Differentiating Absolutely Continuous Functions

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Overview

The background story . . .

- Today we will differentiate functions in nice Banach algebras . . .
- . . . these derivatives are better considered as living in bimodules
- We also consider higher cohomology, which generalise derivations

- For polynomials these higher cohomology groups often vanish
- For Banach algebras vanishing of cohomology is much less common
- Amenable Banach algebras are famous because $\mathcal{H}^n(A, X') = 0$
- Sadly even for $n = 1$, so you can't differentiate them

- When there are derivations they can sometimes be computed . . .
- . . . using bimodule maps from the Kähler module, Ω_A

- The Banach algebra A is called *amenable* if $\mathcal{H}^1(A; X') = 0$ for all dual bimodules X';
  - eponimously $L^1(G)$ is amenable iff G is an amenable l.c. group;
- The Banach algebra A is called *weakly amenable* if $\mathcal{H}^1(A; A') = 0$;
  - A' is a dual module, and so amenable algebras are weakly amenable;
  - for commutative algebras this is equivalent to the condition that $\mathcal{H}^1(A; Y) = 0$ for all commutative bimodules.
- Note, in both cases the bimodule A' plays a special role and this motivates the study of the higher cohomology of this module.
- We call $\mathcal{H}^n(A; A')$ the *simplicial cohomology* of A, and denote these groups by $\mathcal{HH}^n(A)$.

A toy example – $\ell^1(Z_+, \max)$

- In this talk we will consider analogues of the algebra

$$\ell^1(Z_+, \max) = \{f : f = \sum_{n=0}^{\infty} f_n \delta_n, \|f\|_1 < \infty\}$$

where the semigroup operation is given by $n \cdot m = \max(n, m)$

- Recall derivations into commutative bimodules vanish on idempotents $D(e) = D(e^2) = e \cdot De + De \cdot e = 2e \cdot De$ and so $(1 - 2e)De = 0$, but $1 - 2e$ is invertible (an involution) and so we have $De = 0$.
- Hence this algebra clearly is *weakly amenable*.
- It is slightly more difficult to show that it is *not amenable*.
- We will see that it is rather close to being amenable.

Justifying (Higher) Cohomology

We already believe in/appreciate

- *derivations*, which satisfy the 1-*cocycle equation* $(\delta D) = 0$

$$(\delta D)(a, b) := +a \cdot D(b) - D(ab) + D(a) \cdot b$$

- and *inner derivations* which are given as 1-*coboundaries* $\delta x := (a \mapsto a \cdot x - x \cdot a)$ which is a derivation.
- We may then be led to consider *approximate derivations* where

$$\|a \cdot D(b) - D(ab) + D(a) \cdot b\| \leq \epsilon \|a\| \|b\| ,$$

  which is just that $\|\delta D\| \leq \epsilon$.
- Many arguments with derivations work also with approximate derivations

$$(\delta D)(e, e) = e \cdot De - D(e^2) + De \cdot e = (2e - 1)De$$

  and so $De = -(1 - 2e)^{-1}(\delta D)(e, e)$,
- showing that a D can be recovered from δD on idempotents.

Definition of Higher Cohomology

- The approximate derivations $\phi := \delta D$, satisfy an equation known as the 2-*cocycle* identity, $\delta\phi = 0$, where

$$\delta\phi(a,b,c) = a\phi(b,c) - \phi(ab,c) + \phi(a,bc) - \phi(a,b)c$$

- General solutions to this equation are called 2-cocycles, $\mathcal{Z}^2(A;Y)$
- The 'obvious ones' (from $\phi := \delta D$) 2-coboundaries, $\mathcal{B}^2(A;Y)$
- We measure the gap by the *cohomology group* $\mathcal{H}^2(A;Y) := \frac{\mathcal{Z}^2(A;Y)}{\mathcal{B}^2(A;Y)}$

- More generally we define maps between spaces of multilinear maps from an algebra A into a bimodule Y, $\delta^n : L^n(A;Y) \to L^{n+1}(A;Y)$
- and introduce $\mathcal{Z}^n(A;Y)$, $\mathcal{B}^n(A;Y)$ and $\mathcal{H}^n(A;Y) = \frac{\mathcal{Z}^n(A;Y)}{\mathcal{B}^n(A;Y)}$
- We will be particularly interested in the special cases when Y is A'.

