Dirichlet spaces with superharmonic weights

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Classical spaces

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
A=Area measure

Definition (Hardy space H^2)

$$f \in H^2 \iff ||f||^2_{H^2} = |f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) < +\infty.$$

Definition (Dirichlet space $\mathcal{D}$)

$$f \in D \iff \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

Definition (BMOA)

$$f \in BMOA \iff \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \log\frac{|1-\overline{w}z|}{|z-w|} dA(z) < +\infty.$$

Dirichlet spaces with superharmonic weights (A. Aleman, 1993)

- $\omega : \mathbb{D} \mapsto (0, +\infty]$, positive superharmonic function

$$\begin{aligned} \omega(z) &= \int_{\mathbb{D}} \log \left| \frac{1 - \overline{w} z}{z - w} \right| d\mu(w) + \int_{\partial \mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} d\nu(\zeta) \\ &= U_\mu(z) + P_\nu(z), \end{aligned}$$

$$\int_{\mathbb{D}} (1 - |z|) d\mu(z) < +\infty, \qquad \text{and} \qquad \nu(\partial \mathbb{D}) < +\infty.$$

Definition (Weighted Dirichlet space $\mathcal{D}_\omega$)

$$f \in \mathcal{D}_\omega \iff \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < +\infty.$$

- Dirichlet spaces $\mathcal{D}_\nu$ with harmonic weights, $\omega = P_\nu$, (S. Richter, 1991)

We will concentrate on

- Dirichlet spaces $\mathcal{D}_\mu$,
  $\omega(z) = U_\mu(z) = \int_{\mathbb{D}} \log \left| \frac{1-\overline{w}z}{z-w} \right| d\mu(w)$.
- $\lim_{r\to 1} U_\mu(r\zeta) = 0$ for almost every $\zeta \in \partial\mathbb{D}$.
- $$||f||^2_{\mathcal{D}_\mu} = ||f||^2_{H^2} + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z).$$

Examples ($\mathcal{D}_p$ spaces with radial superharmonic weights)

- $\omega_p(z) = (1-|z|^2)^p$, $p \in (0,1)$,
- $d\mu_p = -\Delta((1-|z|^2)^p) dA(z)$,
- $\mu_p(\mathbb{D}) = +\infty$.

Definition (Carleson measures)

For every arc $I \subset \partial\mathbb{D}$ with length $|I|$,

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < r < 1, \zeta \in I\}.$$

μ is Carleson measure if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty.$$

Theorem

- $\mathcal{D}_\mu \subset H^2$, $\forall \mu$,
- *if* $\mu(\mathbb{D}) < +\infty$, $BMOA \subset \mathcal{D}_\mu \subset H^2$,
- *if* $(1 - |z|^2)d\mu(z)$ *is a Carleson measure,* $\mathcal{D} \subsetneq \mathcal{D}_\mu$.

Definition (Balayage)

If $\mu(\mathbb{D}) < +\infty$*, the balayage of* μ *is the function*

$$S_\mu(\zeta) = \frac{1}{2\pi} \int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(z), \qquad \zeta \in \partial\mathbb{D}.$$

Note that every $f \in H^2$ has radial limit $f(\zeta)$ at almost every $\zeta \in \partial\mathbb{D}$.

Definition (Weighted Hardy spaces H^2_μ)

Suppose $\mu(\mathbb{D}) < +\infty$*.*

$$H^2_\mu = \{f \in H^2 : \int_{\partial\mathbb{D}} |f(\zeta)|^2 S_\mu(\zeta)|d\zeta| < +\infty\}.$$

Theorem (with G. Bao and N. G. Göğüş)

If μ *is a Carleson measure, then* $\mathcal{D}_\mu = H^2_\mu$*.*

Corrolary (with G. Bao and N. G. Göğüş)

Let μ be a Carleson measure and let ν be a measure on $\mathbb{D}$. There exists $C > 0$ such that

$$\left(\int_{\mathbb{D}} |f(z)|^2 d\nu(z) \right)^{1/2} \leq C||f||_{\mathcal{D}_\mu}, \qquad f \in \mathcal{D}_\mu,$$

if and only if there exists $C' > 0$ such that

$$\int_{S(I)} |O_\mu|^2 d\nu \leq C'|I|,$$

for every arc $I \subset \partial\mathbb{D}$, where

$$O_\mu(z) = \exp\left(\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \log \frac{1}{\sqrt{S_\mu(\zeta)}} \frac{|d\zeta|}{2\pi} \right), \qquad z \in \mathbb{D},$$

is an outer function with $|O_\mu(\zeta)| = 1/\sqrt{S_\mu(\zeta)}$, at almost every $\zeta \in \partial\mathbb{D}$.

