Isometric weighted composition operators on weighted Bergman spaces

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For $u, \phi \in H(\mathbb{D})$ and $\phi : \mathbb{D} \to \mathbb{D}$ non-constant, define a

Weighted Composition Operator (WCO) $W_{u,\phi} : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$W_{u,\phi}f = u(f \circ \phi)$$


Surjective isometries on the Hardy spaces $H^p$ and Bergman spaces $L^p_\alpha$ when $p \neq 2$ are WCO.

($\phi$ is an automorphism, and $u$ is expressed via $\phi$)

When $p = 2$ there are many other isometries (unitaries).
Weighted Bergman spaces $L^2_a(dm_\alpha)$ on $\mathbb{D}$:

$\alpha > -1, \quad dm_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dm(z),$

$L^2_a(dm_\alpha) = \{ f \in \mathcal{H}(\mathbb{D}); \| f \|^2_\alpha = \int_\mathbb{D} |f(z)|^2 dm_\alpha(z) < \infty \}$

$\alpha = 0$ is the classical Bergman space $L^2_a(dm)$.

$L^2_a(dm_\alpha)$ are Reproducing Kernel Hilbert spaces:

$K_\alpha^\alpha(w, z) = \frac{1}{(1-zw)^{2+\alpha}} = K_z^\alpha(w), \quad < f, K_z^\alpha > = f(z)$

Normalized point evaluation functions at $z \in \mathbb{D}$:

$k_z^\alpha(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-zw)^{2+\alpha}}$
Few facts about WCO $W_{u,\phi}$ on $L^2_a(dm_\alpha)$:

- Bounded, compact WCO determined by Čučković, Zhao (2004)
- If $u \in H^\infty(\mathbb{D})$, then $W_{u,\phi} = M_u C_\phi$ is bounded.
- Necessary condition for boundedness: $u \in L^2_a(dm_\alpha)$.
- $W^*_{u,\phi} K_\phi^\alpha z = \overline{u(z)} K_\phi^\alpha (\phi(z))$
- Unitary WCO determined by Le (2012):

$W_{u,\phi}$ is unitary on $L^2_a(dm_\alpha)$ iff $\phi$ is a disk automorphism and

$$u = c_1 (\phi')^{1+\alpha/2} = c_2 \frac{1}{k_\phi^\alpha \circ \phi}, \quad |c_1| = |c_2| = 1.$$

Question: When is $W_{u,\phi}$ an isometry on $L^2_a(dm_\alpha)$?

- $C_\phi$ is an isometry on $L^2_a(dm_\alpha)$ iff $\phi$ is a rotation.
- $M_u$ is an isometry $L^2_a(dm_\alpha)$ iff $u$ is an unimodular constant.

Example

If $\phi$ is a finite Blaschke product of degree $n$ and $u(z) = \frac{1}{\sqrt{n}} \phi'(z)$, then $W_{u,\phi}$ is an isometry on $L^2_a(dm)$. 
• isometric WCO on $H^2$ determined by Matache (2014):

$W_{u,\phi}$ is an isometry on $H^2$ iff $\phi$ is an inner function and $u \in H^2$ is such that

$$
\int_{\phi^{-1}(E)} |u(\xi)|^2 d\sigma(\xi) = \int_{\phi^{-1}(E)} \frac{1}{(P_{\phi(0)} \circ \phi)(\xi)} d\sigma(\xi),
$$

for all $E \subset \partial \mathbb{D}$ measurable; with $\sigma$ normalized Lebesgue measure on $\partial \mathbb{D}$; $P_{\phi(0)}$ Poisson kernel function at $\phi(0)$.