Simplicially Trivial Algebras

Looking for non-amenable algebras with trivial (simplicial) cohomology

- We call an algebra *simplicially trivial* if $\mathcal{HH}^n(A) = 0$ for $n \geq 1$.
- Clearly this is true of all amenable algebras, e.g. $\ell^1(Z, +)$.
- Also true for the commutative semilattice algebras, $\ell^1(S)$, where S is a commutative semigroup consisting of idempotents, [Y.Choi, 2006];
  - Note there are two extreme cases of such algebras *wide orders*, e.g. $S = 2^X$ with product given by union, and *deep orders* Z_+ with max as the product.
- The simplicial triviality result is true generally for (non-commutative) semigroups consisting of idempotents (so called *bands*) that

Theorem (YC, FMG, MCW, 2012)

Let B be a band semigroup then $\mathcal{HH}^n(\ell^1(B)) = 0$ for $(n \geq 1)$.

From ℓ^1 to L^1

- Blackmore considered simplicial derivations on $L^1(X, \mu)$
- It is these results which our work generalizes.

Theorem (Blackmore, 1997)

Let X is locally compact and totally ordered set equipped with a σ-finite, regular Borel positive measure. Then the algebra $L^1(X, \mu)$ is weakly amenable if and only if the continuous part of the measure is zero.

- Note: there are two extreme cases:
  - $\ell^1(Z_+, \max)$ which has no continuous part to the measure and, as we have seen, is weakly amenable;
  - $L^1(R_+, \max)$ which has only a continuous part to the measure, and so is not weakly amenable.
- In each case it is natural to ask about the higher simplicial cohomology groups $\mathcal{H}^n(A, A')$.

Absolutely Continuous Functions, a.k.a. $L^1(\mathbf{R}_+, \max)$

- At first it is not clear that there is a well defined product (worthy of the name max) on the Banach space $L^1(\mathbf{R}_+, \max)$
- In fact it is given by

$$(f * g)(x) = \int_0^x f(t)g(x)\,dt + \int_0^x f(x)g(t)\,dt,$$

  which (as expected) gives the values of $(f * g)(x)$ as an integral over the set of pairs mapping to x by the product map.
- The characters on this algebra are given by $\hat{f}(x) = \int_0^x f(t)\,dt \quad (0 < x \leq \infty)$ and with this we note that $(f * g)(x) = (\hat{f}g + f\hat{g})(x)$.
- Observe that the product looks rather like a derivation, *because* ...
- You should notice that the Gelfand transform has a (Radon-Nikodym) derivative, and $\hat{f}'(x) = f(x)$.

Pretending L^1 is ℓ^1

- Although the algebra $L^1(\mathbf{R}_+, \max)$ has few idempotents, unlike $\ell^1(Z_+, \max)$, we can still argue as if it did
- eg. consider disjointly supported positive functions with integral 1 (say $e_1 << e_2 << e_3$), then
- $e_1 e_2 = e_2$, etc and so
- $D(e_1 e_2)(e_3) = D(e_2)(e_3 e_1) + D(e_1)(e_2 e_3) = D(e_2)(e_3) + D(e_1)(e_3)$, hence $D(e_1)(e_3) = 0$;
- $D(e_3 e_2)(e_1) = D(e_2)(e_1 e_3) + D(e_3)(e_2 e_1) = D(e_2)(e_3) + D(e_3)(e_2)$, hence $D(e_3)(e_1) = D(e_3)(e_2)$;
- This leads one to suspect the result of Blackmore that the general form of a simplicial derivation on $L^1(R_+, \max)$ is

$$D(f)(g) = \int_0^\infty \int_{x \geq y \geq 0} f(x) g(y) \, dy \, t_D(x) \, dx$$

for some t_D in $L^\infty(R_+)$.