Corrolary (with G. Bao and N. G. Göğüş)

Suppose that $\mu = \sum_{n=1}^{+\infty} a_n \delta_{z_n}$ *is a Carleson measure, where* $z_n \in \mathbb{D}$ *and* $a_n > 0$, $n \in \mathbb{N}$. *The reproducing kernel of* $\mathcal{D}_\mu$ *for* $\lambda \in \mathbb{D}$ *with respect to* $||\cdot||_{\mathcal{D}_\mu}$ *is*

$$K(z,\lambda) = K_0(z,\lambda) + \sum_{n=1}^{+\infty} \frac{a_n K_0(z,z_n) K_0(z_n,\lambda)}{1 - a_n K_0(z_n,z_n)}, \qquad z \in \mathbb{D},$$

where

$$K_0(z,\lambda) = \frac{\overline{T_\mu(\lambda)}}{1 - \overline{\lambda} z} T_\mu(z), \qquad z \in \mathbb{D},$$

and

$$T_\mu(z) = \exp\left(\frac{1}{2\pi}\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \log \frac{1}{\sqrt{1 + S_\mu(\zeta)}} |d\zeta|\right), \qquad z \in \mathbb{D}.$$

Definition

$\phi \in H^2$ is called inner if $|\phi(\zeta)| = 1$ for almost every $\zeta \in \partial\mathbb{D}$.

Theorem (Alexander-Taylor-Ullman inequality)

If $f \in H^2$ with $f(0) = 0$, then

$$||f||^2_{H^2} \leq \frac{A(f(\mathbb{D}))}{\pi}.$$

Equality holds if and only if $f = c\phi$ where $c \in \mathbb{C}$ and ϕ is an inner function satisfying $\phi(0) = 0$.

Theorem (with G. Bao and N. G. Göğüş)

Suppose $\mu(\mathbb{D}) < +\infty$. *If* $f \in \mathcal{D}_\mu$ *with* $f(0) = 0$,

$$||f||^2_{D_\mu} \leq (1 + \mu(\mathbb{D}))\frac{A(f(\mathbb{D}))}{\pi}.$$

Equality holds if and only if the measure μ *is of the form*

$$\mu = a_0\delta_0 + \sum_{n=1}^{+\infty} a_n\delta_{z_n}, \qquad a_n > 0,\ z_n \in \mathbb{D},$$

and f *is of the form* $f = c\phi$, *where* $c \in \mathbb{C}$ *and* ϕ *is an inner function with* $\phi(0) = \phi(z_n) = 0$, *for every* $n \in \mathbb{N}$.

Proof. Fix $w \in \mathbb{D}$.

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^2 \log \frac{|1-\overline{w}z|}{|z-w|} dA(z) &= \int_{f(\mathbb{D})} \sum_{f(a)=x} \log \frac{|1-\overline{w}a|}{|a-w|} dA(x) \\ &\leq \int_{f(\mathbb{D})} G_{f(\mathbb{D})}(x, f(w)) dA(x) \\ &\leq \frac{1}{2} A(f(\mathbb{D}))\end{aligned}$$

and

$$\begin{aligned}&\frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z) \\ = \; &\frac{2}{\pi} \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |f'(z)|^2 \log \frac{|1-\overline{w}z|}{|z-w|} dA(z) \Big) d\mu(w) \\ \leq \; &\frac{2}{\pi} \int_{\mathbb{D}} \frac{1}{2} A(f(\mathbb{D})) d\mu(w) = \frac{\mu(\mathbb{D}) A(f(\mathbb{D}))}{\pi}.\end{aligned}$$

Suppose that equality holds. Then $f = c\phi$ where $c \in \mathbb{C}$ and ϕ is an inner function satisfying $\phi(0) = 0$.

- $$\begin{aligned} |\phi(z)|^2 &= h_\phi(z) - \frac{1}{2\pi}\int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right| \Delta|\phi(w)|^2 dA(w) \\ &= 1 - \frac{2}{\pi}\int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right| |\phi'(w)|^2 dA(w). \end{aligned}$$
- $A(c\phi(\mathbb{D})) = A(c\mathbb{D}) = |c|^2\pi$,
- $$\begin{aligned} \mu(\mathbb{D})|c|^2 &= \frac{2}{\pi}\int_{\mathbb{D}} |c\phi'(z)|^2 U_\mu(z) dA(z) \\ &= |c|^2 \frac{2}{\pi}\int_{\mathbb{D}}\int_{\mathbb{D}} |\phi'(z)|^2 \log\left|\frac{1-\overline{w}z}{z-w}\right| dA(z) d\mu(w) \\ &= |c|^2 \int_{\mathbb{D}} (1-|\phi(w)|^2) d\mu(w) \\ &= \mu(\mathbb{D})|c|^2 - |c|^2 \int_{\mathbb{D}} |\phi(w)|^2 d\mu(w). \end{aligned}$$

$$\int_{\mathbb{D}} |\phi(w)|^2 d\mu(w) = 0,$$

which holds if and only if $\phi = 0$ μ-almost everywhere. Since the zeros of ϕ are isolated, the above equality holds if and only if μ is of the form

$$\mu = a_0\delta_0 + \sum_{n=1}^{+\infty} a_n \delta_{z_n}, \qquad a_n > 0,\ z_n \in \mathbb{D},$$

and the inner function ϕ satisfies $\phi(0) = \phi(z_n) = 0$, for every $n \in \mathbb{N}$.