(Follows Forelli’s characterization of isometric WCO on $H^p$, $p \neq 2$.)
Characterization of isometric WCO on $L^2_a(dm_\alpha)$

For $\alpha$, $u$, $\phi$ as before and $E \subset \mathbb{D}$ Borel measurable, the $u$-weighted, $\phi$ pull-back measure of $dm_\alpha$ is defined by

$$\mu^\alpha_{u,\phi}(E) = \mu^\alpha_u(\phi^{-1}(E)) = \int_{\phi^{-1}(E)} |u(z)|^2 dm_\alpha(z)$$

$$h^\alpha_{u,\phi}(z) = \frac{d\mu^\alpha_{u,\phi}}{dm_\alpha}(z) \quad \text{(Radon-Nikodym derivative)}$$

Recall: for $h \in L^1(dm_\alpha)$ the Toeplitz operator $T_h$ on $L^2_a(dm_\alpha)$ is

$$T_h f(z) = \int_{\mathbb{D}} h(w)f(w)\frac{1}{(1-z\bar{w})^{2+\alpha}} dm_\alpha(w)$$

**Proposition 1.**

Let $\alpha > -1$, $u$, $\phi \in \mathcal{H}(\mathbb{D})$ with $\phi : \mathbb{D} \to \mathbb{D}$ non-constant, such that $W_{u,\phi} : L^2_a(dm_\alpha) \to L^2_a(dm_\alpha)$ is bounded. Then $W^*_u W_{u,\phi} = T_{h^\alpha_{u,\phi}}$. 
Recall (Proposition 1.) $W_{u,\phi}^* W_{u,\phi} = T_{h_{u,\phi}^\alpha}$.

Proof.

(i) Since $h_{u,\phi}^\alpha$ is a non-negative function, $T_{h_{u,\phi}^\alpha}$ is a positive operator. Thus, we need to show that $\forall f \in L^2_a(dm_{\alpha})$, we have $< W_{u,\phi}^* W_{u,\phi} f, f > = < T_{h_{u,\phi}^\alpha} f, f >$. This holds since

$$\|W_{u,\phi} f\|_{\alpha}^2 = \int_D |u(z)|^2 |f(\phi(z))|^2 dm_{\alpha}(z)$$

$$= \int_D |f(w)|^2 d\mu_{u,\phi}(w)$$

$$= \int_D |f(w)|^2 h_{u,\phi}^\alpha(w) dm_{\alpha}(w)$$

$$= < T_{h_{u,\phi}^\alpha} f, f > .$$
The Berezin transform of $T_h$ on $L^2_a(dm_\alpha)$:

$$\widetilde{T}_h(z) = \tilde{h}(z) = \langle T_h k^\alpha_z, k^\alpha_z \rangle = \int_\mathbb{D} h(w) \frac{(1-|z|^2)^{2+\alpha}}{|1-z\overline{w}|^{4+2\alpha}} dm_\alpha(w).$$

**Theorem 1.**

Let $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with $\phi$ a nonconstant self-map of $\mathbb{D}$ such that $W_{u,\phi} : L^2_a(dm_\alpha) \to L^2_a(dm_\alpha)$ is bounded. Then:

(i) $W_{u,\phi}$ is an isometry iff $h^\alpha_{u,\phi} = 1$ almost everywhere on $\mathbb{D}$.

(ii) If $W_{u,\phi}$ is an isometry, then $m(\mathbb{D} \setminus \phi(\mathbb{D})) = 0$.

(iii) $W_{u,\phi}$ is an isometry iff for all $z \in \mathbb{D}$,

$$\widetilde{h}^\alpha_{u,\phi}(z) = \int_\mathbb{D} |u(w)|^2 \frac{(1-|z|^2)^{2+\alpha}}{|1-z\phi(w)|^{4+2\alpha}} dm_\alpha(w) = 1.$$

(iv) $W_{u,\phi}$ is an isometry iff $\|W_{u,\phi}k^\alpha_z\|_\alpha = 1$ for every $z$ in $\mathbb{D}$. 
The boundedness and compactness criteria for WCO on $L^2_a(dm_\alpha)$ given by Čučković, Zhao (2004) uses the integral from Theorem 1, part (iii), which they called "weighted $\phi$-Berezin transform of $|u|^2$".