Resolutions – a quick review

- Cohomology can be computed in other ways, (like Ω_A for derivations)
- Instead of multilinear maps from A to Y, $L^n(A, Y) \cong L(\hat{\otimes}^n A, Y)$
- One can use other bimodules $\{P_n\}_{n=0}^{\infty}$, which behave like $\hat{\otimes}^n A$
- These modules fit together like the $\hat{\otimes}^n A$, and have a map like d
- $A \stackrel{d}{\leftarrow} P_0 \stackrel{d}{\leftarrow} P_1 \stackrel{d}{\leftarrow} P_2 \stackrel{d}{\leftarrow} \cdots$
- Importantly, this is *exact*, i.e. $\mathrm{Im}\ d = Ker\ d$
- Typically the P_n are well-behaved bimodules summands of $\hat{\otimes}^n A$
- We need to be particularly generous with which P_n we can use today
- Our P_n only need duals which are complemented in $(\hat{\otimes}^n A)'$

Theorem

The cohomology groups $\mathcal{H}^n(A; X')$ can be computed using any weakly *admissible biflat resolution of A.*

A Resolution for $\ell^1(Z_+, \max)$

We begin with the diagram

$$0 \leftarrow \ell^1(Z_+) \leftarrow \ell^1(Z_+ \times Z_+) \begin{array}{c} \swarrow \\ \\ \nwarrow \end{array} \begin{array}{c} \ell^1(Z_+ \times_{\geq} Z_+ \times Z_+) \\ \oplus \\ \ell^1(Z_+ \times Z_+ \times_{\leq} Z_+) \end{array} \leftarrow \cdots$$

and then explain the terms.

- Note that the bimodule denoted, $\ell^1(Z_+ \times_{\geq} Z_+ \times Z_+)$ is the image of the bimodule projection $(a, b, c) \mapsto (a \vee b, b, c)$ on $\ell^1(Z_+ \times Z_+ \times Z_+)$ So it is a bimodule summand and so good for us, i.e. biprojective
- We also require that the resolution is: admissible and exact. This is proved by the construction of a contracting homotopy. We set $s(\omega) = e_\omega \otimes \omega$. Then $ds(\omega) = d(e_\omega \otimes \omega) = \omega - e_\omega \otimes d\omega$ and $sd\omega = e_{d\omega} \otimes d\omega$, and hence $(sd + ds)(\omega) = \omega$ on decreasing terms, increasing terms follow similarly using right identities.

Exactness at a tricky place

$$0 \leftarrow \ell^1(Z_+) \leftarrow \ell^1(Z_+ \times Z_+) \begin{array}{c} \swarrow \\ \\ \nwarrow \end{array} \begin{array}{c} \ell^1(Z_+ \times_{\geq} Z_+ \times Z_+) \\ \oplus \\ \ell^1(Z_+ \times Z_+ \times_{\leq} Z_+) \end{array} \leftarrow \cdots$$

- Next we will (half) check exactness at $\ell^1(Z_+ \times Z_+)$:
- Note: terms like $a \otimes b - ab \otimes ab$ span the kernel of the product map δ
- $d(a \otimes a \otimes b) = a \otimes b - a \otimes ab = a \otimes b - ab \otimes ab$, if $ab = a$
- $ab = b$ is similar, but exactness at the next level is slightly longer

The resolution machine now gives us:

Theorem

$\mathcal{H}^n(\ell^1(Z_+, \max), Y) = 0$ *for commutative modules* Y *and* $n > 1$.

The corresponding resolution for $L^1(R_+, \max)$

We begin with the diagram

$$
\begin{array}{ccccccc}
 & & & & L^\infty(R_+ \times_{\geq} R_+ \times R_+) & \rightarrow & \cdots \\
 & & & \nearrow & \oplus & & \oplus \\
0 \rightarrow L^\infty(R_+) \rightarrow & & L^\infty(R_+ \times R_+) & \rightarrow & L^\infty(\hat{R}_+) & \rightarrow & 0 \\
 & & & \searrow & \oplus & & \oplus \\
 & & & & L^\infty(R_+ \times R_+ \times_{\leq} R_+) & \rightarrow & \cdots
\end{array}
$$

and again explain the terms.

