Isometric weighted composition operators on weighted Bergman spaces

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Introduction

For $u, \phi \in H(\mathbb{D})$ and $\phi : \mathbb{D} \to \mathbb{D}$ non-constant, define a

Weighted Composition Operator (WCO) $W_{u,\phi} : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$W_{u,\phi} f = u(f \circ \phi)$$

deLeeuw, Rudin, Wermer (1960), Forelli (1964), Kolaski (1981):

Surjective isometries on the Hardy spaces H^p and Bergman spaces L^p_a when $p \neq 2$ are WCO.
(ϕ is an automorphism, and u is expressed via ϕ)

When $p = 2$ there are many other isometries (unitaries).

Weighted Bergman spaces $L^2_a(dm_\alpha)$ on $\mathbb{D}$:

$\alpha > -1, \quad dm_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dm(z),$

$$L^2_a(dm_\alpha) = \{f \in \mathcal{H}(\mathbb{D});\, ||f||^2_\alpha = \int_{\mathbb{D}} |f(z)|^2 dm_\alpha(z) < \infty\}$$

$\alpha = 0$ is the classical Bergman space $L^2_a(dm)$.

$L^2_a(dm_\alpha)$ are Reproducing Kernel Hilbert spaces:

$$K^\alpha(w,z) = \frac{1}{(1-\bar{z}w)^{2+\alpha}} = K^\alpha_z(w), \quad < f, K^\alpha_z >= f(z)$$

Normalized point evaluation functions at $z \in \mathbb{D}$:

$$k^\alpha_z(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}$$

Few facts about WCO $W_{u,\phi}$ on $L^2_a(dm_\alpha)$:

- Bounded, compact WCO determined by Čučković, Zhao (2004)
- If $u \in H^\infty(\mathbb{D})$, then $W_{u,\phi} = M_u C_\phi$ is bounded.
- Necessary condition for boundedness: $u \in L^2_a(dm_\alpha)$.
- $W^*_{u,\phi} K^\alpha_z = \overline{u(z)} K^\alpha_{\phi(z)}$
- Unitary WCO determined by Le (2012):

  $W_{u,\phi}$ is unitary on $L^2_a(dm_\alpha)$ iff ϕ is a disk automorphism and

$$u = c_1(\phi')^{1+\alpha/2} = c_2 \frac{1}{k^\alpha_{\phi(0)} \circ \phi}, \; |c_1| = |c_2| = 1.$$

Question: When is $W_{u,\phi}$ an isometry on $L^2_a(dm_\alpha)$?

- C_ϕ is an isometry on $L^2_a(dm_\alpha)$ iff ϕ is a rotation.
- M_u is an isometry $L^2_a(dm_\alpha)$ iff u is an unimodular constant.

Example

If ϕ is a finite Blaschke product of degree n and $u(z) = \frac{1}{\sqrt{n}}\phi'(z)$, then $W_{u,\phi}$ is an isometry on $L^2_a(dm)$.

- isometric WCO on H^2 determined by Matache (2014):

$W_{u,\phi}$ is an isometry on H^2 iff ϕ is an inner function and $u \in H^2$ is such that

$$\int_{\phi^{-1}(E)} |u(\xi)|^2 d\sigma(\xi) = \int_{\phi^{-1}(E)} \frac{1}{(P_{\phi(0)} \circ \phi)(\xi)} d\sigma(\xi),$$

for all $E \subset \partial\mathbb{D}$ measurable; with σ normalized Lebesgue measure on $\partial\mathbb{D}$; $P_{\phi(0)}$ Poisson kernel function at $\phi(0)$.

(Follows Forelli's characterization of isometric WCO on $H^p, p \neq 2$.)

