Szego theorems for truncated Toeplitz operators

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Interesting comments concerning Tom Ransford

Content of my talk

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then about objects that have aroused a lot of interest recently :

TRUNCATED TOEPLITZ OPERATORS a.k.a. TTO's and CLARK OPERATORS

and a new 'Szego Theorem' that Dan Timotin, Mohamed Zarrabi and I established recently (J. Approx Theory, Aug 2017) by using Clark operators and Sedlock algebras (which generalize circulants) to establish a relationship between the symbol of a Toeplitz operator and the spectrum of a sequence of 'approximating' TTO's.

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$$\{f = \sum_{n\geq 0} a_n z^n,\ \sum_{n\geq 0} |a_n|^2 < +\infty\}.$$

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using the (well defined) map taking such a function to its boundary values (which exist almost everywhere).
(3) Model spaces ; or orthogonal complements in H^2 of shift invariant subspaces (including (1)).

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(2) As the composition of a multiplication operator M_φ on H^2 with the orthogonal projection from L^2 to H^2 - which can be shown to be bounded if and only if φ is a bounded function. In this case the matrix can be written :

$$\begin{pmatrix} \hat{\varphi}(0) & \hat{\varphi}(-1) & \hat{\varphi}(-2) & \cdots \\ \hat{\varphi}(1) & \hat{\varphi}(0) & \hat{\varphi}(-1) & \ddots \\ \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

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$$C_{(a_0,a_1,\cdots,a_n)} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \quad (1)$$

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The most well-known circulant which interests us is the 'perturbed shift' , of the form above, with $a_n = w$ with $|w| = 1$ and all other entries equal to zero.

The Perturbed Shift

Here is the perturbed shift S_1 :

$$\begin{pmatrix} 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & ... & 1 & 0 \end{pmatrix} \tag{2}$$

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And it is easy to see that :
(1) The eigenvalues of S_1 are the (n)th roots of unity $\zeta_k = e^{i\frac{2k\pi}{n}}$ with eigenvectors $(1, \zeta_k, \zeta_k^2, \zeta_k^{n-1})^t$ and that ;

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(2) The general circulant matrix is just $p(S_1)$ where p is the polynomial $p(z) = a_0 + a_{n-1}z + a_{n-2}z^{n-1} + ...a_1 z^{n-1}$ (the Toeplitz symbol of the matrix) and so :

$$\text{trace}(C_{(a_0,a_1,\cdots,a_{n-1})}) = \sum_{k=0}^{n-2} p(e^{i\frac{2k\pi}{n-1}})$$

and so $\frac{1}{n-1}\text{trace}(C_{(a_0,a_1,\cdots,a_{n-1})})$ is a Riemann sum for the function p(z) around the unit circle.

Notation for the Grenander-Szego theorem

These observations - along with a nice way of approaching Toeplitz matrices by circulant matrices can be used to prove the Grenander-Szego theorem - we give a little necessary notation to be able to write the theorem.

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Let T_φ be a Toeplitz operator with symbol function $\varphi \in L^\infty(\mathbb{T})$. For each $n \in \mathbb{N}$ we write $T_n(\varphi)$ for the $n \times n$ Toeplitz matrix $(\widehat{\varphi}(i-j))_{0 \le i,j \le n-1}$

$$\begin{pmatrix} \hat{\varphi}(0) & \hat{\varphi}(-1) & \cdots & \hat{\varphi}(-n-1) \\ \hat{\varphi}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \hat{\varphi}(n-1) & \hat{\varphi}(n-2) & \cdots & \hat{\varphi}(0) \end{pmatrix}$$

and m for normalized Lebesgue measure on the circle.

