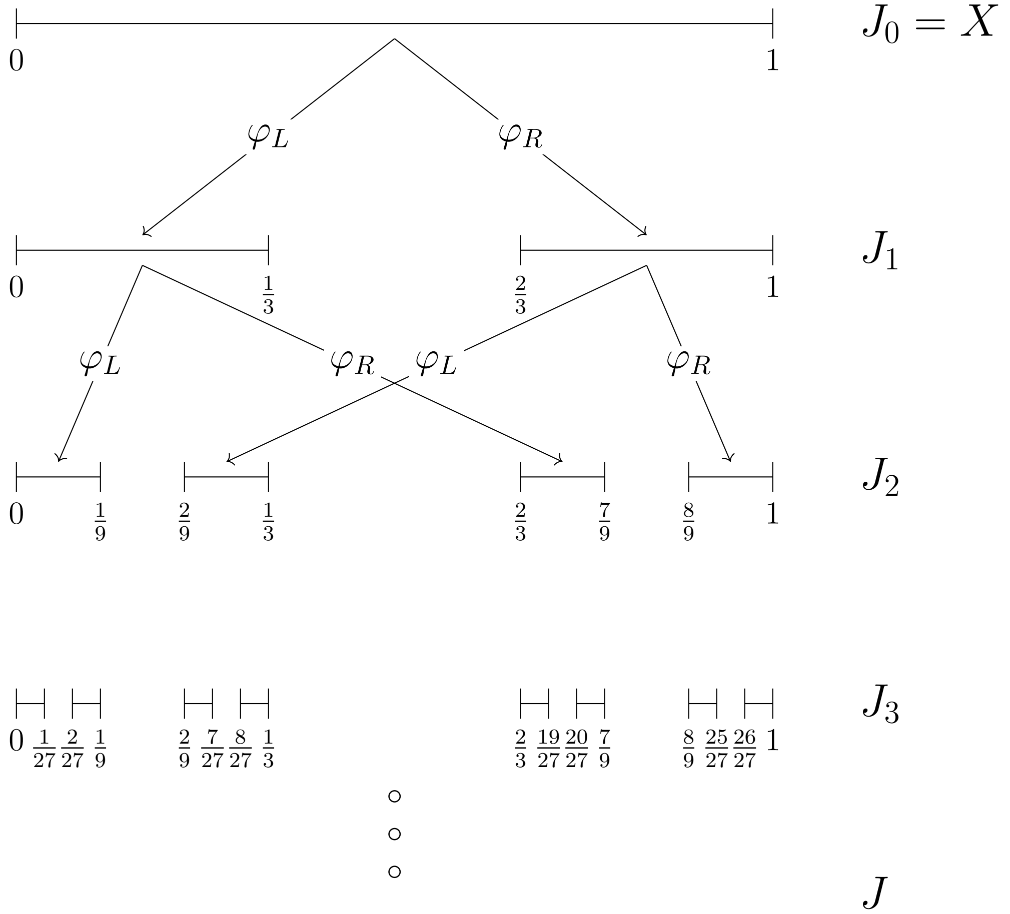
Middle-Third Cantor Set

$$\varphi_L(x) = \frac{x}{3} \quad , \quad \varphi_R(x) = \frac{x}{3} + \frac{2}{3} \quad .$$



$$\dim_H(J) = \frac{\log 2}{\log 3}$$

$E = \{L, R\}$

$\Phi = \{\varphi_L, \varphi_R\}$

$$J = \bigcup_{e \in E} \varphi_e(J)$$

Real Continued Fraction Expansions

Any irrational number in $X = [0,1]$ can be represented as a continued fraction

$$\cfrac{1}{e_1 + \cfrac{1}{e_2 + \cfrac{1}{e_3 + \ddots}}}$$

where $e_i \in E = \mathbb{N}$ for all $i \in \mathbb{N}$.