Characterization of isometric WCO on $L^2_a(dm_\alpha)$

For α, u, ϕ as before and $E \subset \mathbb{D}$ Borel measurable, the u-weighted, ϕ pull-back measure of dm_α is defined by

$$\mu^\alpha_{u,\phi}(E) = \mu^\alpha_u(\phi^{-1}(E)) = \int_{\phi^{-1}(E)} |u(z)|^2 dm_\alpha(z)$$

$h^\alpha_{u,\phi}(z) = \frac{d\mu^\alpha_{u,\phi}}{dm_\alpha}(z)$ (Radon-Nikodym derivative)

Recall: for $h \in L^1(dm_\alpha)$ the Toeplitz operator T_h on $L^2_a(dm_\alpha)$ is

$$T_h f(z) = \int_{\mathbb{D}} h(w) f(w) \frac{1}{(1 - z\overline{w})^{2+\alpha}} dm_\alpha(w)$$

Proposition 1.

Let $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with $\phi : \mathbb{D} \to \mathbb{D}$ non-constant, such that $W_{u,\phi} : L^2_a(dm_\alpha) \to L^2_a(dm_\alpha)$ is bounded. Then $W^*_{u,\phi} W_{u,\phi} = T_{h^\alpha_{u,\phi}}$.

Recall (Proposition 1.) $W_{u,\phi}^* W_{u,\phi} = T_{h_{u,\phi}^\alpha}$.

Proof.

(i) Since $h_{u,\phi}^\alpha$ is a non-negative function, $T_{h_{u,\phi}^\alpha}$ is a positive operator. Thus, we need to show that $\forall f \in L_a^2(dm_\alpha)$, we have $< W_{u,\phi}^* W_{u,\phi} f, f > = < T_{h_{u,\phi}^\alpha} f, f >$. This holds since

$$\begin{aligned}
||W_{u,\phi} f||_\alpha^2 &= \int_{\mathbb{D}} |u(z)|^2 |f(\phi(z))|^2 dm_\alpha(z) \\
&= \int_{\mathbb{D}} |f(w)|^2 d\mu_{u,\phi}^\alpha(w) \\
&= \int_{\mathbb{D}} |f(w)|^2 h_{u,\phi}^\alpha(w) dm_\alpha(w) \\
&= < T_{h_{u,\phi}^\alpha} f, f > .
\end{aligned}$$

□

The Berezin transform of T_h on $L_a^2(dm_\alpha)$:

$$\widetilde{T_h}(z) = \tilde{h}(z) =< T_h k_z^\alpha, k_z^\alpha >= \int_{\mathbb{D}} h(w) \frac{(1-|z|^2)^{2+\alpha}}{|1-z\overline{w}|^{4+2\alpha}} dm_\alpha(w).$$

Theorem 1.

Let $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with ϕ a nonconstant self-map of $\mathbb{D}$ such that $W_{u,\phi} : L_a^2(dm_\alpha) \to L_a^2(dm_\alpha)$ is bounded. Then:

(i) $W_{u,\phi}$ is an isometry iff $h_{u,\phi}^\alpha = 1$ almost everywhere on $\mathbb{D}$.

(ii) If $W_{u,\phi}$ is an isometry, then $m(\mathbb{D} \setminus \phi(\mathbb{D})) = 0$.

(iii) $W_{u,\phi}$ is an isometry iff for all $z \in \mathbb{D}$,

$$\widetilde{h_{u,\phi}^\alpha}(z) = \int_{\mathbb{D}} |u(w)|^2 \frac{(1-|z|^2)^{2+\alpha}}{|1-z\overline{\phi(w)}|^{4+2\alpha}} dm_\alpha(w) = 1.$$

(iv) $W_{u,\phi}$ is an isometry iff $||W_{u,\phi} k_z^\alpha||_\alpha = 1$ for every z in $\mathbb{D}$.

The boundedness and compactness criteria for WCO on $L_a^2(dm_\alpha)$ given by Čučković, Zhao (2004) uses the integral from Theorem 1, part (iii), which they called "weighted ϕ-Berezin transform of $|u|^2$".