Definition (The Möbius invariant function space $M(\mathcal{D}_\mu)$)

The Möbius invariant function space $M(\mathcal{D}_\mu)$ generated by $\mathcal{D}_\mu$ is the class of holomorphic functions f on $\mathbb{D}$, with

$$||f||_{M(\mathcal{D}_\mu)} = \sup_{\phi \in \mathrm{Aut}(\mathbb{D})} ||f \circ \phi - f(\phi(0))||_{\mathcal{D}_\mu} < \infty.$$

Examples

- $M(H^2) = BMOA$,
- $M(\mathcal{D}) = \mathcal{D}$,
- $M(\mathcal{D}_p) = \mathcal{Q}_p$, $p \in (0, 1)$.

Theorem (with G. Bao, J. Mashreghi and H. Wulan)

- *If* $\mu(\mathbb{D}) < +\infty$, $M(\mathcal{D}_\mu) = BMOA$.
- *If* $\mu(\mathbb{D}) = +\infty$, *the following are equivalent:*
  (1) $M(\mathcal{D}_\mu)$ *is not trivial,*
  (2) $\mathcal{D} \subset M(\mathcal{D}_\mu)$,
  (3) $(1 - |z|^2) d\mu(z)$ *is a Carleson measure.*

Which inner functions are contained in $M(\mathcal{D}_\mu)$ $(\mu(\mathbb{D}) = +\infty)$?

Definition (Carleson-Newman Blaschke products)

A Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z}$$

is called Carleson-Newman Blaschke product if $\sum_{k=1}^{\infty}(1 - |a_k|^2)\delta_{a_k}$ *is a Carleson measure.*

Theorem (with G. Bao, J. Mashreghi and H. Wulan)

Suppose that $\mu(\mathbb{D}) = +\infty$ and let I be an inner function.

1. *If $I \in M(\mathcal{D}_\mu)$, I is a Blaschke product.*
2. *Suppose that I is a Carleson-Newman Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$. Then $I \in M(\mathcal{D}_\mu)$ if and only if*

$$\sup_{\phi \in \mathrm{Aut}(\mathbb{D})} \sum_{k=1}^{\infty} \int_{\mathbb{D}} \left(1 - \left|\frac{a_k - \phi(w)}{1 - \overline{a_k}\phi(w)}\right|^2\right) d\mu(w) < \infty.$$

Proof. Let $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $a \in \mathbb{D}$.

$\nu = t\delta_1$, $t > 0$,

$$S_\nu(z) = \exp\left(-t\frac{1+z}{1-z}\right)$$

$$|S_\nu(z)| = \exp\left(-t\frac{1-|z|^2}{|1-z|^2}\right)$$

$$S_\nu \notin M(\mathcal{D}_\mu)$$

Fix $c > 0$. Consider the horodisk

$$D_c = \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{|1 - z|^2} > c \right\},$$

note that

$$|S_\nu| \leq e^{-tc}, \qquad \text{on } D_c,$$

and let

$$\mu_a = \mu \circ \sigma_a, \qquad a \in \mathbb{D}.$$

$$
\begin{aligned}
\int_{\mathbb{D}} |(S_\nu \circ \sigma_a)'(z)|^2 U_\mu(z) dA(z) &= \int_{\mathbb{D}} (1 - |S_\nu(\sigma_a(z))|^2) d\mu(z) \\
&\geq \int_{\sigma_a(D_c)} (1 - |S_\nu(\sigma_a(z))|^2) d\mu(z) \\
&= \int_{D_c} (1 - |S_\nu(z)|^2) d\mu_a(z) \\
&\geq (1 - e^{-2tc}) \mu(\sigma_a(D_c)).
\end{aligned}
$$

Let $\phi_r(z) = -\sigma_r(z)$ and note that $\phi_r(D_c) \nearrow \mathbb{D}$ as $r \to 1$. Then

$$
\lim_{r \to 1} \|S_\nu \circ \phi_r\|^2_{\mathcal{D}_\mu} \geq \lim_{r \to 1} (1 - e^{-2tc}) \mu(\phi_r(D_c)) = (1 - e^{-2tc}) \mu(\mathbb{D}) = +\infty.
$$

$$
S_\nu \notin M(\mathcal{D}_\mu).
$$

Thank you!