Theorem [U. Grenander-G. Szego] :

We have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\lambda_k(T_n(\varphi))^p=\int_0^{2\pi}\varphi(e^{it})^p dm(t),$$

where $\lambda_k(T_n(\varphi)),\ k=1\ldots n$, are the eigenvalues of $T_n(\varphi)$.
Moreover when φ is a real valued function then for every continuous function f on $[\inf_{\mathbb{T}}\varphi,\sup_{\mathbb{T}}\varphi]$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(\lambda_k(T_n(\varphi)))=\int_0^{2\pi}(f\circ\varphi)(e^{it})dm(t).$$

Our Szego theorem

My work with Dan Timotin and Mohamed Zarrabi was a generalization of the Szego type procedure ; we 'approach' the spectrum of our Toeplitz operators by *Truncated Toeplitz operators* using an 'approximation' of the TTOs by elements of what are called Sedlock algebras. This was the clear thing to do, once we realized that the Sedlock algebras were the TTO generalization of circulant matrices.

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We recall the definitions :

Definition :

Let u be an inner function and let $K_u = H^2 \ominus uH^2$ be the model space associated with u. A truncated Toeplitz operator is an operator $T_u[\varphi] : K_u \to K_u$ defined by

$$T_u[\varphi](g) = P_u(\varphi g),$$

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and, of course, Toeplitz matrices are TTO's acting on K_{z^n}.

The intuitive idea behind all this is that a perturbed shift on $\mathbb{C}^n$ or $\mathcal{P}_n$ is obtained in the obvious way by adding a rank 1 operator to the shift to make it unitary :

$$U_w = S + w(1 \otimes z^{n-1})$$

where $|w| = 1$ and :

$$(w \otimes z^{n-1})(f) =< wf, z^{n-1} > 1$$

unitary because the shift 'kills' z^{n-1} and its adjoint 'kills' constants -and the perturbation remedies this. All rank one unitary perturbation of the shift are of this form.

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The operators called Clark operators generalize this idea to model spaces. These operators are rank 1 perturbations of the compression of the shift to the model space K_u, the operator $S_u = T_u[z]$.

Clark operators

For an arbitrary inner function u, 1 is replaced by

$$k_0(z) = 1 - \overline{u(0)}u(z)$$

and z^{n-1} is replaced by

$$K_0(z) = \frac{u(z) - u(0)}{z}$$

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Each Clark operator S_u^w is associated with a *Clark measure* $d\mu_w^u$ on the unit circle Π such that S_u^w is unitarily equivalent to multiplication by z on $L^2(\mu_w^u)$

Simplification

In general the easier way to work with a model space K_u is to begin by assuming that $u(0) = 0$ so that $k_0 = 1$ and $K_0 = \frac{u}{z}$, then using the Crofoot transform to transfer the results to arbitrary inner functions.

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For the rest of this talk I will treat the case $u(0) = 0$; the generalization is straightforward.

Finite Blaschke products : eigenvalues and eigenvectors of S_B^α

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(1) The eigenvalues of S_B^α are simply the solutions of $B(z) = \alpha$ which we shall call $\{\xi_1^\alpha, \cdots \xi_n^\alpha\}$ and the eigenvector associated with ξ_k^α is the reproducing kernel $k_{\xi_k^\alpha}^B$ in ξ_k^B ; this 'kernel ' function $k_{\xi^\alpha}^B$ is defined by :

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(2)The Clark measure associated with S_B^α is the measure

$$\mu_\alpha^B = \sum_{k=1}^n \frac{1}{|B'(\xi_k^\alpha)|}\delta_{\xi_k^\alpha}$$

Generalization of circulants - finite Blaschke products

Now ; if a function f is applied to S_B^α the eigenvalues of of $f(S_B^\alpha)$ will be $(f(\xi_k^\alpha))_{k=1}^n$ and so the trace of $f(S_B^\alpha)$ will be $\sum_{k=1}^n f(\xi_k^\alpha))$. Thus

$$\text{trace}\frac{1}{|B'|}f(S_B^\alpha) \text{ should be } \int_0^{2\pi} f d\mu_\alpha^B.$$