Also, Gallardo-Gutiérrez, Kumar, Partington (2010) showed that this leads to:
$sup_{z\in\mathbb{D}}||W_{u,\phi}k_z^\alpha||_\alpha < \infty \Leftrightarrow W_{u,\phi}$ is bounded on $L_a^2(dm_\alpha)$, and
$\lim_{|z|\to 1}||W_{u,\phi}k_z^\alpha||_\alpha \to 0 \Leftrightarrow W_{u,\phi}$ is compact on $L_a^2(dm_\alpha)$.

Question: Is there a more explicit description of the Radon-Nikodym derivative $h^\alpha_{u,\phi}(z) = \frac{d\mu^\alpha_{u,\phi}}{dm_\alpha}(z)$?

Proposition 2.

Let $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with $\phi : \mathbb{D} \to \mathbb{D}$ non-constant. If ϕ is of multiplicity bounded by N, then the Radon-Nikodym derivative of $\mu^\alpha_{u,\phi}$ with respect to m_α is given by $h^\alpha_{u,\phi}(z) = 0$, if $z \notin \phi(\mathbb{D})$, and otherwise

$$h^\alpha_{u,\phi}(z) = \sum_{n=1}^{N_z} \frac{|u(z_n)|^2(1-|z_n|^2)^\alpha}{|\phi'(z_n)|^2(1-|\phi(z_n)|^2)^\alpha},$$

where for each n, $\phi(z_n) = z$ and $\phi'(z_n) \neq 0$, and $N_z \leq N$.

Proposition 3.

Let $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with $\phi : \mathbb{D} \to \mathbb{D}$. If ϕ is univalent and $W_{u,\phi}$ is an isometry on $L_a^2(dm_\alpha)$, then $m(\mathbb{D} \setminus \phi(\mathbb{D})) = 0$, i.e. ϕ is a full map, and

$$u(z) = c\phi'(z)\frac{(1-\overline{\phi(0)}\phi(z))^\alpha}{(1-|\phi(0)|^2)^{\alpha/2}}, \quad |c| = 1.$$

Note that when $\alpha = 0$ above, then $u = c\phi'$, $|c| = 1$.

Example

Let ϕ be the Riemann map from $\mathbb{D}$ onto $\mathbb{D} \setminus [0,1)$, and let $u = \phi'$. Then $W_{u,\phi}$ is a non-unitary isometry on $L_a^2(dm)$.

Theorem 2.

(i) If ϕ *is a disk automorphism and* $W_{u,\phi}$ *is an isometry on* $L_a^2(dm_\alpha)$*, then* $u = c(\phi')^{1+\alpha/2}$, $|c| = 1$*, and so* $W_{u,\phi}$ *is unitary.*

(ii) If $\alpha = 0$ *and* ϕ *is a univalent full map, then* $W_{u,\phi}$ *is an isometry on* $L_a^2(dm)$ *iff* $u = c\phi'$, $|c| = 1$.

(iii) If $\alpha \neq 0$, ϕ *is univalent,* $\phi(0) = 0$ *and* $W_{u,\phi}$ *is an isometry on* $L_a^2(dm_\alpha)$*, then* ϕ *is a rotation.*

(iv) If $\alpha \neq 0$ *and* ϕ *is univalent, then* $W_{u,\phi}$ *is an isometry on* $L_a^2(dm_\alpha)$ *iff* $W_{u,\phi}$ *is unitary.*

"Geometric aspects" of the isometry criteria

Question(s): If $W_{u,\phi}$ is bounded on $L^2_a(dm_\alpha)$ and $\{z_n\}$ is such that $\phi(z_n) = z$ with $\phi'(z_n) \neq 0$, must

$$\sum_{n=1}^{\infty} \frac{|u(z_n)|^2(1-|z_n|^2)^\alpha}{|\phi'(z_n)|^2(1-|\phi(z_n)|^2)^\alpha} < \infty?$$

What is the "geometric meaning" of this condition?