It is remarkable that the representation by continued fractions can be described by the infinite conformal IFS

$$\Phi = \left\{ \varphi_e : [0,1] \to [0,1] \,\middle|\, \varphi_e(x) = \frac{1}{e+x} \text{ with } e \in E \right\}$$

Dimension Spectrum

$$\mathrm{DS}(\Phi) = \left\{ \dim_H(J_F) \,\middle|\, F \subseteq E \right\} \subseteq [0, \dim_H(J_E)]$$

Texan Conjecture: The Real Continued Fraction CIFS has full Dimension Spectrum.

$$\mathrm{DS}(\Phi) = [0, 1]$$

i.e. for every $0 \leq t \leq 1$ there exists $F_t \subseteq E$ such that the set of irrational numbers whose continued fraction expansion only contains natural numbers from F_t has for Hausdorff dimension t.

Answered positively by Kesseboehmer and Zhu in 2006.

Complex Continued Fraction Expansions

Complex continued fractions can be represented via the infinite conformal IFS

$$\Phi = \left\{ \varphi_e : \overline{B(1/2, 1/2)} \to \overline{B(1/2, 1/2)} \,\middle|\, \varphi_e(z) = \frac{1}{e+z} \text{ for all } e \in E \right\},$$

where

$$E = \left\{ m + ni : m \in I\!N, n \in \boldsymbol{Z} \right\}$$

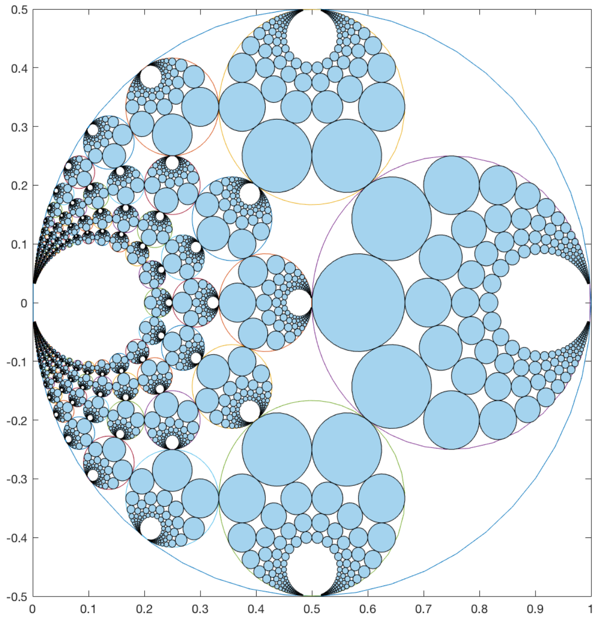


FIGURE 1. An approximation of the limit set of the complex continued fractions IFS after two iterations.

subtler estimates, and, as another new feature, it is also heavily computer assisted. For example we use numerics in order to obtain rigorous estimates for the Hausdorff dimension of certain subsystems of $\mathcal{CF}_{\mathbb{C}}$ which play important role in the proof of Theorem 1.4. This is a rather interesting novelty because it shows that estimates of Hausdorff dimension of limit sets using numerical analysis, as in [9, 10, 19, 20, 30], can be employed in order to obtain theoretical results such as Theorem 1.4.

The paper is organized as follows. In Section 2 we lay down the necessary background from symbolic dynamics and we prove various estimates for the topological pressure of subsystems. In Section 3 we introduce all the relevant concepts related to graph directed Markov systems and we introduce and study new natural parameters which can be realized as variants of the parameter θ. In Section 4 we introduce new dimension spectra for GDMS and study their size and topological properties. In Section 5 we provide an effective tool for calculating the Hausdorff dimension of the limit set of any finitely irreducible and strongly regular conformal GDMS with arbitrarily high accuracy. We thus generalize the main result of [13] to the setting of GDMSs and we simultaneously provide a substantially simpler proof. In Section 6 we narrow our focus to the dimension spectrum of general conformal iterated function systems. The machinery developed in Section 6 is used, among other tools, in Section 7 to prove that the dimension spectrum of complex continued fractions is full.

Texan Conjecture: The Complex Continued Fraction CIFS has full Dimension Spectrum.

$$\mathrm{DS}(\Phi) = [0, \dim_H(J_E)]$$

i.e. for every $0 \leq t \leq \dim_H(J_E) \approx 1.855$ there exists $F_t \subseteq E$ such that the set of irrational numbers whose continued fraction expansion only contains natural numbers from F_t has for Hausdorff dimension t.

Answered positively by Chousionis, Leykekhman and Urbanski in 2018.

Graph Directed Markov Systems (GDMSs)

are based on...