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And, in 2011, Nicolas Sedlock, a student of Richard Rochberg wrote a thesis where he showed that the idea of circulants generalizes to truncated Toeplitz operators. He showed that, for any inner function u with $u(0) = 0$, the maximal subalgebras of K_u are of the form $\mathcal{B}_u^\alpha$, where $A \in \mathcal{B}_u^\alpha$ if and only if :

$$T_u[\phi + \alpha\overline{u}(\phi - \phi(0))] \text{ for some } \phi \in K_u.$$

In order to obtain our results, we viewed these operators in a different way :

Theorem

Let B be an arbitrary inner function stisfying $B(0) = 0$. Suppose that $T = T_B[\phi + \alpha\overline{u}(\phi - \phi(0))] \in \mathcal{B}_u^\alpha$. Then the function ϕ has radial limits almost everywhere with respect to μ_α and, if we denote the limit function by ϕ^* then $\phi^* \in L^\infty(\mu_\alpha)$ and $T = \phi^*(S_B^\alpha)$.

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Thus we see that :

Theorem :

If μ_α is an atomic Clark measure associated with S_B^α with support the sequence (ξ_n) and $T = T_B[\phi + \overline{B}(\phi - \phi(0))]$ then

$$Trace(T^p) = \sum_n \phi^*(\xi_n) \text{and so } Trace(\frac{1}{B'(z)}T^p) = \int_0^{2\pi} \phi^*(t) d\mu_\alpha^B.$$

All of this means that the same type of proof can be used to obtain a Szego theorem for TTOs. For a general sequence of inner function $U_n = u_1 ... u_n$ the factor 1/n becomes the complicated operator Δ_u^α defined below, and then the question is, which type of sequences $(u_1 u_2 ... u_n)$ can replace the sequence (z^n).

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Again writing $k_{\zeta_k}^u$ for the reproducing kernel of K_u at the point ζ_k we define the operator :

$$\Delta_u^\alpha := \sum_{k=1}^{n} \frac{1}{|u'(\zeta_k)|} \left(\frac{k_{\zeta_k}^u}{\|k_{\zeta_k}^u\|} \otimes \frac{k_{\zeta_k}^u}{\|k_{\zeta_k}^u\|} \right).$$

Generalized circulants

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The condition needed to have the Szego theorem for a sequence (B_n) replacing (z^n) is to have $\lim_{n\to\infty} \|\Delta_{B_n}^\alpha\| \to 0$, (easily verified by (z^n))

Theorem 1[Strouse, Timotin, Zarrabi] :

Suppose that $\lambda_j \in \mathbb{D}$, $j \geq 1$ and $\sum(1-|\lambda_j|) = \infty$. Define, for $n \geq 1$, $B_n = \prod_{j=1}^{n} b_{\lambda_j}$, where $b_{\lambda_j}(z) = \frac{|\lambda_j|}{\lambda_j} \frac{\lambda_j - z}{1-\bar{\lambda}_j z}$ is the Blaschke factor corresponding to λ. Then for $\psi \in C(\Pi)$ and $p \in \mathbb{N}$ we have :
(i)

$$\mathrm{Tr}(T_{B_n}[\frac{1}{|B_n'|}](T_{B_n}[\psi])^p) \to \int \psi^p \, dm.$$

If the function ψ is real-valued then, for every continuous function g on $[\inf \psi, \sup \psi]$ we have :
(ii)

$$\mathrm{Tr}(T_{B_n}[\frac{1}{|B_n'|}]g(T_{B_n}[\psi])^p) \to \int g \circ \psi^p \, dm.$$

This theorem works because, if (λ_j) is a Blaschke sequence, then $\lim_{n\to\infty} \|\Delta^{\alpha}_{B_n}\| \to 0$.