If ϕ is of unbounded multiplicity and the series above converges for every $z \in \phi(\mathbb{D})$, then the series does represent $h^\alpha_{u,\phi}(z)$.

Note:

$$h^{\alpha}_{u,\phi}(z) = \sum_{n=1}^{\infty} \frac{|u(z_n)|^2}{|\phi'(z_n)|^{2+\alpha}} (\tau_\phi(z_n))^{\alpha} = \sum_{n=1}^{\infty} \frac{||W^*_{u,\phi} k^{\alpha}_{z_n}||^2_{\alpha}}{\tau_\phi(z_n)^2},$$

where $\tau_\phi(z)$ is the local hyperbolic distortion of ϕ at z:

$$\tau_\phi(z) = \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|^2}.$$

- $\tau_\phi(z) \leq 1$, $\forall z \in \mathbb{D}$ (Schwarz-Pick lemma)
- ϕ is a disk automorphism iff $\exists a \in \mathbb{D}, \tau_\phi(a) = 1$
- ϕ is a finite Blaschke product iff $\lim_{|z|\to 1} \tau_\phi(z) = 1$
- If ϕ has an angular derivative at $\xi \in \partial\mathbb{D}$, then $\tau_\phi(z) \to 1$ as $z \to \xi$ nontangentially.

Hence, if ϕ is of infinite multiplicity and the series above converges, then $\frac{||W^*_{u,\phi} k^\alpha_{z_n}||_\alpha}{\tau_\phi(z_n)} \to 0, n \to \infty$,

(and so furthermore $||W^*_{u,\phi} k^\alpha_{z_n}||^2_\alpha = \frac{|u(z_n)|^2(1-|z_n|^2)^{2+\alpha}}{(1-|\phi(z_n)|^2)^{2+\alpha}} \to 0, n \to \infty$)

Note: $||W^*_{u,\phi} k^\alpha_z||^2_\alpha = \frac{|u(z)|^2(1-|z|^2)^{2+\alpha}}{(1-|\phi(z)|^2)^{2+\alpha}} \to 0, |z| \to 1$ does not even guarantee the boundedness of $W_{u,\phi}$.

Example

$\alpha = 0$, ϕ an infinite Blaschke product in $\mathcal{B}^h_0$ and $u = \phi'$.

Question: If $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, must ϕ be of finite multiplicity (for some α's)?

For $\alpha > 0$, $L_a^2(dm_\alpha) = L_a^2(dA_\alpha)$ with $dA_\alpha(z) = c_\alpha(\log \frac{1}{|z|})^\alpha$.
If $u = \phi'$ and $\{z_n\}$ is such that $\phi(z_n) = z \in \phi(\mathbb{D}) \setminus \{\phi(0)\}$, then

$$\frac{d\check{\mu}^\alpha_{u,\phi}}{dA_\alpha}(z) = \check{h}^\alpha_{u,\phi}(z) = \sum_{n=1}^{\infty} \frac{(\log 1/|z_n|)^\alpha}{(\log 1/|\phi(z_n)|)^\alpha} = \frac{N_{\phi,\alpha}(z)}{(\log 1/|z|)^\alpha}$$

where $N_{\phi,\alpha}(z)$ is the α-Nevanlinna counting function for ϕ.

Recall: If $\alpha = 1$ and ϕ is inner, then $N_{\phi,1}(z) = \log \frac{1}{|\psi_{\phi(0)}(z)|}$, except possibly on a set of logarithmic capacity zero.

Example

Take $\alpha = 1$, ϕ inner with $\phi(0) = 0$, and $u = \phi'$. Then $\check{h}^1_{u,\phi}(z) = \frac{N_{\phi,1}(z)}{\log 1/|z|} = 1$ a.e., and $W_{u,\phi}$ is an isometry on $L_a^2(dA_1)$.

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