Also, Gallardo-Gutiérrez, Kumar, Partington (2010) showed that this leads to:

$$\sup_{z \in \mathbb{D}} ||W_{u,\phi}k_z^\alpha||_\alpha < \infty \iff W_{u,\phi} \text{ is bounded on } L^2_a(dm_\alpha), \text{ and}$$

$$\lim_{|z| \to 1} ||W_{u,\phi}k_z^\alpha||_\alpha \to 0 \iff W_{u,\phi} \text{ is compact on } L^2_a(dm_\alpha).$$
Question: Is there a more explicit description of the Radon-Nikodym derivative \( h_{u,\phi}^\alpha(z) = \frac{d\mu_{u,\phi}^\alpha}{dm_\alpha}(z) \)?

Proposition 2.

Let \( \alpha > -1 \), \( u, \phi \in \mathcal{H}(\mathbb{D}) \) with \( \phi : \mathbb{D} \to \mathbb{D} \) non-constant. If \( \phi \) is of multiplicity bounded by \( N \), then the Radon-Nikodym derivative of \( \mu_{u,\phi}^\alpha \) with respect to \( m_\alpha \) is given by \( h_{u,\phi}^\alpha(z) = 0 \), if \( z \notin \phi(\mathbb{D}) \), and otherwise

\[
h_{u,\phi}^\alpha(z) = \sum_{n=1}^{N_z} \frac{|u(z_n)|^2(1 - |z_n|^2)^\alpha}{|\phi'(z_n)|^2(1 - |\phi(z_n)|^2)^\alpha},
\]

where for each \( n \), \( \phi(z_n) = z \) and \( \phi'(z_n) \neq 0 \), and \( N_z \leq N \).
Proposition 3.

Let $\alpha > -1$, $u, \phi \in \mathcal{H}(\mathbb{D})$ with $\phi : \mathbb{D} \to \mathbb{D}$. If $\phi$ is univalent and $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, then $m(\mathbb{D} \setminus \phi(\mathbb{D})) = 0$, i.e. $\phi$ is a full map, and

$$u(z) = c\phi'(z)\frac{(1 - \overline{\phi(0)}\phi(z))^{\alpha}}{(1 - |\phi(0)|^2)^{\alpha/2}}, \quad |c| = 1.$$

Note that when $\alpha = 0$ above, then $u = c\phi'$, $|c| = 1$.

Example

Let $\phi$ be the Riemann map from $\mathbb{D}$ onto $\mathbb{D} \setminus [0,1)$, and let $u = \phi'$. Then $W_{u,\phi}$ is a non-unitary isometry on $L^2_a(dm)$. 
Theorem 2.

(i) If $\phi$ is a disk automorphism and $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, then $u = c(\phi')^{1+\alpha/2}$, $|c| = 1$, and so $W_{u,\phi}$ is unitary.

(ii) If $\alpha = 0$ and $\phi$ is a univalent full map, then $W_{u,\phi}$ is an isometry on $L^2_a(dm)$ iff $u = c\phi'$, $|c| = 1$.

(iii) If $\alpha \neq 0$, $\phi$ is univalent, $\phi(0) = 0$ and $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$, then $\phi$ is a rotation.

(iv) If $\alpha \neq 0$ and $\phi$ is univalent, then $W_{u,\phi}$ is an isometry on $L^2_a(dm_\alpha)$ iff $W_{u,\phi}$ is unitary.
Question(s): If $W_{u, \phi}$ is bounded on $L^2_a(d\alpha)$ and $\{z_n\}$ is such that $\phi(z_n) = z$ with $\phi'(z_n) \neq 0$, must

$$
\sum_{n=1}^{\infty} \frac{|u(z_n)|^2(1 - |z_n|^2)^\alpha}{|\phi'(z_n)|^2(1 - |\phi(z_n)|^2)^\alpha} < \infty?
$$

What is the ”geometric meaning” of this condition?