- The first observation is that the diagram is more like the dual of the diagram for $\ell^1(Z_+, \max)$
- We need to pass to duals as there are no left identities to use for the contracting homotopy. These maps are now defines using left approximate units and limits

The Tricky Places

- Most of the checks for exactness are the usual ℓ^1 to L^1 changes
- However, computing the kernel of the map from $L^\infty(R_+)$, we get
- $F(x \vee y, z) = F(x, y \vee z)$, for $x \geq y$, OR $y \leq z$
- Giving $F(y, z) = F(x, z)$ for $x \leq y \leq z$,
- i.e., F is constant on horizontal lines above the diagonal
- Similarly, F is constant on vertical lines below the diagonal
- In $\ell^1(Z_+, \max)$ these lines met at (z, z) and so the function $F(x, y)$ factors through some $G(x \vee y)$
- HOWEVER, all the lines above should have said *almost everywhere*
- . . . and the diagonal in 'almost nowhere'
- So we are left with an F in the kernel of the two maps considered
- This is why we need the additional element in the resolution

$$0 \to L^\infty(R_+) \to L^\infty(R_+ \times R_+) \to L^\infty(\hat{R}_+) \to 0$$

Resolving the Tricky Place

- We have not yet stated the bimodule structure of $L^\infty(\hat{R}_+)$
- This is the Banach space $L^\infty(R_+)$, with (bi)module actions $(f.G)(x) = \hat{f}(x)G(x) = (G.f)(x)$
- This is the dual of the similarly defined bimodule $L^1(\hat{R}_+)$
- Before we define the map δ^0 in

$$0 \to L^\infty(R_+) \to L^\infty(R_+ \times R_+) \stackrel{\delta^0}{\to} L^\infty(\hat{R}_+) \to 0$$

  we need several definitions and computations
- The key property which we require of this δ^0 is:
  for a function constant on horizontal lines above and vertical lines below the diagonal, if $\delta^0(F) = 0$, then $F(x, y) = G(x \vee y)$.
- This shows that we have recovered exactness in the L^∞ case, at a small cost

Spreading maps and The Key Lemma

- We define a number of functions:
  $\Delta_\epsilon^N(f)$, $\Delta_\epsilon^S(f)$, $\Delta_\epsilon^W(f)$, $\Delta_\epsilon^E(f) : L^1(\hat{R_+}) \to L^1(R_+^2)$,
  which stretch out f in various directions, ('from the diagonal') e.g.,
  $$\Delta_\epsilon^N(f)(x,y) = f(x)\epsilon^{-1}\chi(x \le y \le x+\epsilon)(x,y)$$
- We set $\Delta_\epsilon = \Delta_\epsilon^N - \Delta_\epsilon^S$ and $\Delta'_\epsilon = \Delta_\epsilon^W - \Delta_\epsilon^E$.
- Now we observe that: $\lim_{\epsilon\to 0} \|\Delta_\epsilon(f) - \Delta'_\epsilon(f)\|_{L^1(R_+^2)} = 0$,
  as stretching North and West (resp. S and E) are almost equal.
- Putting this all together we have an 'almost bimodule map' property

Lemma

Let $f \in L^1(\hat{R_+})$ and $h \in L^1(R_+)$ and so $\Delta_\epsilon(\hat{h}f)$ and $\Delta_\epsilon(f)$ are in $L^1(R_+^2)$.

$$\lim_{\epsilon\to 0} \left\| h * \Delta_\epsilon(f) - \Delta_\epsilon(\hat{h}f) \right\|_{L^1(R_+^2)} = 0 = \lim_{\epsilon\to 0} \left\| \Delta_\epsilon(f) * h - \Delta_\epsilon(f\hat{h}) \right\|_{L^1(R_+^2)}$$

Our map δ_0 is a limit of the duals of the maps Δ_ϵ, and as such it genuinely a bimodule map

Cohomology of $L^1(R_+, \max)$

So what have we actually proved ?

We can use the above admissible resolution by biinjective modules to compute the cohomology of commutative dual modules.

Theorem

$\mathcal{H}^n(L^1(R_+, \max), X') = 0$ *for commutative dual modules* X' *and* $n > 1$.

The resolution also shows us that $\mathcal{H}^1(A, X') = \hom_{A^e}(X, L^\infty(\hat{R}_+))$.
It is in this sense that $L^1(\hat{R}_+)$ plays the role of the Kähler module, Ω_A

We are left wondering:
Which other Banach algebras have well-behaved Kähler modules?