- a directed multigraph (V, E, i, t), where
  - V is a **finite** set of vertices
  - E is a **countable** (finite or infinite) set of edges
  - $i : E \to V$ associates to each edge $e \in E$ its initial vertex $i(e)$
  - $t : E \to V$ associates to each edge $e \in E$ its terminal vertex $t(e)$
- an edge incidence matrix A such that

$$A_{e_1 e_2} = 1 \quad \Longrightarrow \quad t(e_1) = i(e_2)$$

and consist of...

- a non-empty compact subset $X_v \subset \mathbb{R}^d$ attached to each vertex $v \in V$
- a one-to-one contraction
$$\varphi_e : X_{t(e)} \to X_{i(e)}$$
associated to each edge $e \in E$, with contraction ratio at most $0 < s < 1$

Limit Set

- Set of one-sided infinite A-admissible words

$$E_A^\infty := \left\{\omega \in E^\infty : A_{\omega_n \omega_{n+1}} = 1, \forall\, n \in I\!N\right\}$$

- Set of subwords of E_A^∞ of length $n \in I\!N$

$$E_A^n$$

- Set of finite subwords is denoted by

$$E_A^* = \bigcup_{n \in I\!N} E_A^n.$$

- Real Space Dynamics:

For $\omega \in E_A^n$, $n \in I\!N$,

$$\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}$$

For $\omega \in E_A^\infty$,

$$\pi(\omega) := \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton and defines the coding map

$$\pi : E_A^\infty \to \bigoplus_{v \in V} X_v$$

- The limit set of the GDMS is

$$J := \pi(E_A^\infty)$$

Conformal GDMSs (CGDMSs)

A GDMS $\Phi = \{\varphi_e\}_{e \in E}$ is called **conformal** if

(i) For every $v \in V$, the set X_v is a compact connected subset of $I\!R^d$ such that $X_v = \overline{\mathrm{Int}_{I\!R^d}(X_v)}$

(ii) (Open set condition (OSC)) For all $e, f \in E$, $e \neq f$,

$$\varphi_e(\mathrm{Int}(X_{t(e)})) \cap \varphi_f(\mathrm{Int}(X_{t(f)})) = \emptyset$$

(iii) For every vertex $v \in V$, there exists an open connected set W_v such that $X_v \subset W_v \subseteq I\!R^d$ and such that for every $e \in E$ with $t(e) = v$, the map φ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$

(iv) (Cone property)

(v) (Refined Bounded Distortion Property) There are constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\varphi_e'(y)| - |\varphi_e'(x)|\big| \leq L \|(\varphi_e')^{-1}\|_{W_{i(e)}}^{-1} \|y - x\|^\alpha$$

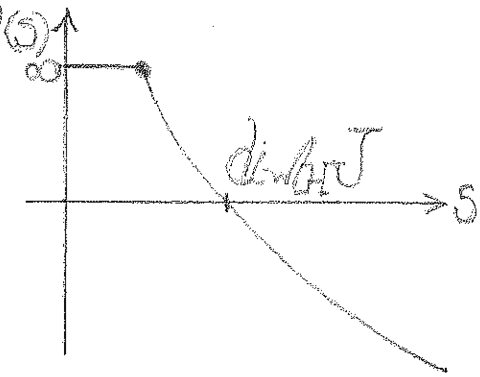
for every $e \in E$ and every pair of points $x, y \in W_{t(e)}$

Pressure Function

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in E_A^n} \|\varphi_\omega'\|^t$$