If $\phi$ is of unbounded multiplicity and the series above converges for every $z \in \phi(\mathbb{D})$, then the series does represent $h^\alpha_{u,\phi}(z)$. 
Note:

\[ h_{u,\phi}(z) = \sum_{n=1}^{\infty} \frac{|u(z_n)|^2}{|\phi'(z_n)|^{2+\alpha}} (\tau_{\phi}(z_n))^{\alpha} = \sum_{n=1}^{\infty} \frac{||W_{u,\phi} k_{z_n}^{\alpha}||^2}{\tau_{\phi}(z_n)^2}, \]

where \( \tau_{\phi}(z) \) is the local hyperbolic distortion of \( \phi \) at \( z \):

\[ \tau_{\phi}(z) = \frac{|\phi'(z)|(1 - |z|^2)}{1 - |\phi(z)|^2}. \]

- \( \tau_{\phi}(z) \leq 1, \ \forall z \in \mathbb{D} \) (Schwarz-Pick lemma)
- \( \phi \) is a disk automorphism iff \( \exists a \in \mathbb{D}, \tau_{\phi}(a) = 1 \)
- \( \phi \) is a finite Blaschke product iff \( \lim_{|z| \to 1} \tau_{\phi}(z) = 1 \)
- If \( \phi \) has an angular derivative at \( \xi \in \partial \mathbb{D} \), then \( \tau_{\phi}(z) \to 1 \) as \( z \to \xi \) nontangentially.
Hence, if $\phi$ is of infinite multiplicity and the series above converges, then \[ \lim_{n \to \infty} \frac{||W_{u, \phi}^* k_{\alpha}^n||}{\tau_{\phi}(z_n)} = 0, \]

(and so furthermore \[ ||W_{u, \phi}^* k_{\alpha}^n||^2 = \frac{|u(z_n)|^2(1-|z_n|^2)^2+\alpha}{(1-|\phi(z_n)|^2)^2+\alpha} \to 0, \ n \to \infty \]

Note: \[ ||W_{u, \phi}^* k_{\alpha}^n||^2 = \frac{|u(z)|^2(1-|z|^2)^2+\alpha}{(1-|\phi(z)|^2)^2+\alpha} \to 0, \ |z| \to 1 \]
does not even guarantee the boundedness of $W_{u, \phi}$.

**Example**

$\alpha = 0$, $\phi$ an infinite Blaschke product in $\mathcal{B}_0^h$ and $u = \phi'$.

**Question:** If $W_{u, \phi}$ is an isometry on $L_a^2(dm_{\alpha})$, must $\phi$ be of finite multiplicity (for some $\alpha$'s)?
For $\alpha > 0$, $L_2^a(dm_\alpha) = L_2^a(dA_\alpha)$ with $dA_\alpha(z) = c_\alpha(\log \frac{1}{|z|})^\alpha$.

If $u = \phi'$ and $\{z_n\}$ is such that $\phi(z_n) = z \in \phi(\mathbb{D}) \setminus \{\phi(0)\}$, then

$$
\frac{d\tilde{\mu}_{u,\phi}}{dA_\alpha}(z) = \tilde{h}_{u,\phi}(z) = \sum_{n=1}^{\infty} \frac{(\log \frac{1}{|z_n|})^\alpha}{(\log \frac{1}{|\phi(z_n)|})^\alpha} = \frac{N_{\phi,\alpha}(z)}{(\log \frac{1}{|z|})^\alpha}
$$

where $N_{\phi,\alpha}(z)$ is the $\alpha$-Nevanlinna counting function for $\phi$.

Recall: If $\alpha = 1$ and $\phi$ is inner, then $N_{\phi,1}(z) = \log \frac{1}{|\psi_{\phi}(0)(z)|}$, except possibly on a set of logarithmic capacity zero.

**Example**

Take $\alpha = 1$, $\phi$ inner with $\phi(0) = 0$, and $u = \phi'$. Then

$$
\tilde{h}_{u,\phi}^1(z) = \frac{N_{\phi,1}(z)}{\log 1/|z|} = 1 \text{ a.e., and } W_{u,\phi} \text{ is an isometry on } L_2^a(dA_1).
$$
References