Theorem 2 [Strouse, Timotin, Zarrabi] :

Suppose that $\alpha \in \Pi$, (λ_j) is a Blaschke sequence, $B_n = \Pi_{k=1}^{n} b_{\lambda_k}$ and $B = \Pi_{k=1}^{\infty} b_{\lambda_k}$. Suppose also that $\lim_{n\to\infty} \|\Delta_{B_n}^{\alpha}\| \to 0$. Then, if $\psi \in (K_{B_N} + \overline{K_{B_N}}) \circ b_{-\lambda_k}$ for some k, N and $p \in \mathbb{N}$ we have :

$$Trace(\Delta_{B_n}^{\alpha}(T_{B_n}[\psi])^p) \to \int \psi^p d\mu_B^{\alpha}.$$

When is or isn't the hypothesis $\|\Delta_{B_n}^{\alpha}\| \to 0$ satisfied ?

To see that the hypothesis is satisfied when the sequence is not Blaschke, we notice that :

$$\|\Delta_{B_n}^{\alpha}\| = \sup_{1\leq j\leq n} \frac{1}{|B_n'(\zeta_j^{(n)})|},$$

so that a sufficient condition for $\|\Delta_{B_n}^{\alpha}\| \to 0$ would be $\inf\{|B_n'(\zeta)| : \zeta \in \mathbb{T}\} \to \infty$. And since, for all $\zeta \in \mathbb{T}$ we have

$$|B_n'(\zeta)| \geq \frac{1}{2}\sum_{j=1}^{n}(1-|\lambda_j|).$$

we see that if $\sum_j (1-|\lambda_j|) = \infty$ then $\|\Delta_{B_n}^{\alpha}\| \to 0$.

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so that a sufficient condition for $\|\Delta^{\alpha}_{B_n}\| \to 0$ would be $\inf\{|B_n'(\zeta)| : \zeta \in \mathbb{T}\} \to \infty$. And since, for all $\zeta \in \mathbb{T}$ we have

$$|B_n'(\zeta)| \geq \frac{1}{2}\sum_{j=1}^{n}(1-|\lambda_j|).$$

we see that if $\sum_j(1-|\lambda_j|) = \infty$ then $\|\Delta^{\alpha}_{B_n}\| \to 0$.

When (λ_j) is a Blaschke sequence, the situation is more complicated.

Example 1 :(WHEN SZEGO HOLDS) Let(λ_j) to be the sequence of points in $\mathbb{D}$ obtained by choosing on each circle of radius $r_m = 1 - \frac{1}{m^4}$ a number of m^2 equidistant points. Then (λ_j) satisfies the Blaschke condition, $(\sum(1 - |\lambda_i|) = \sum m^2(1/m^4) < \infty$ but a concrete calculation shows that $\|\Delta^{\alpha}_{B_n}\| \to 0$ is true, and so the conclusion of theorem 2 holds.

Example 1 :(WHEN SZEGO HOLDS) Let(λ_j) to be the sequence of points in $\mathbb{D}$ obtained by choosing on each circle of radius $r_m = 1 - \frac{1}{m^4}$ a number of m^2 equidistant points. Then (λ_j) satisfies the Blaschke condition, $(\sum(1-|\lambda_i|) = \sum m^2(1/m^4) < \infty$ but a concrete calculation shows that $\|\Delta^{\alpha}_{B_n}\| \to 0$ is true, and so the conclusion of theorem 2 holds.

Example 2 : A straightforward recursive construction gives a Blaschke sequence $(\sum(1-|\lambda_j|) < \infty)$, an integer N, and a function $\psi \in K_{B_N} + \overline{K_{B_N}}$ such that

$$\mathrm{Tr}(\Delta_{B_n} T_{B_n}[\psi]) \not\to \int \psi \, d\mu^B_\alpha.$$

In other words, if condition $\|\Delta^{\alpha}_{B_n}\| \to 0$ is not satisfied, then the assertion (ii) in the above theorem is not necessarily true.

Final comments - what's next ?

We would like to get some characterizations of Blaschke sequences for which Szego type results hold. And figure out how to analyze eigenvalues of big Toeplitz operators using our results. Finally, it would be interesting to obtain a Szego type result for singular inner functions.

Thanks for your attention!