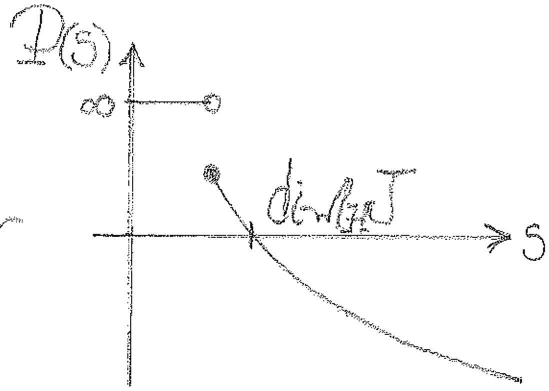
Infinite IFS

Cofinitely regular

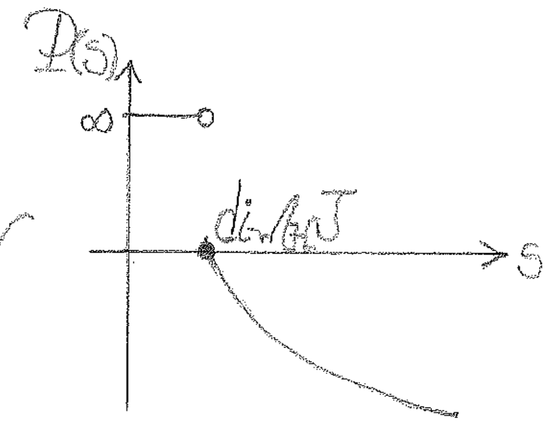


Regular

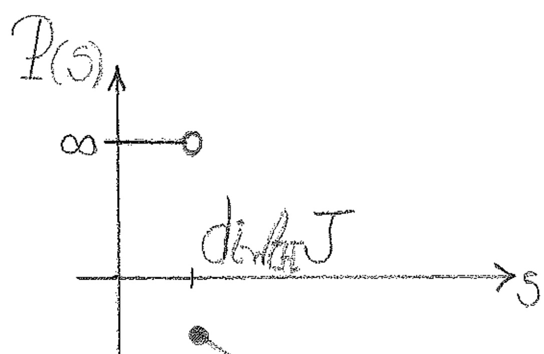
Strongly regular



Critically regular



Irregular



Irreducibility vs. Finite Irreducibility

- A system Φ (or a matrix A) is irreducible if there exists a set $\Omega \subseteq E_A^*$ such that for all $e, f \in E$ there is a word $\omega \in \Omega$ for which $e\omega f \in E_A^*$.

- A system is **finitely** irreducible if there is a **finite** set $\Omega \subset E_A^*$ which makes the system irreducible.

Bowen's Formula for Finitely Irreducible Systems

Theorem. (Mauldin and Urbański)

If Φ is a finitely irreducible CGDMS, then

$$\mathrm{HD}(J) = \sup\big\{\mathrm{HD}(J_F) : F \subseteq E \text{ is finite}\big\} = \inf\big\{t \geq 0 : P(t) \leq 0\big\}.$$

So if $P(t) = 0$, then t is the only zero of the pressure function and $t = \mathrm{HD}(J)$.

Non-Necessarily Irreducible CGDMSs

Strongly Connected Components

- An edge e_1 **leads to** an edge e_2 provided there is an A-admissible word of edges starting with e_1 and ending with e_2.
- A subset $C \subseteq E$ of edges is called a **strongly connected component** of E if for any two edges $e_1, e_2 \in C$ there exists $\omega \in C_A^*$ so that $e_1 \omega e_2 \in C_A^*$ and C is a maximal set (in the sense of inclusion) with this property.
- A strongly connected component C is said to **lead to** an edge e if there is some edge in C which leads to e. A strongly connected component C is said to **follow** an edge e if there is some edge in C which follows e.
- A strongly connected component C_1 is said to **lead to** a strongly connected component C_2 if some edge in C_1 leads to some edge in C_2.

Isolated Edges

- An edge is called **isolated** if it does not belong to any strongly connected component. The set of isolated edges will be denoted by I.

Variation of Bowen's Formula

Theorem. (Roy)

For every CGDMS

- whose strongly connected components are finitely irreducible and form chains that each have a maximal element, and
- which does not admit infinite words consisting only of isolated edges,

we have a variation of Bowen's formula:

$$\begin{aligned}
\mathrm{HD}(J) &= \sup_{\substack{\text{strongly connected}\\ \text{component } C}} \mathrm{HD}(J_C) = \sup_C \sup_{\substack{F \subset C,\\ F \text{ finite}}} \mathrm{HD}(J_F) \\
&= \sup_{\substack{F \subset E,\\ F \text{ finite}}} \mathrm{HD}(J_F) = \inf\Big\{t \geq 0 : \sup_C P_C(t) \leq 0\Big\} \\
&\leq \inf\Big\{t \geq 0 : P(t) \leq 0\Big\}.
\end{aligned}$$

Theorem. (Roy)

At every vertex $v \in V$ of a CGDMS

- whose strongly connected components are finitely irreducible and form chains that each have a maximal element, and
- which does not admit infinite words consisting only of isolated edges,

we have

$$\mathrm{HD}(J_v) = \sup_{C \in \mathcal{C}_v} \mathrm{HD}(J_C),$$

where

$$J_v := \pi(E_v^\infty) := \pi\big(\{\omega \in E_A^\infty : i(\omega_1) = v\}\big)$$

and

$$\mathcal{C}_v = \big\{C \text{ strongly conn. comp.} : \exists\, \omega \in E_v^\infty \text{ such that } \mathrm{Orb}_\sigma(\omega) \cap C^\infty \neq \emptyset\big\}.$$

Proposition. (Roy)

Let Φ be an infinite CGDMS with

- finitely many strongly connected components, all finitely irreducible, and
- which does not admit infinite words consisting only of isolated edges.

Let $h = \mathrm{HD}(J)$. If the strongly connected components of maximal h-pressure do not communicate, then $\mathcal{H}^h(J) < \infty$.

Note that $\mathcal{H}^h(J)$ may be equal to 0 when the alphabet is infinite, even if the entire system is finitely irreducible.

Further Investigation of the Pressure Function

Theorem. (Roy)

Let Φ be a CGDMS such that

- Φ has finitely many strongly connected components, each of which is finitely irreducible
- For every $t \geq 0$, there is a strongly connected component $\overline{C} = \overline{C}(t)$ of maximal t-pressure (i.e. $P_{\overline{C}}(t) = \max_C P_C(t)$) whose partition functions are boundedly supermultiplicative for all $s > \theta_{1,\overline{C}}$
- Words consisting solely of isolated edges are uniformly bounded in length.

Then

$$P(t) = \max_C P_C(t), \ \ \forall\, t > \max\{\theta_{1,I\setminus D}, \max_C \theta_{1,C}\}.$$

Theorem. Let Φ be a CGDMS such that

- Φ has finitely many strongly connected components
- All its strongly connected components are finitely irreducible
- Words consisting solely of isolated edges are uniformly bounded in length.

Then

$$P(t) = \max_C P_C(t), \ \ \forall\, t < \max_C \theta_C \text{ and } \forall\, t > \max\{\theta_{1,I\setminus D}, \max_C \theta_C\}.$$

Under the assumptions of the previous corollary, we have the following three possibilities:

(1) If $\max_C \theta_C > \theta_{1,I\setminus D}$, then

- $P(t) = \max_C P_C(t)$ for all $t \geq 0$
- $\theta = \max_C \theta_C$ and $P(t)$ is right-continuous at θ
- The classical form of Bowen's formula holds

(2) If $\max_C \theta_C = \theta_{1,I\setminus D}$, then

- $P(t) = \max_C P_C(t)$ for all $t \neq \max_C \theta_C$
- $\theta = \max_C \theta_C$
- The classical form of Bowen's formula holds

Moreover,

- If $\max_C P_C(\theta) = \infty$ or $Z_{1,I\setminus D}(\theta) < \infty$, then $P(t) = \max_C P_C(t)$ for all $t \geq 0$ and $P(t)$ is right-continuous at θ
- If $\max_C P_C(\theta) < \infty$ and $Z_{1,I\setminus D}(\theta) = \infty$, then $P(\theta)$ may differ from $\max_C P_C(\theta)$ and may thus not be right-continuous at θ

(3) If $\max_C \theta_C < \theta_{1,I\setminus D}$, then

- $P(t) = \max_C P_C(t)$ for all $t < \max_C \theta_C$ and all $t > \theta_{1,I\setminus D}$
- $\max_C \theta_C \le \theta \le \theta_{1,I\setminus D}$
- The original form of Bowen's formula may not hold:

  $P(t)$ may be strictly greater than $\max_C P_C(t)$ over a subinterval of $(\max_C \theta_C, \theta_{1,I\setminus D})$.

Examples

Example. A CGDS which falls under the purview of (3):

- $V = \{v_1, v_2\}$
- At each vertex v_i lies a self-loop denoted by i
- Add edges $\{i\}_{i \geq 3}$ that start from vertex v_1 and end at vertex v_2
- To each edge, associate a generator in such a way that $\Phi = \{\varphi_i\}_{i \in I\!N}$ is a CGDS such that

$$\sum_{i \geq 3} \|\varphi_i'\|^{1/2} = \infty \quad \text{and} \quad \sum_{i \geq 3} \|\varphi_i'\|^{t} < \infty \ \ \forall\, t > 1/2$$

This CGDS has the following properties.

- The system has two strongly connected components $C_j = \{j\}$, $j = 1, 2$
- $I = \{i\}_{i \geq 3}$, $D = \emptyset$
- $\theta_{1,I\setminus D} = 1/2$
- $P_{C_j}(0) = 0$
- J_{C_j} consists in the fixed point of φ_j
- $\mathrm{HD}(J_{C_j}) = 0$
- By the variation of Bowen's formula, we have

$$\begin{aligned} 0 &= \mathrm{HD}(J) = \max_C \mathrm{HD}(J_C) \\ &= \inf\big\{t \geq 0 : \max_C P_C(t) \leq 0\big\} \leq \inf\big\{t \geq 0 : P(t) \leq 0\big\}. \end{aligned}$$

- This last inequality is strict since $P(t) = \infty$ for all $t \leq 1/2$ b/c

$$Z_n(t) = \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^t \geq \sum_{i=3}^{\infty} \|\varphi'_{1^{n-1}i}\|^t \geq K^{-t} \|\varphi'_{1^{n-1}}\|^t \sum_{i=3}^{\infty} \|\varphi'_i\|^t = \infty$$

So the classical form of Bowen's formula does not hold.

- $\theta = 1/2$ by the previous corollary
- $P(t) = \max_C P_C(t) < 0$ for all $t > 1/2$ by the previous corollary
- $P(t)$ is not right-continuous at θ
- $P(t) > \sup\{P_F(t) : F \text{ is finite}\}$ for all $t \leq 1/2$

Example. A CGDMS that has an irreducible, though not finitely irreducible, strongly connected component which generates, in cooperation with an isolated edge, so much pressure that the classical form of Bowen's formula does not hold:

- $V = \{v_1, v_2\}$
- At vertex v_2 lies a self-loop denoted by e. Associated to it is a similarity φ_e which contracts $X = X_{v_2} = [0, 1]$ into itself
- At vertex v_1 lies a subsystem of the standard continued fractions CIFS: Its self-loops are labeled by $i \geq 100$ and associated to self-loop i is the conformal map $\varphi_i(x) = \frac{1}{(i+x)}$ mapping $X = X_{v_1} = [0, 1]$ into itself
- Add an edge f from vertex v_1 to vertex v_2, and Associate to it a similarity φ_f which contracts $[0, 1]$ into $[1/100, 1]$
- Φ is the system generated by

  $$E = \{i \geq 100\} \cup \{f, e\}$$

  and the matrix A defined by

  - $A_{ij} = 1$ if and only if $|i - j| \leq 1$
  - $A_{if} = 1$ for all i
  - $A_{fe} = A_{ee} = 1$, and 0 otherwise.

This CGDMS has the following properties.

- Two strongly connected components: $C_1 = \{i \geq 100\}$ and $C_2 = \{e\}$
- $I = I \backslash D = \{f\}$
- $\theta_{n,C_1} = 1/(2n)$ while $\theta_{n,C_2} = 0$
- $\theta_{C_1} = 0 = \theta_{C_2}$
- $P(t) = \infty$ for all $t < 1/2$ since

$$Z_n(t) \geq \sum_{i=100}^{\infty} \|\varphi'_{ife^{n-2}}\|^t \geq K^{-t}\|\varphi'_{fe^{n-2}}\|^t \sum_{i=100}^{\infty} \|\varphi'_i\|^t = \infty$$

  for all $n \in I\!N$ and all $t < \theta_{1,C_1} = 1/2$
- $P_{C_1}(3/8) < 0$ as $Z_{2,C_1}(3/8) < 1$
- $\text{HD}(J_{C_1}) \leq 3/8$
- $\text{HD}(J) = \max\{\text{HD}(J_{v_1}), \text{HD}(J_{v_2})\} = \max\{\text{HD}(J_{C_1}), \text{HD}(J_{C_2})\} \leq 3/8$ while $\inf\{t \geq 0 : P(t) \leq 0\} \geq 1/2$. So the classical form of Bowen's formula does not hold.